ON SUBMODULES OF STANDARD MODULES

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Abstract. Consider a standard representation π_{st} of a quasi-split reductive p-adic group G. The generalized injectivity conjecture, posed by Casselman and Shahidi, asserts that any generic irreducible subquotient π of π_{st} is necessarily a subrepresentation of π_{st}. We will prove this conjecture, improving on the verification for many groups by Dijols.

In the process, we first replace it by a more general "standard submodule conjecture", where G does not have to be quasi-split and the genericity of π is replaced by the condition that the Langlands parameter of π is open. (This means that the nilpotent element from the L-parameter belongs to an appropriate open orbit.)

We study this standard submodule conjecture via reduction to Hecke algebras. It does not suffice to pass from G to an affine Hecke algebra, we further reduce to graded Hecke algebras and from there to algebras defined in terms of certain equivariant perverse sheaves. To achieve all these reduction steps one needs mild conditions on the parameters of the involved Hecke algebras, which have been verified for the large majority of reductive p-adic groups and are expected to hold in general.

It is in the geometric setting of graded Hecke algebras from cuspidal local systems on nilpotent orbits that we can finally put the "open" condition on L-parameters to good use. The closure relations between the involved nilpotent orbits provide useful insights in the internal structure of standard modules, which highlight the representations associated with open L-parameters. In the same vein we show that, in the parametrization of irreducible modules of geometric graded Hecke algebras, generic modules always have "open L-parameters".

This leads to a proof of our standard submodule conjecture for graded Hecke algebras of geometric type, which is then transferred to reductive p-adic groups. As a bonus, we obtain that the generalized injectivity conjecture also holds with "tempered" or "essentially square-integrable" instead of generic.

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Introduction

Let $\mathcal{G}$ be a connected reductive group defined over a non-archimedean local field F. We will simply call $\mathcal{G}(F)$ a reductive p-adic group. All our $\mathcal{G}(F)$-representations will by default be smooth and on complex vector spaces.

Recall that any standard $\mathcal{G}(F)$-representation arises in the following way. Start with a parabolic subgroup $\mathcal{P}(F) = \mathcal{M}(F)\mathcal{U}_{\mathcal{P}}(F)$ of $\mathcal{G}(F)$ and an irreducible tempered representation τ of the Levi factor $\mathcal{M}(F)$. Let $\chi \in \mathrm{Hom}(\mathcal{M}(F), \mathbb{R}_{>0})$ be in positive position with respect to $\mathcal{P}(F)$. Let $I_{\mathcal{P}(F)}^{\mathcal{G}(F)}$ be the normalized parabolic induction functor. Then

$$\pi(\mathcal{P}(F), \tau, \chi) = I_{\mathcal{P}(F)}^{\mathcal{G}(F)}(\tau \otimes \chi)$$

is a standard $\mathcal{G}(F)$-representation. Such representations are interesting for several reasons:

- By the Langlands classification [Ren, §VII.4.2], every standard representation has a unique irreducible quotient. This yields a bijection from the set of standard $\mathcal{G}(F)$-representations (up to isomorphism) to $\mathrm{Irr}(\mathcal{G}(F))$, the set of irreducible $\mathcal{G}(F)$-representations up to isomorphism.
- Standard representations interpolate between $\mathrm{Irr}(\mathcal{G}(F))$ and the set of irreducible tempered representations of Levi subgroups of $\mathcal{G}(F)$.
- Standard representations are often easier to handle than irreducible representations. For instance, one can vary χ in $\pi(\mathcal{P}(F), \tau, \chi)$ and then one obtains a holomorphic family of $\mathcal{G}(F)$-representations (even an algebraic family if we forget about the positivity of χ), which can all be defined on the same vector space.
- There also exist standard representations (or modules) in related settings like real reductive groups, semisimple or affine Lie algebras and affine Hecke algebras. Especially for semisimple Lie algebras, the Jordan–Hölder content of a standard module has interesting geometric interpretations. This goes via the Kazhdan–Lusztig conjecture, we refer to [KaLu1] and [Ach, §7.3.10] for more background.

