STANDARD MODULES AND INTERTWINING OPERATORS FOR REDUCTIVE p-ADIC GROUPS

Maarten Solleveld
IMAPP, Radboud Universiteit Nijmegen
Heyendaalseweg 135, 6525AJ Nijmegen, the Netherlands
email: m.solleveld@science.ru.nl

Abstract. Consider a reductive group G over a non-archimedean local field. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on the category of smooth complex G-representations. We prove that this action stabilizes the class of standard $\mathbb{C}G$-modules. This generalizes and relies on an analogous result from [KSV] about essentially square-integrable representations.

Other important objects in the proof of our main result are intertwining operators between parabolically induced G-representations, and the associated Harish-Chandra μ-functions. We determine an explicit formula for the μ-function of any irreducible representation of any Levi subgroup of G.

Contents

Date: December 26, 2025.
2010 *Mathematics Subject Classification.* 20G25, 22E50, 22E30.

1. Introduction

This paper is a sequel to [KSV]. That project started with the question: which classes of representations of reductive p-adic groups G are stable under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$? By default, the representations that we consider here are smooth and on complex vector spaces. The motivation for such questions is twofold.

Firstly, it relates to L-functions. One may hope to prove statements of the kind

$$\text{if } L(s,\pi)=0 \text{ for some } s\in\frac{1}{2}\mathbb{Z}, \text{ then } L(s,\gamma\cdot\pi)=0 \text{ for } \gamma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q}).$$

This could apply to representations π of reductive groups over local fields or of adelic reductive groups (and of course one needs reductive groups for which L-functions of irreducible representations are defined). For general linear groups, this has been studied in [KrCl].

Secondly, in algebro-geometric investigations related to reductive p-adic groups it is often beneficial to use representations not over $\mathbb{C}$ but over $\overline{\mathbb{Q}_\ell}$ for a prime number $\ell\neq p$. Here we are thinking in particular of the Fargues–Scholze program [FaSc], of the generalized Springer correspondence [Lus1] and of geometric graded Hecke algebras [AMS]. One may wonder whether certain results about $\mathbb{C}$-representations obtained via $\overline{\mathbb{Q}_\ell}$-representations depend on ℓ or on the choice of a field isomorphism $\mathbb{C}\cong\overline{\mathbb{Q}_\ell}$. Any two such field isomorphisms differ by composition with an element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, so one wants to understand which properties of $\mathbb{C}$-representations preserved by $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

It is clear that the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on G-representations preserves irreducibility, and it is easy to see that it preserve cuspidality. However, this action does in general not preserve analytic notions like unitarity, temperedness or square-integrability modulo center. The main results of [KSV] say that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ stabilizes

- the class of essentially square-integrable G-representations,
- the class of elliptic (virtual) G-representations.

In this paper we focus on a larger class of representations, that of standard $\mathbb{C}G$-modules. Let $Q=MU_Q$ be a parabolic subgroup of G and let τ be an irreducible tempered M-representation. Let $\nu\in\mathrm{Hom}(M,\mathbb{R}_{>0})$ be strictly positive with respect to Q (for the precise condition see Section 4). By definition, a standard $\mathbb{C}G$-module is a G-representation of the form $I_Q^G(\tau\otimes\nu)$, with (Q,τ,ν) as above. The importance of standard modules stems from the Langlands classification (which for p-adic groups is not due to Langlands):

Theorem A. [Ren, §VII.4]

(*a*) *Every standard $\mathbb{C}G$-module $I_Q^G(\tau\otimes\nu)$ has a unique irreducible quotient, which we call $\mathcal{L}(Q,\tau\otimes\nu)$.*

(*b*) *Every irreducible G-representation π arises as the quotient of a standard $\mathbb{C}G$-module π_{st}.*

(*c*) *If $I_{Q'}^G(\tau'\otimes\nu')$ is a standard module and $\mathcal{L}(Q,\tau\otimes\nu)\cong\mathcal{L}(Q',\tau'\otimes\nu')$, then there exists a $g\in G$ such that $gQg^{-1}=Q'$, $gMg^{-1}=M'$ and $g(\tau\otimes\nu)\cong\tau'\otimes\nu'$.*

(*d*) *The maps $I_Q^G(\tau\otimes\nu)\mapsto\mathcal{L}(Q,\tau\otimes\nu)$ and $\pi\mapsto\pi_{st}$ set up a bijection between $\mathrm{Irr}(\mathbb{C}G)$ and the set of standard $\mathbb{C}G$-modules (up to isomorphism).*

(*e*) *The set of standard $\mathbb{C}G$-modules (up to isomorphism) forms a $\mathbb{Z}$-basis of the Grothendieck group of the category of finite length G-representations.*

It is expected that in categorical versions of the local Langlands correspondence, standard modules behave better than irreducible G-representations. The reason should be that non-elliptic standard modules always come in families (because ν can vary continuously), which does not hold for irreducible representations.

1.1. Main results.

Theorem B. *The action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *on the category of smooth complex G-representations stabilizes the class of standard $\mathbb{C}G$-modules.*

Theorem B enables us to define standard $\overline{\mathbb{Q}_\ell}G$-modules in an unambiguous way. Namely, we call a G-representation π_ℓ on a $\overline{\mathbb{Q}_\ell}$-vector space standard if the complex G-representation obtained from π_ℓ via any field isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$ is a standard $\mathbb{C}G$-module.

Essential ingredients for Theorem B are Harish-Chandra's intertwining operators

$$J_{P'|P}(\pi) : I_P^G(\pi) \to I_{P'}^G(\pi) \quad \text{for finite length } L\text{-representations } \pi.$$

In fact we need more properties than can be found in the literature, so we further develop the theory of intertwining operators. Let $\pi \in \mathrm{Irr}(L)$ be an irreducible L-representation. The invertibility of $J_{P'|P}(\pi)$ is governed by Harish-Chandra's μ-function $\mu_{G,L}(\pi)$ [Wal]. More precisely, $\mu_{G,L}(\pi \otimes \chi)$ is a rational function of an unramified character $\chi \in X_{\mathrm{nr}}(L)$, and $J_{P'|P}(\pi)$ is invertible if $\mu_{G,L}(\pi) \in \mathbb{C}^\times$. Usually $J_{P'|P}(\pi)$ is not invertible if $\mu_{G,L}(\pi) \in \{0, \infty\}$.

Theorem C. (see Proposition 3.3 and Theorem 3.6)
Let $M \subset L$ be a Levi subgroup and let $\sigma \in \mathrm{Irr}(M)$ be such that $\pi \in \mathrm{Irr}(L)$ is a subquotient of $I_{MU}^L(\sigma)$, for some parabolic subgroup MU of L.

(a) *There exists an explicit $c \in \mathbb{R}_{>0}$ such that*

$$\mu_{G,L}(\pi \otimes \chi) = c\, \mu_{G,M}(\sigma \otimes \chi) \mu_{L,M}(\sigma \otimes \chi)^{-1} \qquad \chi \in X_{\mathrm{nr}}(L).$$

(b) *Suppose in addition that σ is cuspidal. Then*

$$\mu_{G,L}(\pi \otimes \chi) = c \prod\nolimits_{M_\alpha} \mu_{M_\alpha,M}(\sigma \otimes \chi) \qquad \chi \in X_{\mathrm{nr}}(L),$$

where the product runs over the Levi subgroups $M_\alpha \subset G$ which contain M as minimal Levi subgroup but are not contained in L. Moreover each term $\mu_{M_\alpha,M}(\sigma \otimes \chi)$ admits an explicit formula as a rational function of χ.

1.2. Structure of the main proof.

The initial step towards Theorem B is an alternative construction of standard modules, from [Sol1]. Let $P = LU_P$ be a parabolic subgroup of G and let δ be an irreducible essentially square-integrable L-representation. We say that δ is positive with respect to P if the absolute value of the central character of δ is so. In that case $I_P^G(\delta)$ is a direct sum of standard $\mathbb{C}G$-modules $I_P^G(\delta)_\kappa$. (See Paragraph 3.4 for meaning of κ.) Moreover every standard $\mathbb{C}G$-module arises in this way, from essentially unique (P, L, δ).

Without the positivity condition on δ, $I_P^G(\delta)$ is a direct sum of so-called quasi-standard $\mathbb{C}G$-modules $I_P^G(\delta)_\kappa$ (Definition 4.2). Any quasi-standard $\mathbb{C}G$-module $I_P^G(\delta)_\kappa$ can be made into a standard $\mathbb{C}G$-module by adjusting P, but in general that changes the isomorphism class of the module. Since $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ preserves essential square-integrability [KSV], $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ stabilizes the class of quasi-standard $\mathbb{C}G$-modules (Lemma 5.4).

From this point on we present two proofs of Theorem B, both of interest in their own way. The first method relies on an invariant $\mathcal{N}$ of G-representations π, which measures a distance from π to the set of parabolic inductions of unitary cuspidal representations of Levi subgroups of G (see Paragraph 4.2). It is known from [Sol1] that $\mathcal{L}(Q, \tau \otimes \nu)$ is the unique irreducible subquotient of $I_Q^G(\tau \otimes \nu)$ which has the same $\mathcal{N}$-value as $I_Q^G(\tau \otimes \nu)$. This enables us to characterize standard $\mathbb{C}G$-modules as those quasi-standard $\mathbb{C}G$-modules which have an irreducible quotient with the appropriate $\mathcal{N}$-value (Theorem 4.9). In contrast to the original definition, this characterization of standard modules uses neither temperedness not positivity of characters.

We show that this configuration is preserved when we let any $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ act on a standard $\mathbb{C}G$-module. That leads to our first proof of Theorem B, in Proposition 5.5. However, this proof is conditional: we assume that γ preserves the $\mathcal{N}$-values of all essentially square-integrable representations of Levi subgroups of G. That property is not yet known, but it follows from the rationality of q-parameters for related Hecke algebras. Such rationality has been conjectured by Lusztig [Lus2], and has been checked in the large majority of all cases [Sol3, Oha].

Our second proof of Theorem B uses that the parabolic subgroup P in a quasi-standard $\mathbb{C}G$-module $I_P^G(\delta)_\kappa$ is often not unique. Namely, for any other parabolic subgroup P' with the same Levi factor L, there exists an intertwining operator

$$J_{P'|P}(\delta) : I_P^G(\delta) \to I_{P'}^G(\delta). \tag{1}$$

Under mild conditions (1) is an isomorphism, which entails that $I_P^G(\delta)_\kappa$ is isomorphic to a quasi-standard direct summand of $I_{P'}^G(\delta)$.

With Theorem C one can reduce questions about intertwining operators and μ-functions to the cases of cuspidal representations, which can be analysed more easily. For instance, consider the corank one intertwining operator $J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi)$, where $MU_{-\alpha}$ and MU_α are the parabolic subgroups of M_α with Levi factor M. It was already known that, if $\mu_{M_\alpha,M}(\sigma) = 0$, then $J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi)$ can be normalized to an operator

$$J'_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi) : I_P^G(\sigma \otimes \chi) \to I_{P'}^G(\sigma \otimes \chi),$$

which is invertible for χ in a neighborhood of 1 in $X_{\mathrm{nr}}(M)$. In particular $I_{MU_{-\alpha}}^{M_\alpha}(\sigma)$ is isomorphic to $I_{MU_\alpha}^{M_\alpha}(\sigma)$ whenever $\mu_{M_\alpha,M}(\sigma) \neq \infty$. More generally, we prove in Theorem 3.9 that (with the notations from Theorem C)

$$I_P^G(\pi) \cong I_{P'}^G(\pi) \text{ unless } \mu_{M_\alpha,M}(\sigma) = \infty \text{ for some } M_\alpha \text{ with } P \supset MU_\alpha \not\subset P'. \tag{2}$$

This is used in our second proof of Theorem B.

For a given $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and (P, L, δ) as above, we construct a particular $P' = LU_{P'}$ such that $\gamma \cdot \delta$ is positive with respect to P'. An explicit analysis of the corank one situation (Proposition 4.5) reveals an asymmetry between $I_{MU_\alpha}^{M_\alpha}(\sigma)$ and $I_{MU_{-\alpha}}^{M_\alpha}(\sigma)$ when $\mu_{M_\alpha,M}(\sigma) = \infty$, the roles of the unique quotient and the unique subrepresentation differ. Using that with as σ a representative of the cuspidal support of δ, we can arrange that all the M_α with $\mu_{M_\alpha,M}(\sigma) = \infty$ satisfy $MU_\alpha \subset P \cap P'$ or $MU_{-\alpha} \subset P \cap P'$. With (2), it follows that a normalized version $J'_{P'|P}(\delta)$ of $J_{P'|P}(\delta)$ gives isomorphisms

$$I_P^G(\delta) \cong I_{P'}^G(\delta) \quad \text{and} \quad \gamma \cdot I_P^G(\delta) \cong \gamma \cdot I_{P'}^G(\delta). \tag{3}$$

From there, we show in Theorem 5.8 that $\gamma \cdot I_P^G(\delta)_\kappa$ is a standard $\mathbb{C}G$-module.