Conjectures about standard modules and generic representations
A natural follow-up to the Langlands classification is the question: when is a standard $\mathcal{G}(F)$-representation irreducible? Although they are almost always irreducible, the cases where they are not are usually more interesting. For $\pi(\mathcal{P}(F), \tau, \chi)$ with $\mathcal{G}(F)$ quasi-split and τ generic, this was the subject of the standard module conjecture, posed by Casselman–Shahidi [CaSh] and proven in [HeMu, HeOp]. It says that $\pi(\mathcal{P}(F), \tau, \chi)$ is irreducible if and only if it is generic.

Next one may wonder: what are the irreducible subrepresentations, or more generally the irreducible subquotients of a standard representation? The multiplicities with which irreducible $\mathcal{G}(F)$-representations appear as constituents of a standard representation are predicted by the Kazhdan–Lusztig conjecture, formulated for p-adic groups by Vogan [Vog, Conjecture 8.11]. It is phrased in terms of the geometry of Langlands parameters, and has been proven in many cases in [Sol8]. However, that does not yet tell us which of these constituents occur as subrepresentations. One aspect of that issue is:

Conjecture A. (Generalized injectivity conjecture [CaSh])
Let $\mathcal{G}(F)$ be quasi-split and τ generic. Then any generic irreducible subquotient of the standard representation $\pi(\mathcal{P}(F), \tau, \chi)$ is a subrepresentation.

We remark that in this setting it is known that $\pi(\mathcal{P}(F), \tau, \chi)$ has a unique generic irreducible subquotient. Conjecture A has been verified whenever $\mathcal{G}(F)$ has no simple factors of exceptional type (and for many Bernstein blocks for other groups as well) by Dijols [Dij]. The proof is a tour de force with L-functions, root data and combinatorics. Unfortunately, it did not give the author of these lines much feeling for why Conjecture A should hold.

Based on comparisons with other known results and conjectures, we believe that Conjecture A can be regarded as a special case of a more common phenomenon. Before formulating this more general conjecture, we will discuss some of its background.

Let $\mathcal{B}$ be a minimal parabolic F-subgroup of $\mathcal{G}$ and let $\mathcal{U}$ be the unipotent radical of $\mathcal{B}$. Fix a character $\xi : \mathcal{U}(\mathcal{F}) \to \mathbb{C}^\times$ which is nondegenerate, that is, nontrivial on every root subgroup $\mathcal{U}_\alpha(F) \subset \mathcal{U}(F)$ for a simple root α. We recall that a $\mathcal{G}(F)$-representation π is generic, or more precisely $(\mathcal{U}(F), \xi)$-generic, if $\mathrm{Hom}_U(\pi, \xi)$ is nonzero. An equivalent condition is

$$\mathrm{Hom}_{\mathcal{G}(F)}\big(\pi, \mathrm{Ind}_{\mathcal{U}(F)}^{\mathcal{G}(F)}(\xi)\big) \neq 0,$$

where Ind means smooth induction. Following [BuHe]

$$\text{we call } \pi \text{ simply generic if } \dim \mathrm{Hom}_{\mathcal{G}(F)}\big(\pi, \mathrm{Ind}_{\mathcal{U}(F)}^{\mathcal{G}(F)}(\xi)\big) = 1.$$

It has been known for a long time that every irreducible generic representation of a quasi-split group is simply generic [Rod, Shal]. In our view, this multiplicity one property is the essence of genericity for quasi-split groups, for it enables the normalization of several objects, in particular of intertwining operators between parabolically induced representations. Therefore it does not come as a surprise that many properties of generic representations of quasi-split groups also hold for simply generic representations of arbitrary reductive p-adic groups.