2. Notations

F: non-archimedean local field
G: F-rational points of a connected reductive group $\mathcal{G}$ defined over F
G_{der}: F-rational points of derived subgroup of $\mathcal{G}$
$Z(G)$: center of G
A_G: maximal F-split torus in $Z(G)$
$\text{Rep}(G)$: category of smooth complex G-representations
$\text{Irr}(G)$: set of irreducible objects in $\text{Rep}(G)$, up to isomorphism
G^1: subgroup of G generated by all compact subgroups of G
$X_{\text{nr}}(G) = \text{Hom}(G/G^1, \mathbb{C}^\times)$: group of unramified characters of G
L, M: Levi subgroups of G
$P = LU_P$: parabolic subgroup of G with Levi factor L and unipotent radical U_P
$\overline{P} = LU_{\overline{P}}$: parabolic subgroup opposite to $P = LU_P$
I_P^G: normalized parabolic induction functor $\text{Rep}(L) \to \text{Rep}(G)$

3. Intertwining operators and Harish-Chandra's μ-functions

We recall the definition of Harish-Chandra's intertwining operators. Consider two parabolic subgroups $P = LU_P$ and $P' = LU_{P'}$ with a common Levi factor L. Let (π, V_π) be a L-representation. All the representations $I_P^G(\pi \otimes \chi)$ with $\chi \in X_{\text{nr}}(L)$ can be realized on the same vector space, namely $\text{ind}_{P \cap K_0}^{K_0} V_\pi$ for a good maximal compact subgroup K_0 of G. This makes it possible to speak of objects on $I_P^G(\pi \otimes \chi)$ that vary regularly or rationally as functions of $\chi \in X_{\text{nr}}(L)$. Consider the intertwining operators

$$\begin{array}{llll} J_{P'|P}(\pi \otimes \chi): & I_P^G(\pi \otimes \chi) & \to & I_{P'}^G(\pi \otimes \chi) \\ & f & \mapsto & \left[g \mapsto \int_{U_P \cap U_{P'} \backslash U_{P'}} f(ug) \, \mathrm{d}u \right] \end{array}. \tag{3.1}$$

For π of finite length, this is well-defined as a family of G-homomorphisms depending rationally on $\chi \in X_{\text{nr}}(L)$ [Wal, Théorème IV.1.1]. There is an alternative construction of (3.1), in [Wal, proof of Théorème IV.1.1]. That construction works for representations with coefficients in any algebraically closed field of characteristic not p, which has been exploited recently in [MoTr] to define intertwining operators in more general settings.

Let $\bar{P} = LU_{\bar{P}}$ be the parabolic subgroup opposite to $P = LU_P$. We assume that π is irreducible and we consider the composition

$$J_{P|\bar{P}}(\pi \otimes \chi) J_{\bar{P}|P}(\pi \otimes \chi) : I_P^G(\pi \otimes \chi) \to I_P^G(\pi \otimes \chi) \qquad \chi \in X_{\text{nr}}(L). \tag{3.2}$$

This depends rationally on χ, and for generic χ the representation $I_P^G(\pi \otimes \chi)$ is irreducible [Sau, Théorème 3.2]. Therefore (3.2) is a scalar operator [Wal, §IV.3], say

$$j_{G,L}(\pi \otimes \chi) \, \text{id} \quad \text{with} \quad j_{G,L} : X_{\text{nr}}(L)\pi \to \mathbb{C} \cup \{\infty\}. \tag{3.3}$$

For purposes of harmonic analysis, the reciprocal of $j_{G,L}$ is often more convenient than $j_{G,L}$ itself. It usually rescaled by numbers $\gamma(G|L), c(G|L) \in \mathbb{Q}_{>0}$ defined in [Wal, p. 241]. By definition [Wal, §V.2] Harish-Chandra's μ-function is

$$\mu_{G,L}(\pi \otimes \chi) = c(G|L)^2 \gamma(G|L)^2 j_{G,L}(\pi \otimes \chi)^{-1}. \tag{3.4}$$

This μ-function is especially important for essentially square-integrable representations π, because then $\mu_{G,L}(\pi \otimes \chi)$ describes how the Plancherel density on $\{I_P^G(\pi \otimes \chi) : \chi \in X_{\rm nr}(L)\}$ varies as a function of χ [Wal].

Let A_L be the maximal split torus in $Z(L)$. The set of nonzero weights by which A_L acts on the Lie algebra of G is not necessarily a root system, but it is always a generalized root system in the sense of [DiFi]. In particular notions like basis, positive roots and reduced roots still make sense. Let $\Phi(G, A_L)$ be the set of reduced roots of (G, A_L) and let $\Phi(G, A_L)^+ = \Phi(U_P, A_L)$ be the subset of roots appearing in the Lie algebra of P. For $\alpha \in \Phi(G, A_L)^+$, let U_α (resp. $U_{-\alpha}$) be the root subgroup of G for all positive (resp. negative) multiples of α. Let L_α be the Levi subgroup of G generated by $L \cup U_\alpha \cup U_{-\alpha}$. Then L is a maximal proper Levi subgroup of L_α. Now [Wal, IV.3.(5) and Lemma V.2.1] say that

$$
\begin{aligned}
j_{G,L}(\pi) &= \prod\nolimits_{\alpha \in \Phi(G,A_L)^+} j_{L_\alpha,L}(\pi), \\
\mu_{G,L}(\pi) &= \prod\nolimits_{\alpha \in \Phi(G,A_L)^+} \mu_{L_\alpha,L}(\pi).
\end{aligned} \tag{3.5}
$$

With these μ-functions one can check easily whether certain intertwining operators are invertible.

Lemma 3.1. *Suppose that $\mu_{L_\alpha,L}(\pi) \notin \{0,\infty\}$ (or equivalently $j_{L_\alpha,L}(\pi) \notin \{0,\infty\}$) for all $\alpha \in \Phi(U_P, A_L) \cap \Phi(U_{\overline{P'}}, A_L)$. Then $J_{P'|P}(\pi) : I_P^G(\pi) \to I_{P'}^G(\pi)$ is invertible.*

Proof. As noticed on [Wal, p. 279], there exists a sequence of parabolic subgroups $P = P_0, P_1, \cdots, P_d = P'$, all with Levi factor L, such that $\Phi(P_i, A_L)$ and $\Phi(P_{i-1}, A_L)$ differ by only one root and $d = |\Phi(U_P, A_L) \cap \Phi(U_{\overline{P'}}, A_L)|$. In this situation [Wal, IV.1.(12)] says that

$$
J_{P'|P}(\pi) = J_{P_d|P_{d-1}}(\pi) \circ \cdots \circ J_{P_1|P_0}(\pi). \tag{3.6}
$$

It suffices to show that each $J_{P_i|P_{i-1}}(\pi)$ is invertible

Therefore we may assume that $\Phi(U_{P'}, A_L) \cap \Phi(U_{\overline{P'}}, A_L)$ consists of a single root α. By [Wal, IV.1.(14)] we may identify

$$
J_{P',P}(\pi) = I_{L_\alpha P}^G\big(J_{LU_{-\alpha}|LU_\alpha}(\pi)\big) : I_{L_\alpha P}^G(I_{LU_\alpha}^L(\pi)) \to I_{L_\alpha P}^G(I_{LU_{-\alpha}}^L(\pi)). \tag{3.7}
$$

By assumption

$$
J_{LU_\alpha|LU_{-\alpha}}(\pi) J_{LU_{-\alpha}|LU_\alpha}(\pi) = j_{L_\alpha,L}(\pi)\,\mathrm{id} \in \mathbb{C}^\times\,\mathrm{id}.
$$

Hence $J_{LU_{-\alpha}|LU_\alpha}(\pi)$ is invertible and (3.7) is invertible as well. □

3.1. Silberger's formulas for the μ-functions.

In [Sil3, Sil4] the functions $\mu_{G,L}(\pi)$ were determined, for essentially square-integrable representations. We will provide a different argument to arrive at the same formula in larger generality.

Let M be a Levi subgroup of L and $Q = MU_Q$ be a parabolic subgroup of G with Levi factor M, such that $Q \subset P$. Then $Q \cap L$ is a parabolic subgroup of L with Levi factor M, $P = QL$ and $\overline{P} = L\overline{Q}$. We note that, since $P = L \ltimes U_P$:

$$
U_Q = U_{Q \cap L} \ltimes U_P \qquad \text{and} \qquad U_{\overline{Q}} = U_{\overline{Q} \cap L} \ltimes U_{\overline{P}}. \tag{3.8}
$$

Lemma 3.2. *Suppose that $\sigma \in \mathrm{Irr}(M)$ and that π is a subquotient of $I_{Q \cap L}^L(\sigma)$. Then $\mu_{G,L}\big(I_{Q \cap L}^L(\sigma) \otimes \chi\big)$ is defined for $\chi \in X_{\rm nr}(L)$, and equals $\mu_{G,L}(\pi \otimes \chi)$.*

Proof. There is a natural isomorphism $I^L_{Q\cap L}(\sigma)\otimes\chi \cong I^L_{Q\cap L}(\sigma\otimes\chi|_M)$. For $\chi'\in X_{\mathrm{nr}}(M)$ in generic position, $I^L_Q(\chi')$ is irreducible [Sau, Théorème 3.2]. In the same way as in (3.2) and (3.3) we see that

$$(3.9)\quad J_{P|\bar P}\big(I^L_{Q\cap L}(\sigma\otimes\chi')\big)J_{\bar P|P}\big(I^L_{Q\cap L}(\sigma\otimes\chi')\big) = j_{G,L}\big(I^L_{Q\cap L}(\sigma\otimes\chi')\big)\mathrm{id}\quad \chi'\in X_{\mathrm{nr}}(M).$$

This shows that $j_{G,L}\big(I^L_{Q\cap L}(\sigma\otimes\chi')\big)$ and $\mu_{G,L}\big(I^L_{Q\cap L}(\sigma\otimes\chi')\big)$ are well-defined. We note that the formulas for $J_{\bar P|P}\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big)$ and $J_{\bar P|P}(\pi\otimes\chi)$ are essentially the same, only applied to different representations.

Write $\pi=\pi_1/\pi_2$ where π_1,π_2 are subrepresentations of $I^L_{Q\cap L}(\sigma)$. One can obtain $J_{\bar P|P}(\pi): I^G_P(\pi)\to I^G_{\overline{P}}(\pi)$ from $J_{\bar P|P}\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big)$ by first restriction to $J_{\overline{P}|P}(\pi_1)$ and then taking the induced homomorphism on $I^G_P(\pi)\cong I^G_P(\pi_1)/I^G_P(\pi_2)$. Since (3.9) with $\chi'=\chi\in X_{\mathrm{nr}}(L)$ is a scalar operator, it follows that $J_{P|\bar P}(\pi\otimes\chi)J_{\bar P|P}(\pi\otimes\chi)$ is also a scalar operator, with the same scalar. In other words,

$$(3.10)\qquad j_{G,L}\big(I^L_{Q\cap L}(\sigma)\otimes\chi\big) = j_{G,L}(\pi\otimes\chi).$$

This argument applies initially for every $\chi\in X_{\mathrm{nr}}(L)$ such that $j_{G,L}(\pi\otimes\chi)\neq\infty$, and then it extends to all $\chi\in X_{\mathrm{nr}}(L)$ because both j-functions are rational in χ. From (3.10) and (3.4) we see that $\mu_{G,L}\big(I^L_{Q\cap L}(\sigma)\otimes\chi\big)=\mu_{G,L}(\pi\otimes\chi)$. □

The following result generalizes [Sil4, Theorem 1].

Proposition 3.3. *In the setting of Lemma 3.2 we have, for $\chi\in X_{\mathrm{nr}}(L)$:*

(a) $j_{G,L}(\pi\otimes\chi) = j_{G,M}(\sigma\otimes\chi)j_{L,M}(\sigma\otimes\chi)^{-1}$,

(b) $\mu_{G,L}(\pi\otimes\chi) = \dfrac{\mu_{G,M}(\sigma\otimes\chi)}{\mu_{L,M}(\sigma\otimes\chi)}\dfrac{c(G|L)^2c(L|M)^2}{c(G|M)^2}$.

Proof. (a) In view of (3.10), we may replace $\pi\otimes\chi$ by $I^L_{Q\cap L}(\sigma)\otimes\chi\cong I^L_{Q\cap L}(\sigma\otimes\chi)$. Then all the involved expressions are defined for any $\chi\in X_{\mathrm{nr}}(M)$.

Consider the operator

$$(3.11)\quad I^G_{\overline{P}}(J_{\overline{Q\cap L}|Q\cap L}(\sigma\otimes\chi))\circ J_{\overline{P}|P}\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big): I^G_P\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big)\to I^G_{\overline{P}}\big(I^L_{\overline{Q\cap L}}(\sigma\otimes\chi)\big).$$

For $u\in G$ and a function f on G we write $(\lambda_u f)(g)=f(u^{-1}g)$. Then the effect of (3.11) is

$$f\mapsto \int_{U_{\overline{P}}}(\lambda_{u_1}f)\,\mathrm{d}u_1 \mapsto \int_{U_{\overline{Q}\cap L}}\int_{U_{\overline{P}}}\lambda_{u_2}(\lambda_{u_1}f)\,\mathrm{d}u_1\mathrm{d}u_2.$$

By (3.8), that is the same as $f\mapsto\int_{U_{\overline{Q}}}(\lambda_{u_3}f)\mathrm{d}u_3$. The transitivity of parabolic induction [Ren, Lemme VI.1.4] says that there are natural isomorphisms

$$(3.12)\qquad I^G_P\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big)\cong I^G_Q(\sigma\otimes\chi)\quad\text{and}\quad I^G_{\overline{P}}\big(I^L_{\overline{Q}\cap L}(\sigma\otimes\chi)\big)\cong I^G_{\overline{Q}}(\sigma\otimes\chi).$$

Therefore (3.11) can be identified with

$$J_{\overline{Q}|Q}(\sigma\otimes\chi): I^G_Q(\sigma\otimes\chi)\to I^G_{\overline{Q}}(\sigma\otimes\chi).$$

In the same way one can check that

$$(3.13)\quad J_{P|\overline{P}}\big(I^L_{Q\cap L}(\sigma\otimes\chi)\big)\circ I^G_{\overline{P}}(J_{Q|\overline{Q}}(\sigma\otimes\chi)) = J_{Q|\overline{Q}}(\sigma\otimes\chi): I^G_{\overline{Q}}(\sigma\otimes\chi)\to I^G_Q(\sigma\otimes\chi).$$

Combining (3.13) and the two expressions for (3.11), we compute

$$
\begin{aligned}
&j_{G,M}(\sigma \otimes \chi)\mathrm{id} = J_{Q|\overline{Q}}(\sigma \otimes \chi) J_{\overline{Q}|Q}(\sigma \otimes \chi) = \\
&J_{P|\overline{P}}(I^L_{Q\cap L}(\sigma\otimes\chi)) I^G_{\overline{P}}(J_{Q\cap L|\overline{Q}\cap L}(\sigma \otimes \chi)) I^G_{\overline{P}}(J_{\overline{Q}\cap L|Q\cap L}(\sigma \otimes \chi)) J_{\overline{P}|P}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big) \\
&\qquad = J_{P|\overline{P}}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big) I^G_{\overline{P}}\big(j_{L,M}(\sigma \otimes \chi)\mathrm{id}\big) J_{\overline{P}|P}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big) \\
&\qquad = j_{L,M}(\sigma \otimes \chi) J_{P|\overline{P}}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big) J_{\overline{P}|P}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big) \\
&\qquad = j_{L,M}(\sigma \otimes \chi) j_{G,L}\big(I^L_{Q\cap L}(\sigma \otimes \chi)\big)\mathrm{id}.
\end{aligned}
$$

(b) Recall that $\mu_{G,L} = c(G|L)^2 \gamma(G|L)^2 j_{G,L}^{-1}$. It follows from [Wal, p. 241] that $\gamma(G|L) = \gamma(G|M)\gamma(M|L)$, but $c(G|L)$ need not satisfy such a relation. Thus Lemma 3.2 and part (a) entail

$$
\frac{\mu_{G,L}(\pi \otimes \chi)}{c(G|L)^2} = \frac{\mu_{G,L}(I^L_{Q\cap L}(\sigma \otimes \chi))}{c(G|L)^2} = \frac{\mu_{G,M}(\sigma \otimes \chi)}{\mu_{L,M}(\sigma \otimes \chi)} \frac{c(L|M)^2}{c(G|M)^2}. \qquad \square
$$

Propositon 3.3 enables us to reduce the computation of μ-functions to the case of cuspidal representations, which is already well-understood.

Let $\sigma \in \mathrm{Irr}(M)$ be cuspidal. For $\alpha \in \Phi(G, A_M)^+$, let $h_\alpha^\vee \in M/M^1$ be as in [Sol2, Appendix] and [FlSo]. This element $h_\alpha^\vee$ depends on $X_{\mathrm{nr}}(M)\sigma$ and plays a role similar to a coroot $\alpha^\vee$. If $N_{M_\alpha}(M) \neq M$, we pick an element $s_\alpha \in N_{M_\alpha}(M) \setminus M$.

Theorem 3.4. [Sil3, Theorem 1.6] and [FlSo, Theorem 1.2]

(a) *If $j_{M_\alpha,M}$ does not have a pole on $X_{\mathrm{nr}}(M)\sigma$, then it equals a constant function $c_\alpha \in \mathbb{R}_{>0}$ on $X_{\mathrm{nr}}(M)\sigma$. This happens whenever $N_{M_\alpha}(M) = M$ or $N_{M_\alpha}(M) \neq M$ and s_α does not stabilize $X_{\mathrm{nr}}(M)\sigma$.*

(b) *Suppose that $j_{M_\alpha,M}$ has a pole on $X_{\mathrm{nr}}(M)\sigma$. By moving σ inside $X_{\mathrm{nr}}(M)\sigma$, we can arrange that σ is unitary, $j_{G,M}(\sigma) = \infty$ and s_α fixes σ. Then there exist $c_\alpha \in \mathbb{R}_{>0}, q_\alpha \in \mathbb{R}_{>1}, q_{\alpha*} \in \mathbb{R}_{\geq 1}$ such that*

$$
j_{M_\alpha,M}(\sigma \otimes \chi) = c_\alpha \frac{(1 - q_\alpha \chi(h_\alpha^\vee))(1 - q_\alpha \chi(h_\alpha^\vee)^{-1})}{(1 - \chi(h_\alpha^\vee))(1 - \chi(h_\alpha^\vee)^{-1})} \frac{(1 + q_{\alpha*} \chi(h_\alpha^\vee))(1 + q_{\alpha*} \chi(h_\alpha^\vee)^{-1})}{(1 + \chi(h_\alpha^\vee))(1 + \chi(h_\alpha^\vee)^{-1})}
$$

for all $\chi \in X_{\mathrm{nr}}(M)$.

In Theorem 3.4.b, $q_{\alpha*} = 1$ if 2α is not a root of (G, A_M). Theorem 3.4.a can be described by the same formula as part b, namely with $q_\alpha = q_{\alpha*} = 1$.

Consider a cuspidal Bernstein component $X_{\mathrm{nr}}(M)\sigma'$ in $\mathrm{Irr}(M)$. Let $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$ be the set of those $\alpha \in \Phi(G, A_M)$ for which $\mu_{M_\alpha,M}$ has a zero (or equivalently is not constant) on $X_{\mathrm{nr}}(M)\sigma'$. By [Hei, Proposition 1.3], $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$ is a reduced root system whose Weyl group embeds canonically in $N_G(M)/M$. The following result helps us to apply Theorem 3.4 simultaneously to several roots from $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$.

Lemma 3.5. *There exists a unitary $\sigma \in X_{\mathrm{nr}}(M)\sigma'$ such that $\mu_{M_\alpha,M}(\sigma) = 0$ and $s_\alpha \cdot \sigma \cong \sigma$ for all $\alpha \in \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$.*

Proof. A parabolic subgroup $P' = MU_{P'} \subset G$ determines which roots in $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$ are positive and which are simple. The simple roots are linearly independent so, as already observed in [Hei], one can find a unitary $\sigma \in X_{\mathrm{nr}}(M)\sigma'$ such that $\mu_{M_\alpha,M}(\sigma) = 0$ for all simple $\alpha \in \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$. By [Sil2, §5.4.2],

$s_\alpha \sigma \cong \sigma$ for all such α. Hence $W\big(\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')\big)$ fixes σ (up to isomorphism). Given any $\beta \in \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$, there exists a

$$w \in W\big(\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')\big) \subset N_G(M)/M$$

such that $\beta = w(\alpha)$ for a simple root α. Via an isomorphism $w^{-1}\sigma \cong \sigma$, we can identify

$$\begin{aligned} J_{MU_{-\beta}|MU_\beta}(\sigma \otimes \chi) &= w \circ J_{MU_{-\alpha}|MU_\alpha}(w^{-1}(\sigma \otimes \chi)) \circ w^{-1} \\ &= w \circ J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes w^{-1}\chi) \circ w^{-1}. \end{aligned}$$

The same holds for $-\beta$, which entails that

$$j_{M_\beta, M}(\sigma \otimes \chi) = j_{M_\alpha, M}(\sigma \otimes w^{-1}\chi) \quad \text{for all} \quad \chi \in X_{\mathrm{nr}}(M). \tag{3.14}$$

By (3.4) and (3.14) we have $\mu_{M_\beta, M}(\sigma) = \mu_{M_\alpha, M}(\sigma) = 0$. □

We are ready to state an explicit formula for Harish-Chandra's function $\mu_{G,L}$, for any irreducible L-representation.

Theorem 3.6. *Let $\pi \in \mathrm{Irr}(L)$. Suppose that $(M, \sigma \otimes \chi_\pi)$ represents the cuspidal support of π, where σ is as in Lemma 3.5 and $\chi_\pi \in X_{\mathrm{nr}}(M)$. Then there exists $c \in \mathbb{R}_{>0}$, depending only on $X_{\mathrm{nr}}(M)\sigma$ and G, such that*

$$\mu_{G,L}(\pi \otimes \chi) = c \prod_{\alpha \in \Phi(G,A_M)^+ \setminus \Phi(L,A_M)^+} \frac{(1 - (\chi_\pi\chi)(h_\alpha^\vee))(1 - (\chi_\pi\chi)(h_\alpha^\vee)^{-1})}{(1 - q_\alpha(\chi_\pi\chi)(h_\alpha^\vee))(1 - q_\alpha(\chi_\pi\chi)(h_\alpha^\vee)^{-1})} \cdot \frac{(1 + (\chi_\pi\chi)(h_\alpha^\vee))(1 + (\chi_\pi\chi)(h_\alpha^\vee)^{-1})}{(1 + q_{\alpha*}(\chi_\pi\chi)(h_\alpha^\vee))(1 + q_{\alpha*}(\chi_\pi\chi)(h_\alpha^\vee)^{-1})}$$

as rational functions of $\chi \in X_{\mathrm{nr}}(L)$.

Proof. By Proposition 3.3 and (3.5) we have

$$\mu_{G,L}(\pi \otimes \chi) = \frac{c(G|L)^2 c(L|M)^2}{c(G|M)^2} \frac{\prod_{\alpha \in \Phi(G,A_M)^+} \mu_{M_\alpha, M}(\sigma \otimes \chi_\pi\chi)}{\prod_{\alpha \in \Phi(l,A_M)^+} \mu_{M_\alpha, M}(\sigma \otimes \chi_\pi\chi)}.$$

Combine that with Theorem 3.4 and (3.4). Lemma 3.5 guarantees that σ is in the position required in Theorem 3.4.b, for any $\alpha \in \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma')$. □

Remark 3.7. Consider a finite central cover $\tilde{G}$ of the topological group G. The results in this paragraph hold just as well for $\tilde{G}$. The reason is that every unipotent subgroup of G admits a canonical lifting to $\tilde{G}$ [MWa, §A.1], so that one can reason in $\tilde{G}$ with unipotent subgroups exactly like in G. Therefore our proofs apply also to $\tilde{G}$. Theorem 3.4 was already proven in that generality in [FlSo].

3.2. **Normalized intertwining operators.**

Consider a cuspidal $\sigma \in \mathrm{Irr}(M)$ and $\alpha \in \Phi(G, A_M)$ such that $\mu_{M_\alpha, M}(\sigma) = 0$. We define a normalized version of $J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi)$ by

$$J'_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi) = (\chi(h_\alpha^\vee) - 1) J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi) \qquad \chi \in X_{\mathrm{nr}}(M).$$

According to [Hei, Lemme 1.8],

(3.15) $J'_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \chi)$ is invertible for χ in a neighborhood of 1 in $X_{\mathrm{nr}}(M)$.

More generally, let Q and Q' be parabolic subgroups of G with Levi factor M. We define the normalization of $J_{Q'|Q}(\sigma \otimes \chi)$ as

$$J'_{Q'|Q}(\sigma \otimes \chi) = J_{Q'|Q}(\sigma \otimes \chi) \prod_{\alpha \in \Phi(U_Q, A_M) \cap \Phi(U_{\overline{Q'}}, A_M) : \mu_{M_\alpha, M}(\sigma) = 0} (\chi(h_\alpha^\vee) - 1).$$

By reduction to (3.15) one shows:

Proposition 3.8. *Suppose that $\mu_{M_\alpha,M}(\sigma) \neq \infty$ for all $\alpha \in \Phi(U_Q, A_M) \cap \Phi(U_{\overline{Q'}}, A_M)$. Then there exists a neighborhood V_1 of 1 in $X_{\mathrm{nr}}(M)$, such that*

$$J'_{Q'|Q}(\sigma \otimes \chi) : I_Q^G(\sigma \otimes \chi) \to I_{Q'}^G(\sigma \otimes \chi)$$

is an isomorphism of G-representations for all $\chi \in V_1$.

There are also normalized intertwining operators for non-cuspidal representations. Let $\pi \in \mathrm{Irr}(L)$ be a subquotient of $I^Q_{Q \cap L}(\sigma)$ and write $P = QL$. For another parabolic subgroup $P' \subset G$ with Levi factor L we define

$$J'_{P'|P}(\pi \otimes \chi) = J'_{P'|P}(\pi \otimes \chi) \prod_{\alpha \in \Phi(U_P, A_M) \cap \Phi(U_{\overline{P'}}, A_M) : \mu_{M_\alpha, M}(\sigma) = 0} (\chi(h_\alpha^\vee) - 1). \tag{3.16}$$

Theorem 3.9. *Suppose that $\mu_{M_\alpha,M}(\sigma) \neq \infty$ for all $\alpha \in \Phi(U_P, A_M) \cap \Phi(U_{\overline{P'}}, A_M)$. Then there exists a neighborhood V_1' of 1 in $X_{\mathrm{nr}}(L)$, such that*

$$J'_{P'|P}(\pi \otimes \chi) : I_P^G(\pi \otimes \chi) \to I_{P'}^G(\pi \otimes \chi)$$

is an isomorphism of G-representations, for all $\chi \in V_1'$.