The enhanced L-parameters of generic irreducible representations of quasi-split groups are known (among others) for classical groups [Art], for unipotent representations [Ree] and for principal series representations [Sol10]. All these L-parameters are open, in the following sense.

Let ${}^L\mathcal{G} = \mathcal{G}^\vee \rtimes \mathbf{W}_F$ be the L-group of $\mathcal{G}(F)$ and write $\mathfrak{g}^\vee = \mathrm{Lie}(\mathcal{G}^\vee)$. Consider a L-parameter for $\mathcal{G}(F)$ in Weil–Deligne form, so a pair (ϕ, N) with

- $\phi : \mathbf{W}_F \to {}^L\mathcal{G}$ is a semisimple smooth homomorphism,
- $N \in \mathfrak{g}^\vee$ is nilpotent and belongs to

$$\mathfrak{g}^\vee_\phi = \{X \in \mathfrak{g}^\vee : \mathrm{Ad}(\phi(w))X = \|w\| X \text{ for all } w \in \mathbf{W}_F\}.$$

It is known from [KaLu2] that $Z_{\mathcal{G}^\vee}(\phi)^\circ$ acts on the variety $\mathfrak{g}^\vee_\phi$ with finitely many orbits, of which one is Zariski-open. In this setup

$$(\phi, N) \text{ is called open when } \mathrm{Ad}(Z_{\mathcal{G}^\vee}(\phi))N \text{ is open in } \mathfrak{g}^\vee_\phi. \tag{1}$$

We encountered this terminology in [CFZ, §0.6], where it is mentioned that bounded L-parameters are open. In the same vein, discrete L-parameters are open. Meanwhile these claims have been proven in [CDFZ], although not yet for all discrete L-parameters. In Proposition 7.2 we show, in an alternative way, that bounded or discrete L-parameters are always open. For the (conjectural) local Langlands correspondence, this means that the L-parameters of tempered representations and of essentially square-integrable representations should be open.

Further, we recall that Shahidi [Shah, Conjecture 9.4] has conjectured that every tempered L-packet for a quasi-split group has a generic member. Based on the above, on [GrPr, Conjecture 2.6], on [GrRe, Conjecture 7.1.(3)] and on the results in this paper, we pose:

Conjecture B. *Let π be an irreducible representation of a reductive p-adic group $\mathcal{G}(F)$. Assume that a local Langlands correspondence exists for the Bernstein block of* $\mathrm{Rep}(\mathcal{G}(F))$ *containing π.*

(*a*) *If π is simply generic, then its L-parameter is open.*

(*b*) *Suppose that $\mathcal{G}(F)$ is quasi-split and that the local Langlands correspondence is normalized with respect to the Whittaker datum $(\mathcal{U}(F), \xi)$, in the following sense: every irreducible $(\mathcal{U}(F), \xi)$-generic representation is matched with an enhanced L-parameter such that the enhancement is the trivial representation.*

Then π is $(\mathcal{U}(F), \xi)$-generic if and only if its L-parameter (ϕ, N) is open and its enhancement is the trivial representation of $\pi_0(Z_{\mathcal{G}^\vee}(\phi, N))$.

Part (b), or very similar statements, has been known to several experts. The authors of [CDFZ] have, independently from the current paper, arrived at the same formulation. We remark that for irreducible representations of non-quasi-split groups $\mathcal{G}(F)$, the trivial representation should never occur as the enhancement of a Langlands parameter, because it should already correspond to a representation of the quasi-split inner form of $\mathcal{G}(F)$.

It seems to us that the reason why Conjecture A should hold in larger generality is not so much the genericity of π, but rather that the L-parameter of π is open (as in Conjecture B). One can say that we replace the analytic motivation for Conjecture A from [CaSh] by algebro-geometric motivation. Our expectations come together in the next "standard submodule conjecture".