Proof. The set of roots $\Phi(U_{P'}, A_M) \cup \Phi(U_{Q \cap L}, A_M)$ is a positive system in $\Phi(G, A_M)$. That gives a parabolic subgroup Q' of G with Levi factor M, such that $P' = Q'L$ and $Q' \cap L = Q \cap L$. Now

$$\Phi(U_P, A_M) \cap \Phi(U_{\overline{P'}}, A_M) = \Phi(U_Q, A_M) \cap \Phi(U_{\overline{Q'}}, A_M),$$

which means that the normalization factors $\prod_\alpha (\chi(h_\alpha^\vee) - 1)$ are the same for $J_{Q'|Q}(\sigma \otimes \chi)$ and $J_{P'|P}\big(I^L_{Q \cap L}(\sigma \otimes \chi)\big)$. With an argument like in (3.11)–(3.13) we obtain

$$J'_{Q'|Q}(\sigma \otimes \chi) = J'_{QL'|QL}\big(I^L_{Q \cap L}(\sigma \otimes \chi)\big) : I_Q^G(\sigma \otimes \chi) \to I_{Q'}^G(\sigma \otimes \chi).$$

By Proposition 3.8, $J'_{Q'|Q}(\sigma \otimes \chi)$ is an isomorphism for $\chi \in V_1 \subset X_{\mathrm{nr}}(M)$. Hence

$$J'_{P'|P}\big(I^L_{Q \cap L}(\sigma) \otimes \chi\big) = J'_{Q'L|QL}\big(I^L_{Q \cap L}(\sigma \otimes \chi)\big)$$

is an isomorphism for $\chi \in V_1' := \{\chi \in X_{\mathrm{nr}}(L) : \chi|_M \in V_1\}$. Pick subrepresentations π_1, π_2 of $I^L_{Q \cap L}(\sigma)$ such that $\pi = \pi_1 / \pi_2$. By the above

$$J'_{P'|P}(\pi_i \otimes \chi) : I_P^G(\pi_i \otimes \chi) \to I_{P'}^G(\pi_i \otimes \chi) \qquad i = 1, 2, \chi \in V_1'$$

are isomorphisms. The map for $i = 1$ extends the map for $i = 2$, and passing to the quotient $\pi = \pi_1 / \pi_2$ we find that

$$J'_{P'|P}(\pi \otimes \chi) : I_P^G(\pi \otimes \chi) \to I_{P'}^G(\pi \otimes \chi) \qquad \chi \in V_1'$$

is also an isomorphism. □

3.3. Residual points of the μ-functions.

From (3.4) and (3.5) we see that $\mu_{G,M}$ can only have a pole at $\sigma \otimes \chi$ if at least one of the functions $j_{M_\alpha,M}$ has a zero at $\sigma \otimes \chi$. Then

$$J_{MU_\alpha | MU_{-\alpha}}(\sigma \otimes \chi) J_{MU_{-\alpha} | MU_\alpha}(\sigma \otimes \chi) = 0$$

and both factors are nonzero, so both $J_{MU_\alpha | MU_{-\alpha}}(\sigma \otimes \chi)$ and $J_{MU_{-\alpha} | MU_\alpha}(\sigma \otimes \chi)$ are not injective. We write

$$r_M = \dim_F Z(M) - \dim_F Z(G) = \dim_F A_M - \dim_F A_G .$$

Definition 3.10. A representation $\sigma \otimes \chi$ at which $\mu_{G,M}$ has a pole of order r_M is called a residual point of $\mu_{G,M}$.

The function $\mu_{G,M}$ has no poles of order $> r_M$ [Hei1, Corollaire 8.6].

Theorem 3.11. [Hei1, Théorème 8.6 and Corollaire 8.7]
The representation $I_Q^G(\sigma \otimes \chi)$ has an essentially square-integrable subquotient if and only if $\mu_{G,M}$ has a pole of order r_M at $\sigma \otimes \chi$. Such a subquotient is square-integrable modulo centre if and only if $|cc(\sigma \otimes \chi)|_{Z(G)} = 1$.

Theorem 3.11 says that the cuspidal supports of essentially square-integrable representations are precisely the residual points of the Harish-Chandra μ-functions. We note that $X_{\mathrm{nr}}(M)\sigma$ does not always contain residual points. A necessary condition is that $\Phi(G, M, X_{\mathrm{nr}}(M)\sigma)$ has rank r_M [Opd, Proposition A.3.(1)].

Lemma 3.12. (*a*) *There are only finitely many $X_{\mathrm{nr}}(G)$-orbits of residual points for $\mu_{G,M}$ in $X_{\mathrm{nr}}(M)\sigma$.*

(*b*) *Suppose that $Z(G)$ is compact and that σ is as in Lemma 3.5. Then every residual point $\sigma \otimes \chi$ satisfies a collection of equations*

$$\chi(h_\alpha^\vee) = q, \quad \text{where} \quad q \in \{\pm q_\alpha, \pm q_\alpha^{-1}, \pm q_{\alpha *}, \pm q_{\alpha *}^{-1}, \pm 1\}$$

in the notation of Theorem 3.4 for $M_\alpha \supset M$, and α runs through a subset of $\Phi(G, M, X_{\mathrm{nr}}(M)\sigma)^+$ whose $\mathbb{Q}$-span has dimension r_M.

Conversely, this collection of equations determines χ up to a finite subgroup of $X_{\mathrm{nr}}(M)$.

(*c*) *In the setting of part (b), $|cc(\sigma \otimes \chi)|$ is determined by a collection of equations*

$$\big| cc(\sigma' \otimes \chi)(h_\alpha^{\vee N_\alpha}) \big| = \chi\big(h_\alpha^{\vee N_\alpha}\big) \in \{q_\alpha^{N_\alpha}, q_\alpha^{-N_\alpha}, q_{\alpha *}^{N_\alpha}, q_{\alpha *}^{-N_\alpha}, 1\},$$

with the same α as in (b) and some $N_\alpha \in 2\mathbb{Z}_{>0}$.

Proof. (a) This is a special case of [Opd, Corollary A.2].
(b) As we saw above, $\Phi(G, M, X_{\mathrm{nr}}(M)\sigma)$ must have rank $r_M = \dim_{\mathbb{C}} X_{\mathrm{nr}}(M)\sigma$. By part (a) and the compactness of $Z(G)$, $\mu_{G,M}$ has only finitely many residual points in $X_{\mathrm{nr}}(M)\sigma$. By (3.4) and [Opd, Theorem A.7], there exist r_M linearly independent roots $\alpha \in \Phi(G, M, X_{\mathrm{nr}}(M)\sigma)^+$ such that

$$\mu_{M_\alpha, M}(\sigma) = 0 \quad \text{and} \quad j_{M_\alpha, M}(\sigma \otimes \chi) = \infty .$$

By Theorem 3.4 and (3.4), these equations imply that $\chi(h_\alpha^\vee)$ or $\chi(h_\alpha^\vee)^{-1}$ lies in $\{\pm q_\alpha, \pm q_{\alpha *}\}$. There may be further $\alpha \in \Phi(G, M, X_{\mathrm{nr}}(M)\sigma)^+$ with $\chi(h_\alpha^\vee)$ or $\chi(h_\alpha^\vee)^{-1}$ in $\{\pm q_\alpha, \pm q_{\alpha *}, \pm 1\}$, in the notations from Theorem 3.4 for $j_{M_\alpha,M}$. We include those as equations for $\sigma \otimes \chi$.

The elements $h_\alpha^\vee$ for the α as above span a finite index sublattice of M/M^1. Therefore the values $\chi(h_\alpha^\vee)$ determine χ up to a finite subgroup of $X_{\mathrm{nr}}(M)$.

(c) Recall from Lemma 3.5 that σ is unitary. Since $Z(M)M^1$ has finite index in M, we can find $N_\alpha \in 2\mathbb{Z}_{>0}$ such that $h_\alpha^{\vee N_\alpha} \in Z(M)M^1/M^1$. For any representative $h \in Z(M)$ of $h_\alpha^{\vee N_\alpha}$, part (b) shows that

$$|cc(\sigma' \otimes \chi)(h)| = |\chi(h)| = \big|\chi\big(h_\alpha^{\vee N_\alpha}\big)\big| = \chi\big(h_\alpha^{\vee N_\alpha}\big) \in \{q^{\pm N_\alpha}, q'^{\pm N_\alpha}, 1\}. \tag{3.17}$$

We may define this number to be $\big|cc(\sigma' \otimes \chi)(h_\alpha^{\vee N_\alpha})\big|$. The numbers (3.17), for all α as in part (b), determine $|cc(\sigma' \otimes \chi)| \in \mathrm{Hom}(Z(M), \mathbb{R}_{>0})$ on a finite index sublattice of $Z(M)/Z(M)^1$. Since all n-th roots are unique in $\mathbb{R}_{>0}$, that determines $|cc(\sigma' \otimes \chi)|$ completely. $\square$

3.4. **Analytic R-groups.**

Let $P = LU_P$ be a parabolic subgroup of G. The group $N_G(L)$ acts naturally on $\mathrm{Irr}(L)$, by $(n \cdot \pi)(l) = \pi(n^{-1} l n)$. This descends to an action of

$$W_L := N_G(L)/L$$

on $\mathrm{Irr}(L)$, which sends $X_{\mathrm{nr}}(L)$ to $X_{\mathrm{nr}}(L)$. Let $W_{L,\pi}$ be the stabilizer of $\pi \in \mathrm{Irr}(L)$ in W_L.

Let $\delta \in \mathrm{Irr}(L)$ be essentially square-integrable. Consider the set of reduced roots α of (G, A_L) such that Harish-Chandra's function $\mu_{L_\alpha, L}$ has a zero at δ. These roots form a finite integral root system [Sil1], say $\Phi(G, A_L, \delta)$. The group $W_{L,\delta}$ acts on $\Phi(G, A_L, \delta)$ and contains the Weyl group $W(\Phi(G, A_L, \delta))$ as a normal subgroup. Let $\Phi(G, A_L, \delta)^+$ be the positive system of roots appearing in the Lie algebra of P. The analytic R-group R_δ is defined as the stabilizer of $\Phi(G, A_L, \delta)^+$ in $W_{L,\delta}$. Since $W(\Phi(G, A_L, \delta))$ acts simply transitively on the collection of positive systems in $\Phi(G, A_L, \delta)$, we have a decomposition

$$W_{L,\delta} = W(\Phi(G, A_L, \delta)) \rtimes R_\delta. \tag{3.18}$$

This is a generalization of the R-groups from [Art] because we allow non-tempered representations δ, but apart from that it is the same definition.

Every $w \in W_{L,\delta}$ gives rise to an intertwining operator $J_\delta(w) \in \mathrm{Aut}_G(I_P^G(\delta))$ [ABPS, Lemma 1.3], unique up to scalars. It arises from the normalized intertwining operators (3.16) by a further normalization (to make it unitary if δ is tempered) and translation along w. By results of Knapp–Stein [Sil1], and by [ABPS, Lemma 1.5] in the non-tempered cases, $J_\delta(w)$ is a scalar multiple of the identity if and only if $w \in W(\Phi(G, A_L, \delta))$. Therefore it suffices to consider the intertwining operators $J_\delta(r)$ with $r \in R_\delta$. These operators span a twisted group algebra $\mathbb{C}[R_\delta, \natural_\delta]$, for some 2-cocycle $R_\delta \times R_\delta \to \mathbb{C}^\times$. In other words, J_δ yields a projective representation of R_δ on $I_P^G(\delta)$. By [ABPS, Theorem 1.6] there is a decomposition of $\mathbb{C}[R_\delta, \natural_\delta] \times \mathbb{C}G$-modules

$$\begin{aligned} I_P^G(\delta) &= \bigoplus\nolimits_{\kappa \in \mathrm{Irr}\, \mathbb{C}[R_\delta, \natural_\delta]} \kappa \otimes I_P^G(\delta)_\kappa, \\ I_P^G(\delta)_\kappa &= \mathrm{Hom}_{\mathbb{C}[R_\delta, \natural_\delta]}(\kappa, I_P^G(\delta)). \end{aligned} \tag{3.19}$$

If δ is square-integrable modulo centre (so in particular tempered), then $\mathbb{C}[R_\delta, \natural_\delta]$ equals $\mathrm{End}_G(I_P^G(\delta))$ and all the representations $I_P^G(\delta)_\kappa$ are irreducible [Sil1].

4. Quasi-standard modules

Le $L \subset G$ be a Levi subgroup and let $S \subset L$ be a maximal F-split torus. Then S is the maximal split central torus in the Levi subgroup $Z_G(S)$, and $\Phi(G, S)$ is the set of reduced roots of (G,S). This is a reduced integral root system in $X^*(S)$, and there is a coroot system $\Phi(G,S)^\vee$ in $X_*(S)$. We recall from [Ren, §V.3.13] that there are canonical decompositions

$$\begin{array}{lll} X^*(S)\otimes_{\mathbb Z}\mathbb R &=& X^*(S\cap L_{\mathrm{der}})\otimes_{\mathbb Z}\mathbb R \,\oplus\, X^*(A_L)\otimes_{\mathbb Z}\mathbb R, \\ X_*(S)\otimes_{\mathbb Z}\mathbb R &=& X_*(S\cap L_{\mathrm{der}})\otimes_{\mathbb Z}\mathbb R \,\oplus\, X_*(A_L)\otimes_{\mathbb Z}\mathbb R. \end{array} \tag{4.1}$$

Every $\alpha\in\Phi(G,A_L)\subset X^*(A_L)$ can be extended to an element $\alpha_S\in\Phi(G,S)$, usually in several ways. We define $\alpha^\vee$ as the projection of $\alpha_S^\vee$ to $X_*(A_L)\otimes_{\mathbb Z}\mathbb R$ via (4.1). This does not depend on the choice of α_S because $X^*(S\cap L_{\mathrm{der}})$ is orthogonal to $X_*(A_L)$. It does not depend on S either, because all maximal F-split tori of L are conjugate.

Let $P=LU_P$ be a parabolic subgroup of G with Levi factor L and let $\nu\in\mathrm{Hom}(L,\mathbb R_{>0})$, so $\log\nu\in\mathrm{Hom}(L,\mathbb R)\cong X^*(A_L)\otimes_{\mathbb Z}\mathbb R$. We say that ν is strictly positive with respect to P if $\langle\alpha^\vee,\log\nu\rangle>0$ for all $\alpha\in\Phi(P,A_L)=\Phi(U_P,A_L)$. This condition is equivalent to:

$$\langle\alpha_S^\vee,\log\nu\rangle>0\quad\forall\alpha_S\in\Phi(G,S)\text{ with }\alpha_S|_{A_L}\in\Phi(U_P,A_L). \tag{4.2}$$

Let $\delta\in\mathrm{Irr}(L)$ be essentially square-integrable and let $cc(\delta):Z(L)\to\mathbb C^\times$ be its central character. We note that $|cc(\delta)|$ is determined by its restriction to A_L, because $L_{\mathrm{der}}A_L$ is cocompact in L.

Definition 4.1. We call (P,L,δ) an induction datum for G. We say that (P,L,δ) is positive if $\langle\alpha^\vee,\log|cc(\delta)|\rangle\geq 0$ for all roots α of (P,A_L). If $\tilde\delta\cong\delta$ then $(P,L,\tilde\delta)$ is considered as equivalent to (P,L,δ).

Recall the R-group R_δ, the twisted group algebra $\mathbb C[R_\delta,\natural_\delta]$ and the decomposition of $I_P^G(\delta)$ from (3.19).

Definition 4.2. Let (P,L,δ) be an induction datum and let $\kappa\in\mathrm{Irr}\,\mathbb C[R_\delta,\natural_\delta]$. A $\mathbb CG$-module is called quasi-standard if it has the form $I_P^G(\delta)_\kappa$ as in (3.19).

This terminology is motivated by the following result.

Theorem 4.3. [Sol1, §2.4] and [ABPS, §1]
Let (P,L,δ) be an induction datum and let $\kappa\in\mathrm{Irr}\,\mathbb C[R_\delta,\natural_\delta]$.

(a) *If (P,L,δ) is positive, then $\mathbb C[R_\delta,\natural_\delta]=\mathrm{End}_G(I_P^G(\delta))$ and $I_P^G(\delta)_\kappa$ is a standard $\mathbb CG$-module.*
(b) *Every standard $\mathbb CG$-module arises as in part (a), from (P,L,δ,κ) which are unique up to G-conjugation.*
(c) *Let $\pi\in\mathrm{Irr}(G)$. There exists a positive induction datum, unique up to G-conjugation, such that π is a quotient of $I_P^G(\delta)$.*

Let $L_\delta\supset L$ be the largest Levi subgroup of G such that $|cc(\delta)|=1$ on $Z(L)\cap L_{\delta,\mathrm{der}}$. By [ABPS, Theorem 1.6], $I_{L_\delta\cap P}^{L_\delta}(\delta)$ is completely reducible and decomposes as

$$I_{L_\delta\cap P}^{L_\delta}(\delta)=\bigoplus\nolimits_{\kappa\in\mathrm{Irr}\,\mathbb C[R_\delta,\natural_\delta]}\kappa\otimes I_{L_\delta\cap P}^{L_\delta}(\delta)_\kappa. \tag{4.3}$$

Moreover, each $I_{L_\delta\cap P}^{L_\delta}(\delta)_\kappa$ can be written as $\tau\otimes\nu$ where τ is an irreducible tempered L_δ-representation and $\nu\in\mathrm{Hom}(L_\delta,\mathbb R_{>0})$. By the definition of L_δ, ν does not extend to a character of any Levi subgroup of G strictly containing L_δ. We note that, by the transitivity of parabolic induction

$$I_P^G(\delta)_\kappa\cong I_{L_\delta P}^G\big(I_{L_\delta\cap P}^{L_\delta}(\delta)_\kappa\big)\cong I_{L_\delta P}^G(\tau\otimes\nu). \tag{4.4}$$

Therefore one can characterize quasi-standard $\mathbb{C}G$-modules as representations of the form $I_Q^G(\tau \otimes \nu)$, where $Q = MU_Q$ is a parabolic subgroup of G, $\tau \in \mathrm{Irr}(M)$ is tempered and $\nu \in \mathrm{Hom}(M, \mathbb{R}_{>0})$ does not extend to a character of any strictly larger Levi subgroup of G. The difference with standard $\mathbb{C}G$-modules is that ν need not be positive with respect to Q.

Let Q_ν be the parabolic subgroup of G with Levi factor M and unipotent radical generated by root subgroups α with $\langle \alpha^\vee, \log \nu \rangle > 0$. Then $I_{Q_\nu}^G(\tau \otimes \nu)$ is standard. In the same way every induction datum (P, L, δ) can be made positive by changing only P.

We say that two induction data (P, L, δ) and (P', L', ω) are G-associate if there exists a $g \in G$ such that $gLg^{-1} = L'$ and $g \cdot \delta \cong \omega$. It is known from [Sol1, Lemma 2.13] that every induction datum is G-associate to a positive induction datum, unique up to equivalence.

For two associate induction data as above we have

$$I_P^G(\delta) \cong I_{gPg^{-1}}^G(g \cdot \delta) \cong I_{gPg^{-1}}^G(\omega). \tag{4.5}$$

By (4.5) and [ABPS, Lemma 1.1]

$$I_P^G(\delta) \quad \text{and} \quad I_{P'}^G(\omega) \quad \text{have the same Jordan–Hölder content.} \tag{4.6}$$

We proceed to make this statement more precise. The group $gL_\delta g^{-1} = L'_\omega$ has the same properties as L_δ, only for (P', L', ω). By [ABPS, Lemma 1.1] the L'_ω-representations $g \cdot I_{L_\delta \cap P}^{L_\delta}(\delta) \cong I_{L'_\omega \cap gPg^{-1}}^{L'_\omega}(\omega)$ and $I_{L'_\omega \cap P'}^{L'_\omega}(\omega)$ have the same Jordan–Hölder content. Since they are both completely reducible, we conclude that

$$g \cdot I_{L_\delta \cap P}^{L_\delta}(\delta) \cong I_{L'_\omega \cap P'}^{L'_\omega}(\omega). \tag{4.7}$$

Conjugation by g induces a group isomorphism $R_\delta \cong R_\omega$ and a bijection

$$\mathrm{Irr}\, \mathbb{C}[R_\delta, \natural_\delta] \to \mathrm{Irr}\, \mathbb{C}[R_\omega, \natural_\omega] : \kappa \mapsto \kappa'.$$

Together with (4.3) and (4.7) that implies

$$g(\kappa \otimes I_{L_\delta \cap P}^{L_\delta}(\delta)_\kappa) \cong \kappa' \otimes I_{L'_\omega \cap P'}^{L'_\omega}(\omega)_{\kappa'}. \tag{4.8}$$

Lemma 4.4. *In the setting of (4.8), the representations $I_P^G(\delta)_\kappa$ and $I_{P'}^G(\omega)_{\kappa'}$ have the same Jordan–Hölder content. Moreover, there exists a nonzero G-intertwining operator $I_P^G(\delta)_\kappa \to I_{P'}^G(\omega)_{\kappa'}$.*

Proof. We abbreviate $\tau' = I_{L_\delta \cap P}^{L_\delta}(\delta)_\kappa$, so that $I_P^G(\delta)_\kappa \cong I_{L_\delta P}^G(\tau')$. By (4.8) there are isomorphisms

$$I_{P'}^G(\omega)_{\kappa'} \cong I_{L'_\omega P}^G\big(I_{L'_\omega \cap P'}^{L'_\omega}(\omega)_{\kappa'}\big) \cong I_{L'_\omega P'}^G(g \cdot \tau') \cong I_{L_\delta g^{-1} P' g}^G(\tau'). \tag{4.9}$$

By [ABPS, Lemma 1.1] $I_{L_\delta P}^G(\tau')$ and $I_{L_\delta g^{-1}P'g}^G(\tau')$ have the same Jordan–Hölder content. We recall Harish-Chandra's intertwining operators

$$J_{L_\delta g^{-1}P'g | L_\delta P}(\tau' \otimes \chi) : I_{L_\delta P}^G(\tau' \otimes \chi) \to I_{L_\delta g^{-1}Pg}^G(\tau' \otimes \chi) \quad \chi \in X_{\mathrm{nr}}(L_\delta). \tag{4.10}$$

from (3.1). As we saw in (3.6)–(3.7), $J_{L_\delta g^{-1}P'g|L_\delta P}(\tau' \otimes \chi)$ is a composition of intertwining operators from a corank one setting. Let $J_{P_2|P_1}(\tau' \otimes \chi)$ be a such a simpler intertwining operator and let L_{12} be the derived group of the group generated by $P_1 \cup P_2$. By Theorem 3.4 or [Wal, p. 283], $J_{P_2|P_1}(\tau' \otimes \chi)$ can only have a pole at $\chi = 1$ if the nontrivial element s_α of the associated Weyl group (a subgroup of $N_G(L_\delta)/L_\delta$

of order at most two) stabilizes τ'. That would imply that $|cc(\tau')| = |cc(\delta)|_{Z(L_\delta)}$ is trivial on $Z(L_{12}) \supsetneq Z(L_{\delta,\mathrm{der}})$. But that would contradict the construction of L_δ, so $J_{P_2|P_1}(\tau' \otimes \chi)$ is regular at $\chi = 1$.

Hence $J_{L_\delta g^{-1} P' g | L_\delta P}(\tau' \otimes \chi)$ is regular at $\chi = 1$ and (4.10) is well-defined. Then [Wal, p. 283] shows that (4.10) is nonzero. Finally, we compose (4.10) with the isomorphism (4.9) (from right to left). □

4.1. **A rank one case.**

We work out quasi-standard modules in a relevant simple case, which is known but for which we could not find a good reference.

Proposition 4.5. *Let $\sigma \in \mathrm{Irr}(M)$ be unitary supercuspidal. Let $\nu \in \mathrm{Hom}(M, \mathbb{R}_{>0})$ and $\alpha \in \Phi(G, A_M)^+$.*

(a) *If $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq \infty$, then $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is completely reducible and has no essentially square-integrable subquotients. It has length two if and only if $s_\alpha(\sigma \otimes \nu) \cong \sigma \otimes \nu$ and $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq 0$. Otherwise $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is irreducible.*

(b) *If $\mu_{M_\alpha,M}(\sigma \otimes \nu) = \infty$, then $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ has length two and is indecomposable. If $\langle \alpha^\vee, \log \nu \rangle > 0$, then the irreducible quotient of $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is not tempered and the irreducible subrepresentation of $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is essentially square-integrable. If $\langle \alpha^\vee, \log \nu \rangle < 0$, then these properties of the quotient and the subrepresentation are exchanged.*

Proof. It is known from [Ren, Théorème VI.5.4] that $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ has length at most two. We recall from Theorem 3.11 that $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ has an essentially square-integrable subquotient if and only if $\mu_{M_\alpha,M}(\sigma \otimes \nu) = \infty$.

Case I: $\langle \alpha^\vee, \log \nu \rangle = 0$. Then $M_{\sigma \otimes \nu} = M_\alpha$ and, as we saw in (4.3), $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is completely reducible. Theorem 3.4 shows that $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq \infty$. Moreover (4.3) shows that $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is irreducible whenever $R_{\sigma \otimes \nu}$ is trivial. If $R_{\sigma \otimes \nu}$ is nontrivial, then its only nontrivial element is s_α, and $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq 0$ by the definition of $R_{\sigma \otimes \nu}$.

Case II: $\langle \alpha^\vee, \log \nu \rangle > 0$. If $N_{M_\alpha}(M)/M$ has a nontrivial element, then that does not fix ν, so in any case $W_{M_\alpha, \sigma \otimes \nu} = \{e\}$. By Theorem 4.3.a, $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is a standard module, so by the Langlands classification it has a unique irreducible quotient. As it has length at most two, it also has a unique irreducible subrepresentation.

Suppose that $\mu_{M_\alpha,M}(\sigma \otimes \nu) = \infty$. Then

$$J_{MU_\alpha | MU_{-\alpha}}(\sigma \otimes \nu) \circ J_{MU_{-\alpha} | MU_\alpha}(\sigma \otimes \nu) = j_{M_\alpha,M}(\sigma \otimes \nu)\, \mathrm{id} = 0. \tag{4.11}$$

Both J-operators in (4.11) are nonzero, so both are not injective. It follows that $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ is reducible. By the uniqueness in the Langlands classification, its irreducible quotient $\mathcal{L}(MU_\alpha, \sigma \otimes \nu)$ is not tempered. More precisely, $\mathcal{L}(MU_\alpha, \sigma \otimes \nu)$ is not a tensor product of a tempered representation and a character of M_α, because in that case its standard module would be $\mathcal{L}(MU_\alpha, \sigma \otimes \nu)$ itself. This also entails that the essentially square-integrable subquotient of $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)$ must be its irreducible subrepresentation.

Suppose next that $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq \infty$, or equivalently $j_{M_\alpha,M}(\sigma \otimes \nu) \neq 0$. From Theorem 3.4 we see that $j_{M_\alpha,M}(\sigma \otimes \nu) \neq \infty$, so both $J_{MU_\alpha | MU_{-\alpha}}(\sigma \otimes \nu)$ and $J_{MU_{-\alpha} | MU_\alpha}(\sigma \otimes \nu)$ are invertible. By construction $\mathcal{L}(MU_\alpha, \sigma \otimes \nu)$ is the image of $J_{MU_{-\alpha}|MU_\alpha}(\sigma \otimes \nu)$, see [Ren, Lemme VII.4.1]. Hence

$$\mathcal{L}(MU_\alpha, \sigma \otimes \nu) = I_{MU_{-\alpha}}(\sigma \otimes \nu) \cong I_{MU_\alpha}(\sigma \otimes \nu).$$

Case III: $\langle \alpha^\vee, \log \nu \rangle < 0$**.** By [Ren, (IV.2.1.2)], the smooth contragredient representation $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)^\vee$ is isomorphic to

$$I_{MU_\alpha}^{M_\alpha}((\sigma \otimes \nu)^\vee) \cong I_{MU_\alpha}^{M_\alpha}(\sigma^\vee \otimes \nu^{-1}).$$

The representation $\sigma^\vee$ is again unitary supercuspidal, so this brings us back to case II. According to [Wal, Lemme V.2.1], which is proven in [FlSo, Theorem 3.5],

$$\mu_{M_\alpha,M}(\sigma^\vee \otimes \nu^{-1}) = \mu_{M_\alpha,M}(\sigma \otimes \nu).$$

When $\mu_{M_\alpha,M}(\sigma \otimes \nu) \neq \infty$, we know from case II that $I_{MU_\alpha}^{M_\alpha}(\sigma^\vee \otimes \nu^{-1})$ is irreducible but not essentially square-integrable. Then its contragredient $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)^\vee$ has the same two properties.

When $\mu_{M_\alpha,M}(\sigma \otimes \nu) = \infty$, we know from case II that $I_{MU_\alpha}^{M_\alpha}(\sigma^\vee \otimes \nu^{-1})$ has an essentially square-integrable subrepresentation (say δ) and a non-tempered irreducible quotient $\mathcal{L}(MU_\alpha, \sigma^\vee \otimes \nu^{-1})$. Then its contragredient $I_{MU_\alpha}^{M_\alpha}(\sigma \otimes \nu)^\vee$ has the essentially square-integrable representation $\delta^\vee$ as quotient and the non-tempered representation $\mathcal{L}(MU_\alpha, \sigma^\vee \otimes \nu^{-1})^\vee$ as subrepresentation. □

4.2. An alternative characterization of standard modules.

We characterize standard modules as quasi-standard modules with some extra properties. In this way one can avoid the use of temperedness or positivity of characters in the description of standard modules.

We need some information about the irreducible constituents of a standard module which are not quotients. All these are larger than the irreducible quotient, a claim that we will quantify with an invariant from [Sol1].

We fix a maximal F-split torus S in G and a $W(G,S)$-invariant inner product on $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. We may assume that all Levi subgroups in our constructions are standard, in the sense that they contain $Z_G(S)$. Alternatively we can pass to another maximal split torus S', and then the inner product transfers canonically to $X^*(S') \otimes_{\mathbb{Z}} \mathbb{R}$ by its $W(G,S)$-invariance.

As before, let $\delta \in \mathrm{Irr}(L)$ be essentially square-integrable. Let $(\tilde{L}, \sigma)$ be a representative of the cuspidal support $\mathrm{Sc}(\delta)$ and consider $\mathrm{cc}(\sigma) : Z(\tilde{L}) \to \mathbb{C}^\times$. As $Z(\tilde{L})\tilde{L}^1$ is cocompact in $\tilde{L}$,

$$\log |cc(\sigma)| : Z(\tilde{L}) \to \mathbb{R}$$

extends uniquely to a group homomorphism from $\tilde{L}$ to $\mathbb{R}$. Then $\log |cc(\sigma)| : \tilde{L} \to \mathbb{R}$ determines an element of

$$\mathrm{Hom}(S, \mathbb{R}) \cong X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

As $\tilde{L}^1 S$ is cocompact in $\tilde{L}$, that element still determines $\log |cc(\sigma)|$. In these terms, the restriction of $\log |cc(\sigma)|$ to $\tilde{L} \cap L_{\mathrm{der}}$ can be described by restriction from S to $S \cap L_{\mathrm{der}}$, so by an element of $X^*(S \cap L_{\mathrm{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$. The canonical decomposition (4.1) provides $X^*(S \cap L_{\mathrm{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$ with a $W(L,S)$-invariant inner product.

Let $I_P^G(\delta)_\kappa$ be a quasi-standard summand of $I_P^G(\delta)$. We define

$$\mathcal{N}(I_P^G(\delta)_\kappa) = \mathcal{N}(I_P^G(\delta)) = \mathcal{N}(\delta) = \| \log |cc(\sigma)|_{\tilde{L} \cap L_{\mathrm{der}}} \|, \tag{4.12}$$

where the norm comes from the inner product on $X^*(S \cap L_{\mathrm{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$. The $W(L,S)$-invariance of this inner product implies that (4.12) does not depend on the choice of a representative of the cuspidal support of δ. The invariant $\mathcal{N}$ measures the distance from $\delta|_{L_{\mathrm{der}}}$ to the parabolic induction of a unitary cuspidal $\tilde{L}$-representation.

Clearly $|cc(\sigma)|_{\tilde{L} \cap L_{\mathrm{der}}}$ depends only on $\delta|_{L_{\mathrm{der}}}$. Therefore $\mathcal{N}(I_P^G(\delta)_\kappa)$ depends only on $\delta|_{L_{\mathrm{der}}}$, which is a direct sum of finitely many square-integrable representations δ_1 [Tad, Lemma 2.1 and Proposition 2.7]. Then $\mathrm{Sc}(\delta_1)$ can be represented by a subrepresentation of $\sigma|_{\tilde{L} \cap L_{\mathrm{der}}}$. Therefore $\mathcal{N}(I_P^G(\delta)_\kappa)$ can be computed as

$$\mathcal{N}(I_P^G(\delta)_\kappa) = \mathcal{N}(I_P^G(\delta)) = \| \log |cc(\mathrm{Sc}(\delta_1))| \, \| = \mathcal{N}(\delta_1). \tag{4.13}$$

We note that $(P, L, \delta) \mapsto \mathcal{N}(I_P^G(\delta))$ is constant on G-conjugacy classes of induction data, by the $W(G,S)$-invariance of the inner product on $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. It is even constant on G-association classes of induction data, because P is inessential in (4.12). This enables us to define, for any standard $\mathbb{C}G$-module π_{st} with irreducible quotient π:

$$\mathcal{N}(\pi) := \mathcal{N}(\pi_{st}).$$

Lemma 4.6. [Sol1, Lemma 2.12]
Let $I_Q^G(\tau \otimes \nu)$ be a standard $\mathbb{C}G$-module with an irreducible constituent π different from $\mathcal{L}(Q, \tau \otimes \nu)$. Then $\mathcal{N}(\pi) > \mathcal{N}(I_Q^G(\tau \otimes \nu))$.

Easier, earlier versions of Lemma 4.6 have been used to prove that the standard $\mathbb{C}G$-modules form a $\mathbb{Z}$-basis of the Grothendieck group finite length G-representations. We can also use Lemma 4.6 to improve on Theorem 4.3.c.

Lemma 4.7. *Suppose that a standard module π_{st} with quotient π is a direct summand of $I_P^G(\delta)$, for a positive induction datum (P, L, δ). Then π_{st} is, up to isomorphism, the only indecomposable summand of $I_P^G(\delta)$ in which π appears.*

Proof. By Theorem 4.3.a every indecomposable direct summand of $I_P^G(\delta)$ is a standard module, say $I_Q^G(\tau \otimes \nu)$. Let π be a subquotient of $I_Q^G(\tau \otimes \nu)$. By definition we have equalities

$$\mathcal{N}(\pi) = \mathcal{N}(\pi_{st}) = \mathcal{N}(I_P^G(\delta)) = \mathcal{N}(\mathcal{L}(Q, \tau \otimes \nu)).$$

Lemma 4.6 shows that π must be the irreducible quotient of $I_Q^G(\tau \otimes \nu)$. Then $I_Q^G(\tau \otimes \nu)$ is a standard module with quotient π, so by Theorem A.c $I_Q^G(\tau \otimes \nu)$ is isomorphic to π_{st}. $\square$

Next we generalize Lemma 4.7 to not necessarily positive induction data.

Theorem 4.8. *Let $\pi \in \mathrm{Irr}(\mathbb{C}G)$. There exists an induction datum (P, L, δ) and $\kappa \in \mathrm{Irr}\, \mathbb{C}[R_\delta, \natural_\delta]$, unique up to G-association, such that:*

- *π is a constituent of $I_P^G(\delta)_\kappa$,*
- *$\mathcal{N}(I_P^G(\delta))$ is maximal for the previous property.*

Moreover, in this case $\mathcal{N}(\pi) = \mathcal{N}(I_P^G(\delta))$.

Proof. Without κ, this is a reformulation of [Sol1, Theorem 2.15]. The additional claims about κ follow from Lemma 4.7 and (4.8). $\square$

We are ready to characterize standard modules without temperedness or positivity. We abbreviate the previous $\tau \otimes \nu$ to τ'.

Theorem 4.9. *Let $I_Q^G(\tau')$ be a quasi-standard $\mathbb{C}G$-module which has a unique irreducible quotient π and satisfies $\mathcal{N}(I_Q^G(\tau')) = \mathcal{N}(\pi)$. Then $I_Q^G(\tau')$ is a standard $\mathbb{C}G$-module.*

Proof. By Theorem 4.3, there exists a positive induction datum (P, L, δ) and $\kappa \in \mathrm{Irr}\, \mathbb{C}[R_\delta, \natural_\delta]$ such that $\pi_{st} \cong I_P^G(\delta)_\kappa$. By the definition of quasi-standard, there exists an induction datum (P', L', ω) and $\kappa' \in \mathrm{Irr}\, \mathbb{C}[R_\omega, \natural_\omega]$ such that $I_Q^G(\tau') \cong I_{P'}^G(\omega)_{\kappa'}$. The condition $\mathcal{N}(I_Q^G(\tau')) = \mathcal{N}(\pi)$ and Theorem 4.8 imply that $I_Q^G(\tau')$ has maximal $\mathcal{N}$-value among the quasi-standard modules with π as constituent. As $\mathcal{N}(\pi) = \mathcal{N}(\pi_{st})$, the same holds for $I_P^G(\delta)_\kappa$. By the uniqueness in Theorem 4.8, (P, L, δ, κ) and $(P', L', \omega, \kappa')$ are G-associate. By Lemma 4.4, there exists a nonzero G-intertwining operator

$$J : I_P^G(\delta)_\kappa \to I_{P'}^G(\omega)_{\kappa'}.$$

Let $q : I_P^G(\delta)_\kappa \cong \pi_{st} \to \pi$ and $q' : I_{P'}^G(\omega)_{\kappa'} \cong I_Q^G(\tau') \to \pi$ be the quotient maps. The kernel of J is not the whole of $I_P^G(\delta)_\kappa \cong \pi_{st}$, so it is contained in $\ker q$ (because π is the unique irreducible quotient of π_{st}). Thus J induces an injection

$$\pi \cong I_P^G(\delta)_\kappa / \ker q \xrightarrow{\bar{J}} I_Q^G(\omega)_{\kappa'} / J(\ker q).$$

By Lemmas 4.6 and 4.4, π appears with multiplicity one in $I_P^G(\delta)_\kappa$ and in $I_Q^G(\omega)_{\kappa'}$. Since $\bar{J}$ is injective, $J(\ker q)$ does not contain $\pi \cong \bar{J}(\pi)$ as subquotient. Hence $J(\ker q) \subset \ker q'$ and $q' \circ \bar{J}$ is nonzero. It follows that the image of J is a subrepresentation of $I_Q^G(\omega)_{\kappa'}$ not contained in $\ker q'$. As π is the unique irreducible quotient of $I_Q^G(\omega)_{\kappa'}$, J is surjective. Further $I_P^G(\delta)_\kappa$ and $I_Q^G(\omega)_{\kappa'}$ have the same Jordan–Hölder content (Lemma 4.4), so J is an isomorphism. □

5. The action of the Galois group on representations

Let $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ be the automorphism group of the field extension $\mathbb{C}/\mathbb{Q}$. Strictly speaking this is not a Galois extension because it is not algebraic, but for brevity we still speak of the Galois group of this extension.

For $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, let $\mathbb{C}_\gamma$ be $\mathbb{C}$ as $\mathbb{C}$-$\mathbb{C}$-bimodule with action

$$z_1 \cdot v \cdot z_2 = z_1 v \gamma(z_2) \qquad z_i \in \mathbb{C}, v \in \mathbb{C}_\gamma.$$

For a G-representation (π, V_π) we define ${}^\gamma V_\pi = \mathbb{C}_\gamma \otimes_{\mathbb{C}} V_\pi$. This means that as an abelian group ${}^\gamma V_\pi$ can identified with V_π, but with the adjusted scalar multiplication

$$z(1 \otimes v) = z \otimes v = 1 \otimes \gamma^{-1}(z) v \qquad z \in \mathbb{C}, v \in V_\pi.$$

Definition 5.1. $\gamma \cdot \pi$ is the G-representation on ${}^\gamma V_\pi$ given by

$$(\gamma \cdot \pi)(g)(z \otimes v) = z \otimes \pi(g) v \qquad g \in G, z \in \mathbb{C}_\gamma, v \in V_\pi.$$

If λ lies in the dual space V_π^*, then $z \otimes v \mapsto z\gamma(\lambda(v))$ lies in $({}^\gamma V_\pi)^*$. For a matrix coefficient $m_{v,\lambda} : g \mapsto \lambda(\pi(g)v)$ of π, the corresponding matrix coefficient of $\gamma \cdot \pi$ is $g \mapsto \gamma\big(\lambda(\pi(g)v)\big)$. Thus, for any finite dimensional representation π', $\gamma \cdot \pi'$ can be obtained from π' by applying γ to the matrices that define π'.

The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on G-representations preserves irreducibility and cuspidality [KSV, Theorem 2.3.(1)]. In general it does not preserve unitarity or temperedness, as can already be seen in the case $G = GL_1(F)$.

It is easy to check that the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on representations of G or L commutes with unnormalized parabolic induction. However, that is not entirely true for normalized parabolic induction. Consider the modular function δ_P of $P = LU_P$. It takes values in $q_F^{\mathbb{Z}}$, where q_F denotes the cardinality of the residue field of F. In particular $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ fixes δ_P. But $\delta_P^{1/2}$ takes values in $(q_F^{1/2})^{\mathbb{Z}}$, and if $q_F^{1/2} \notin \mathbb{Q}$, then some elements of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ send $q_F^{1/2}$ to $-q_F^{1/2}$. It follows that for every $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ there exists a unique quadratic character $\epsilon_{P,\gamma} : L = P/U_P \to \{\pm 1\}$ such that

$$\gamma \cdot \delta_P^{1/2} = \delta_P^{1/2} \otimes \epsilon_{P,\gamma}.$$

Proposition 5.2. [KSV]

(a) *As a character of L, $\epsilon_{P,\gamma}$ depends on L and G, but not on the choice of the parabolic subgroup P with Levi factor L.*
(b) *The group $N_G(L)$ fixes $\epsilon_{P,\gamma}$.*
(c) *For any L-representation π, there is an isomorphism $\gamma \cdot I_P^G(\pi) \cong I_P^G(\gamma \cdot \pi \otimes \epsilon_{P,\gamma})$.*
(d) *For any finite length L-representation (π, V_π):*

$$J_{P'|P}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma})(z \otimes v) = z \otimes J_{P'|P}(\pi)(v) \qquad z \in \mathbb{C}_\gamma, v \in I_P^G(V_\pi).$$

(e) *For any $\pi \in \mathrm{Irr}(L)$, $\mu_{G,L}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma}) = \prod_{\alpha \in \Phi(G,A_L)^+} \mu_{L_\alpha,L}(\gamma \cdot \pi \otimes \epsilon_{LU_\alpha,\gamma})$.*

Proof. (a) and (b) are [KSV, Lemma 5.11] and (c) is [KSV, (5.12)].
(d) This follows directly from part (c) and the definitions of $\gamma \cdot I_P^G(\pi)$ and $J_{P'|P}$.
(e) Part (d) and the definition of $\mu_{G,L}$ in (3.1)–(3.4) show that

$$\begin{aligned} \mu_{G,L}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma}) &= J_{P|\bar{P}}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma}) \circ J_{\bar{P}|P}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma}) \\ &= \gamma\big(J_{P|\bar{P}}(\pi) \circ J_{\bar{P}|P}(\pi)\big) = \gamma(\mu_{G,L}(\pi)). \end{aligned} \tag{5.1}$$

From (3.5) and (5.1) we deduce

$$\begin{aligned} &\mu_{G,L}(\gamma \cdot \pi \otimes \epsilon_{P,\gamma}) = \gamma(\mu_{G,L}(\pi)) \\ &\qquad = \gamma\Big(\prod_{\alpha \in \Phi(G,A_L)^+} \mu_{L_\alpha,L}(\pi)\Big) = \prod_{\alpha \in \Phi(G,A_L)^+} \mu_{L_\alpha,L}(\gamma \cdot \pi \otimes \epsilon_{LU_\alpha,\gamma}). \qquad \square \end{aligned}$$

5.1. The Galois action on quasi-standard modules.

We would like to understand how $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on quasi-standard $\mathbb{C}G$-modules. A crucial step is the following result, which for semisimple groups over p-adic fields is due to Clozel (unpublished).

Theorem 5.3. [KSV, Theorem 4.6]
Let δ be an essentially square-integrable L-representation and let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then $\gamma \cdot \delta$ is also essentially square-integrable.

We preserve the notations from Theorem 5.3. Recall from (3.18) and (3.19) that for $r \in R_\delta$ we have $J_\delta(r) \in \mathrm{End}_{\mathbb{C}G}(I_P^G(\delta))$, and that these operators span a twisted group algebra $\mathbb{C}[R_\delta, \natural_\delta]$. Theorem 5.3 tells us that $\gamma \cdot \delta \otimes \epsilon_{P,\gamma}$ is essentially square-integrable. By [KSV, Proposition 5.12] we have $R_{\gamma\delta\otimes\epsilon_{P,\gamma}} = R_\delta$, and we may define

$$\begin{aligned} &J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(r) \in \mathrm{End}_{\mathbb{C}G}\big(I_P^G(\gamma \cdot \pi \otimes \epsilon_{P,\gamma})\big) = \mathrm{End}_{\mathbb{C}G}\big(\mathbb{C}_\gamma \otimes_{\mathbb{C},\gamma} I_P^G(\delta)\big) \\ &J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(r)(z \otimes v) = z \otimes J_\delta(r)(v). \end{aligned}$$

By construction $J_\delta(r) J_\delta(r') = \natural_\delta(r, r') J_\delta(rr')$, which implies that

$$J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(r) J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(r') = \gamma(\natural_\delta(r, r')) J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(rr').$$

Hence the operators $J_{\gamma\delta\otimes\epsilon_{P,\gamma}}(r)$ span a twisted group algebra

$$\mathbb{C}[R_{\gamma\delta\otimes\epsilon_{P,\gamma}}, \natural_{\gamma\delta\otimes\epsilon_{P,\gamma}}] = \mathbb{C}[R_\delta, \gamma\natural_\delta]. \tag{5.2}$$

It is easily seen that there is a canonical bijection

$$\mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta]) \to \mathrm{Irr}(\mathbb{C}[R_\delta, \gamma\natural_\delta]) : \kappa \mapsto \gamma \cdot \kappa.$$

Lemma 5.4. *For any quasi-standard $\mathbb{C}G$-module $I_P^G(\delta)_\kappa$ and any $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, there is an isomorphism*

$$\gamma \cdot I_P^G(\delta)_\kappa \cong I_P^G\big(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}\big)_{\gamma\cdot\kappa}.$$

In particular the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *on* $\mathrm{Rep}(G)$ *stabilizes the set of quasi-standard $\mathbb{C}G$-modules.*

Proof. One step in the construction of $I_P^G(\delta)_\kappa$ is the representation $I_{L_\delta\cap P}^{L_\delta}(\delta)_\kappa$ from (4.3), to get $I_P^G(\delta)_\kappa$ we parabolically induce that. Recall that both parabolic induction and its normalized version are transitive [Ren, Lemme VI.1.4], and that an ingredient for the latter is the equality of modular functions $\delta_P = \delta_{L_\delta P}\delta_{L_\delta\cap P}$. This equality entails that

$$\epsilon_{P,\gamma} = \epsilon_{L_\delta P,\gamma}\, \epsilon_{L_\delta\cap P,\gamma}. \tag{5.3}$$

With Proposition 5.2.c and (5.2) we compute

$$\gamma\cdot I_P^G(\delta)_\kappa \cong I_{L_\delta P}^G\big(\gamma\cdot I_{L_\delta\cap P}^{L_\delta}(\delta)_\kappa\otimes\epsilon_{P,\gamma}\big) \cong I_{L_\delta P}^G\big(I_{L_\delta\cap P}^{L_\delta}(\gamma\cdot\delta\otimes\epsilon_{L_\delta\cap P,\gamma})_{\gamma\cdot\kappa}\otimes\epsilon_{L_\delta P,\gamma}\big). \tag{5.4}$$

Notice that these expressions are well-defined because $\gamma \cdot \delta \otimes \epsilon_{L_\delta\cap P,\gamma}$ is essentially square-integrable (Theorem 5.3). Tensoring by $\epsilon_{L_\delta P,\gamma} \in X_{\mathrm{nr}}(L_\delta)$ commutes with all the operations involved in $I_{L_\delta\cap P}^{L_\delta}(\gamma \cdot \delta \otimes \epsilon_{L_\delta\cap P,\gamma})_{\gamma\cdot\kappa}$. By that and (5.3), the right hand side of (5.4) is isomorphic to

$$I_{L_\delta P}^G\big(I_{L_\delta\cap P}^{L_\delta}(\gamma\cdot\delta\otimes\epsilon_{P,\gamma})_{\gamma\cdot\kappa}\big) \cong I_{L_\delta P}^G\big(\mathrm{Hom}_{\mathbb{C}[R_\delta,\gamma\natural_\delta]}\big(\gamma\cdot\kappa, I_{L_\delta\cap P}^{L_\delta}(\gamma\cdot\delta\otimes\epsilon_{P,\gamma})\big)\big). \tag{5.5}$$

By the transitivity of normalized parabolic induction, the right hand side of (5.5) equals the quasi-standard $\mathbb{C}G$-module $I_P^G\big(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}\big)_{\gamma\cdot\kappa}$. □

Recall $\mathcal{N}$ from (4.12). Although $\gamma \cdot \delta \otimes \epsilon_{P,\gamma}$ is essentially square-integrable and $\mathcal{N}(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}) = \mathcal{N}(\gamma \cdot \delta)$, it is not obvious whether $\mathcal{N}(\gamma \cdot \delta)$ equals $\mathcal{N}(\delta)$ for all $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Via Theorem 3.11 and Lemma 3.12, that can be reduced to the question:

(5.6) are the numbers $q_\alpha^2, q_{\alpha*}^2$ from Theorem 3.4 always rational?

In [Oha, Sol3] it has been shown that $q_\alpha, q_{\alpha*}$ belong to $(q_F^{1/2})^{\mathbb{Z}}$ in the large majority of all cases. Nevertheless there is no general proof for (5.6). This means that currently it is known that $\mathcal{N}(\gamma \cdot \delta) = \mathcal{N}(\delta)$ for most essentially square-integrable representations, but at the same time that remains open in general.

Proposition 5.5. *Assume that the action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *preserves the $\mathcal{N}$-values of all essentially square-integrable representations of Levi subgroups of G. Then* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ *stabilizes the set of standard $\mathbb{C}G$-modules.*

Proof. Consider any quasi-standard $\mathbb{C}G$-module $I_P^G(\delta)_\kappa$. By (4.12) and the assumptions:

$$\mathcal{N}\big(I_P^G(\delta)_\kappa\big) = \mathcal{N}(\delta) = \mathcal{N}(\gamma \cdot \delta) = \mathcal{N}(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}) = \mathcal{N}\big(I_P^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})\big) = \mathcal{N}\big(\gamma \cdot I_P^G(\delta)_\kappa\big). \tag{5.7}$$

Next we consider any standard $\mathbb{C}G$-module π_{st}, with irreducible quotient π. We know from Lemma 5.4 that $\gamma \cdot \pi_{st}$ is a quasi-standard $\mathbb{C}G$-module. By (5.7) and Theorem 4.8 we have

$$\mathcal{N}(\gamma \cdot \pi) \geq \mathcal{N}(\gamma \cdot \pi_{st}) = \mathcal{N}(\pi_{st}) = \mathcal{N}(\pi). \tag{5.8}$$

We may als apply this to γ^{-1} acting on $\gamma \cdot \pi$, then we find

$$\mathcal{N}(\pi) = \mathcal{N}(\gamma^{-1} \cdot \gamma \cdot \pi) \geq \mathcal{N}(\gamma \cdot \pi) \geq \mathcal{N}(\pi). \tag{5.9}$$

We conclude that $\mathcal{N}(\gamma \cdot \pi)$ equals $\mathcal{N}(\pi)$.

From (5.8) and (5.9) we see that $\mathcal{N}(\gamma \cdot \pi_{st}) = \mathcal{N}(\gamma \cdot \pi)$. As π is a quotient of π_{st}, $\gamma \cdot \pi$ is a quotient of $\gamma \cdot \pi_{st}$. Now we are in the right position to apply Theorem 4.9, which guarantees that $\gamma \cdot \pi_{st}$ is a standard $\mathbb{C}G$-module. $\square$

5.2. The Galois action on standard modules.

We proceed to establish an unconditional version of Proposition 5.5. Let (P, L, δ) be a positive induction datum and let $\kappa \in \mathbb{C}[R_\delta, \natural_\delta]$. Recall from Theorem 4.3 that $I_P^G(\delta)_\kappa$ is a standard $\mathbb{C}G$-module and that every standard $\mathbb{C}G$-module has this form. Let $M \subset L$ be a Levi subgroup and let $\sigma \in \mathrm{Irr}(M)$ be such that (M, σ) represents the cuspidal support of (δ, V_δ).

We write $\delta = \delta_u \otimes \nu_\delta$ where $\delta_u \in \mathrm{Irr}(L)$ is square-integrable modulo center and $\nu_\delta \in \mathrm{Hom}(L, \mathbb{R}_{>0})$. We note that ν_δ is determined by $\nu_\delta|_{A_L} = |cc(\delta)|_{A_L}$. Similarly we write $\sigma = \sigma_u \otimes \nu_\sigma$ with $\sigma \in \mathrm{Irr}(M)$ unitary supercuspidal and $\nu_\sigma \in \mathrm{Hom}(M, \mathbb{R}_{>0})$. Then ν_σ decomposes as $(\nu_\sigma \nu_\delta^{-1}|_M)\, \nu_\delta|_M$ where $\nu_\delta|_M$ is trivial on $M \cap L_{\mathrm{der}}$ and $\nu_\sigma \nu_\delta^{-1}|_M$ is trivial on $Z(L)$. For $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we have

$$\gamma \cdot \delta = (\gamma \cdot \delta)_u \otimes \nu_{\gamma\delta} \quad \text{with} \quad \nu_{\gamma\delta}|_{A_L} = |cc(\gamma \cdot \delta)|_{A_L} = |\gamma \cdot cc(\delta)|_{A_L},$$

and analogously for σ. Moreover

$$\nu_{\gamma\sigma}\nu_{\gamma\delta}^{-1}|_M \text{ is trivial on } Z(L) \quad \text{and} \quad \nu_{\gamma\delta}|_M \text{ is trivial on } M \cap L_{\mathrm{der}}, \tag{5.10}$$

However, in general $(\gamma \cdot \delta)_u \not\cong \gamma \cdot \delta_u$ and $(\gamma \cdot \sigma)_u \not\cong \gamma \cdot \sigma_u$.

Lemma 5.6. *Let $\alpha \in \Phi(G, A_M)$.*

(a) If $\mu_{M_\alpha, M}(\sigma) = \infty$, then $\mu_{M_\alpha, M}(\gamma \cdot \sigma \otimes \epsilon_{MU_\alpha,\gamma}) = \infty$ and

$$\langle \alpha^\vee, \log(\nu_\sigma) \rangle \langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle > 0.$$

(b) If $\mu_{M_\alpha, M}(\sigma) = 0$, then $\mu_{M_\alpha, M}(\gamma \cdot \sigma \otimes \epsilon_{MU_\alpha,\gamma}) = 0$ and

$$\langle \alpha^\vee, \log(\nu_\sigma) \rangle = \langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle = 0.$$

Proof. (a) Proposition 5.2.e says that $\mu_{M_\alpha, M}(\gamma \cdot \sigma \otimes \epsilon_{MU_\alpha,\gamma}) = \infty$. By Theorem 5.3,

$$\gamma \cdot I_Q^G(\sigma) \cong I_Q^G(\gamma \cdot \sigma \otimes \epsilon_{MU_\alpha,\gamma}) = I_Q^G\big((\gamma \cdot \sigma)_u \otimes \epsilon_{MU_\alpha,\gamma} \otimes \nu_{\gamma\sigma}\big)$$

has the irreducible essentially square-integrable subquotient $\gamma \cdot \delta$. More precisely, $\gamma \cdot \delta$ is a quotient (resp. a subrepresentation) if and only if δ is a quotient (resp. a subrepresentation) of $I_Q^G(\sigma)$. Now Proposition 4.5 says that $\langle \alpha^\vee, (\log \nu_\sigma) \rangle$ and

$\langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle$ have the same sign (which is nonzero by Theorem 3.4.b).
(b) Theorem 3.4 shows that $\langle \alpha^\vee, \log(\nu_\sigma) \rangle = 0$. From Proposition 5.2.d we see that

$$\mu_{M_\alpha,M}\big((\gamma \cdot \sigma)_u \otimes \epsilon_{MU_\alpha,\gamma} \otimes \nu_{\gamma\sigma} \big) = \mu_{M_\alpha,M}(\gamma \cdot \sigma \otimes \epsilon_{MU_\alpha,\gamma}) = 0.$$

Then Theorem 3.4 proves also that $\langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle = 0$. □

Let $Q \subset G$ be a parabolic subgroup with Levi factor M, such that $P = QL$. Recall that $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma)$ is the set of $\alpha \in \Phi(G, A_M)$ for which $\mu_{M_\alpha,M}$ is not constant on $X_{\mathrm{nr}}(M)\sigma$ (or equivalently has a zero on $X_{\mathrm{nr}}(M)\sigma$). By [Hei, Proposition 1.3], this is a reduced root system in $X^*(A_M)$. The same holds with L instead of G, the crucial point is that σ is cuspidal.

The Weyl group $W\big(\Phi(L, A_M, X_{\mathrm{nr}}(M)\sigma)\big)$ is contained in $N_L(M)/M$ and acts on $\mathrm{Irr}(M)$. For any $w_\sigma \in W\big(\Phi(L, A_M, X_{\mathrm{nr}}(M)\sigma)\big)$, $I^L_{Q\cap L}(w_\sigma \cdot \sigma)$ has the same irreducible subquotients as $I^L_{Q\cap L}(\sigma)$, in particular δ. Furthermore $w_\sigma \cdot \nu_\delta = \nu_\delta$ because $(w_\sigma \cdot \nu_\delta)|_{Z(L)} = \nu_\delta|_{Z(L)}$.

We pick w_σ such that $w_\sigma \cdot \sigma$ lies in the (closed) positive Weyl chamber for $\Phi(L, A_M, X_{\mathrm{nr}}(M)\sigma)$, with respect to the positive roots from $Q \cap L$. Next we replace σ by $w_\sigma \cdot \sigma$, which is allowed because our main interest is not σ but δ. Thus

(5.11) $\log(\nu_\sigma)$ is positive with respect to $\Phi(L, A_M, X_{\mathrm{nr}}(M)\sigma) \cap \Phi(U_Q, A_M)$,

but maybe not with respect to other elements of $\Phi(U_Q, A_M)$.

Recall from Proposition 5.2.c that

$$\gamma \cdot I_Q^G(\sigma) \cong I_Q^G(\gamma \cdot \sigma \otimes \epsilon_{Q,\gamma}) = I_Q^G\big((\gamma \cdot \sigma)_u \otimes \epsilon_{Q,\gamma} \otimes \nu_{\gamma\sigma} \big).$$

By (5.1) we have

$$\Phi\big(G, A_M, X_{\mathrm{nr}}(M)(\gamma \cdot \sigma \otimes \epsilon_{Q,\gamma})\big) = \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma).$$

We pick a set of positive roots $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma)'^{+}$ such that $\log(\nu_{\gamma\sigma}) \in X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$ lies in the corresponding (closed) positive Weyl chamber.

For $\alpha \in \Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma)'^{+}$ we find, using the definition of $(\alpha|_{A_L})^\vee$ from (4.1)

$$\langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle \geq \langle (\alpha|_{A_L})^\vee, \log(\nu_{\gamma\sigma}) \rangle = \langle (\alpha|_{A_L})^\vee, \log(\nu_{\gamma\delta}) \rangle \geq 0. \tag{5.12}$$

This enables us to extend $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma)'^{+}$ to a set of positive roots $\Phi(G, A_M)'^{+}$ of $\Phi(G, A_M)$ such that

- (i) if $\alpha \in \Phi(G, A_M)'^{+} \cap \Phi(L, A_M)$, then $\langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle \geq 0$,
- (ii) if $\alpha \in \Phi(G, A_M)'^{+}, \alpha \notin \Phi(L, A_M)$, then $\langle (\alpha|_{A_L})^\vee, \log(\nu_{\gamma\delta}) \rangle \geq 0$.

Let $Q' \subset G$ be the parabolic subgroup with Levi factor M and

$$\Phi(U_{Q'}, A_M) = \Phi(G, A_M)'^{+}. \tag{5.13}$$

Then (ii) says that

(5.14) $\gamma \cdot \delta$ and $\log(\nu_{\gamma\delta})$ are positive with respect to $Q'L$.

Lemma 5.7. *$J'_{Q'L|QL}(\delta) : I^G_{QL}(\delta) \to I^G_{Q'L}(\delta)$ is an isomorphism.*

Proof. We take σ and $\Phi(G, A_M, X_{\mathrm{nr}}(M)\sigma)'^{+}$ as above. Suppose that $\alpha \in \Phi(U_Q, A_M)$ and $\mu_{M_\alpha,M}(\sigma) = \infty$. Then $\langle \alpha^\vee, \log(\nu_\sigma) \rangle \geq 0$ by (5.11). Lemma 5.6.a says that

$$\langle \alpha^\vee, \log(\nu_\sigma) \rangle > 0 \text{ and } \langle \alpha^\vee, \log(\nu_{\gamma\sigma}) \rangle > 0.$$

Now (5.14) guarantees that

$$\alpha \in \Phi(G, M, X_{\mathrm{nr}}(M)\sigma)'^{+} \subset \Phi(G, A_M)'^{+} = \Phi(U_Q, A_M),$$

and in particular $\alpha \notin \Phi(U_{\overline{Q'}}, A_M)$. As $U_{QL} \subset U_Q$ and $U_{Q'L} \subset U_{Q'}$, we find that $\mu_{M_\alpha, M}(\sigma) \neq \infty$ for all $\alpha \in \Phi(U_{QL}, A_M) \cap \Phi(U_{\overline{Q'}}, A_M)$. Now Theorem 3.9 says that $J'_{Q'L|QL}(\delta)$ is an isomorphism. □

In addition to $P = QL$, we write $P' = Q'L$ with Q' as in (5.13).

Theorem 5.8. *Let (P, L, δ) be a positive induction datum and let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.*
(a) $\gamma \cdot I_P^G(\delta) \cong I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})$ and $(P', L, \gamma \cdot \delta \otimes \epsilon_{P,\gamma})$ is a positive induction datum.
(b) For any $\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])$, there exists $\kappa' \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_{\gamma\cdot\delta} \otimes \epsilon_{P,\gamma}])$ such that

$$\gamma \cdot I_P^G(\delta)_\kappa \cong I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})_{\kappa'}.$$

(c) The action on $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathrm{Rep}(G)$ stabilizes the set of standard $\mathbb{C}G$-modules.

Proof. (a) From Proposition 5.2.c we know that

$$\gamma \cdot I_P^G(\delta) \cong I_P^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}).$$

By Lemma 5.7 $J'_{P'|P}(\delta) : I_P^G(\delta) \to I_{P'}^G(\delta)$ is an isomorphism. The operator $J'_{P'|P}(\delta)$ is a normalized version of $J_{P'|P}(\delta)$, and Proposition 5.2.c entails that

$$J'_{P'|P}(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})(z \otimes v) = z \otimes J'_{P'|P}(\delta)v \qquad z \in \mathbb{C}_\gamma, v \in I_P^G(V_\delta).$$

It follows that

$$J'_{P'|P}(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}) : I_P^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}) \to I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})$$

is an isomorphism. From Theorem 5.3 we see that $(P', L, \gamma \cdot \delta \otimes \epsilon_{P,\gamma})$ is an induction datum, and (5.14) says that it is positive.
(b) Recall from (3.19) and Theorem 4.3.a that every indecomposable direct summand of $I_P^G(\delta)$ is isomorphic to $I_P^G(\delta)_\kappa$ for some $\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])$. By part (a) that applies also to $\gamma \cdot I_P^G(\delta)$, and with (5.2) we can simplify it a little to

$$I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}) \cong \bigoplus_{\kappa' \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_{\gamma\cdot\delta} \otimes \epsilon_{P,\gamma}])} \kappa' \otimes I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})_{\kappa'}. \tag{5.15}$$

As $I_P^G(\delta)_\kappa$ is isomorphic to an indecomposable direct summand of $I_P^G(\delta)$, the representation $\gamma \cdot I_P^G(\delta)_\kappa$ is isomorphic to an indecomposable direct summand of

$$\gamma \cdot I_P^G(\delta) \cong I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma}).$$

By (5.15), the latter has the form $I_{P'}^G(\gamma \cdot \delta \otimes \epsilon_{P,\gamma})_{\kappa'}$ for some κ'.
(c) Recall from Theorem 4.3.b that every standard $\mathbb{C}G$-module has the form $I_P^G(\delta)_\kappa$. By parts (a) and (b) and Theorem 4.3.a, $\gamma \cdot I_P^G(\delta)_\kappa$ is (isomorphic to) a standard $\mathbb{C}G$-module. □

References

[Art] J. Arthur, "On elliptic tempered characters", Acta. Math. **171** (1993), 73–138
[ABPS] A.-M. Aubert, P.F. Baum, R.J. Plymen, M. Solleveld, "On the local Langlands correspondence for non-tempered representations", Münster J. Math. **7** (2014), 27–50
[AMS] A.-M. Aubert, A. Moussaoui, M. Solleveld, "Graded Hecke algebras for disconnected reductive groups", pp. 23–84 in: *Geometric aspects of the trace formula, W. Müller, S. W. Shin, N. Templier (eds.)*, Simons Symposia, Springer, 2018
[DiFi] I. Dimitrov, R. Fioresi, "Generalized root systems", arXiv:2308.06852, 2023

[FlSo] J. Flikkema, M. Solleveld, "Intertwining operators for representations of covering groups of reductive p-adic groups", arXiv:2502.18128v2, 2025

[Hei1] V. Heiermann, "Décomposition spectrale et représentations spéciales d'un groupe réductif p-adique", J. Inst. Math. Jussieu **3** (2004), 327–395

[Hei] V. Heiermann, "Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique - le cas des groupes classiques", Selecta Math. **17.3** (2011), 713–756

[KSV] D. Kazhdan, M. Solleveld, Y. Varshavksy, "Rationality properties of complex representations of reductive p-adic groups", arXiv:2510.24201, 2025

[FaSc] L. Fargues, P. Scholze, "Geometrization of the local Langlands correspondence", arXiv:2102.13459, 2021

[KrCl] A. Kret, L. Clozel, "On the central value of Rankin L-functions for self-dual algebraic representations of linear groups over totally real fields", Tunis. J. Math. **7.3-4** (2025), 637–669

[Lus1] G. Lusztig, "Intersection cohomology complexes on a reductive group", Invent. Math. **75.2** (1984), 205–272

[Lus2] G. Lusztig, "Open problems on Iwahori–Hecke algebras", EMS Newsletter (2020)

[MWa] C. Mœglin, J.-L. Waldspurger, *Décomposition spectrale et séries d'Eisenstein*, Progress in Mathematics **113**. Birkhäuser Verlag, Basel, 1994

[MoTr] G. Moss, J. Trias, "The universal Harish-Chandra j-function", arXiv:2509.20169, 2025

[Oha] K. Ohara, "On parameters of Hecke algebras for p-adic groups", arXiv:2505.16040, 2025

[Opd] E.M. Opdam, "On the spectral decomposition of affine Hecke algebras", J. Inst. Math. Jussieu **3.4** (2004), 531–648

[Ren] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours spécialisés **17**, Société Mathématique de France, 2010

[Sau] F. Sauvageot, "Principe de densité pour les groupes réductifs", Compositio Math. **108** (1997), 151–184

[Sil1] A.J. Silberger, "The Knapp-Stein dimension theorem for p-adic groups", Proc. Amer. Math. Soc. **68.2** (1978), 243–246 and Correction Proc. Amer. Math. Soc. **76.1** (1979), 169–170

[Sil2] A.J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Mathematical Notes **23**, Princeton University Press, Princeton NJ, 1979

[Sil3] A.J. Silberger, "Special representations of reductive p-adic groups are not integrable", Ann. Math. **111** (1980), 571–587

[Sil4] A.J. Silberger, "On Harish-Chandra's μ-function for p-adic groups", Trans. Amer. Math. Soc. **260.1** (1980), 113–121.

[Sol1] M. Solleveld, "Periodic cyclic homology of reductive p-adic groups", J. Noncommut. Geom. **3.4** (2009), 501–558

[Sol2] M. Solleveld, "Endomorphism algebras and Hecke algebras for reductive p-adic groups", J. Algebra **606** (2022), 371–470, and correction in arXiv:2005.07899v3 (2023)

[Sol3] M. Solleveld, "Parameters of Hecke algebras for Bernstein components of p-adic groups", Indag. Math. **36.1** (2025), 124–170

[Tad] M. Tadić, "Notes on representations of non-archimedean $SL(n)$", Pacific J. Math. **152.2** (1992), 375–396

[Wal] J.-L. Waldspurger, "La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)", J. Inst. Math. Jussieu **2.2** (2003), 235–333