Conjecture C. *Let π_{st} be a standard representation of a reductive p-adic group $\mathcal{G}(F)$ and let π be an irreducible subquotient of π_{st}. Suppose that (a), (b), (c) or (d) holds:*

(*a*) *a local Langlands correspondence exists for the Bernstein block of* $\mathrm{Rep}(\mathcal{G}(F))$ *containing π_{st}, and the L-parameter of π is open,*

(*b*) *π is tempered,*

(*c*) *π is essentially square-integrable,*

(*d*) *π and π_{st} are simply generic.*

Then π is a subrepresentation of π_{st}.

We note that part (d) of our standard submodule conjecture contains conjecture A. As we mentioned before, it is expected that under either of the assumptions (b), (c), (d), the L-parameter of π is open. Hence part (a) is the most general case of Conjecture C. On the other hand, it is not clear how to formulate assumption (a) purely in terms of $\mathcal{G}(F)$-representations, and even less so when no local Langlands correspondence is known for the involved representations. That is an advantage of parts (b), (c), (d) over part (a).

To state our main result, we focus on a Bernstein block $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ with $\mathfrak{s} = [\mathcal{M}(F), \omega]$. It was shown in [Sol4, Corollary 9.4] that $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ is closely related to the module category of a certain affine Hecke algebra $\mathcal{H}_{\mathfrak{s}}$. Lusztig [Lus7] has conjectured that the q-parameters of $\mathcal{H}_{\mathfrak{s}}$ are always of a special kind, namely they are parameters that also arise from a Bernstein block of unipotent representations. Meanwhile, Lusztig's conjecture has been verified in [Sol7], for all reductive p-adic groups $\mathcal{G}(F)$ that do not have any simple factors of following kinds:

- Lie type $E_7, {}^2E_7$ or E_8,
- isogenous to a special orthogonal or symplectic group of quaternionic type.

Theorem D. (see Theorems 6.6 and 6.9)
Consider a Bernstein block $\mathrm{Rep}(\mathcal{G}(F))^{[\mathcal{M}(F),\omega]}$ *in the category of smooth complex representations of a reductive p-adic group* $\mathcal{G}(F)$. *Suppose one of the following:*

(a) *Lusztig's conjecture* [Lus7] *about Hecke algebra parameters holds for* $\mathrm{Rep}(\mathcal{G}(F))^{[\mathcal{M}(F),\omega]}$,
(b) $\mathcal{G}(F)$ *is quasi-split and* ω *is generic.*

Then parts (b), (c), (d) of Conjecture C hold.

In particular, Theorem D.b proves the generalized injectivity conjecture from [CaSh], via Conjecture C.d.

To obtain results about Conjecture B or Conjecture C.a with our techniques, we need to suppose that a good local Langlands correspondence, constructed via Hecke algebras, is available. The precise assumptions are formulated in Condition 7.3. Currently this condition has been shown to hold in the following cases:

- inner forms of general/special linear groups, [ABPS] and [AMS3, §5],
- pure inner forms of quasi-split classical F-groups [Hei, MoRe, AMS4],
- principal series representations of quasi-split F-groups [Sol10],
- unipotent representations (of arbitrary connected reductive groups over F) [Lus5, Lus6, Sol5, Sol6],
- G_2 [AuXu].

Theorem E. (see Theorem 7.4)
Suppose that Condition 7.3 holds for $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$, *so for instance we are in one the cases listed above.*

(a) *Suppose that* $\pi \in \mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ *is tempered or essentially square-integrable, or that* ω *(from* $\mathfrak{s}$*) is simply generic and* π *generic. Then the L-parameter of* π *is open.*
(b) *Suppose that* $\pi \in \mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ *has an open L-parameter and is a subquotient of a standard* $\mathcal{G}(F)$*-representation* π_{st}. *Then* π *is a subrepresentation of* π_{st}.

In particular Conjectures B.a and C hold for $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$.

To prove instances of Conjecture B.b, one would have to know in addition that the L-packet of an open L-parameter always contains at least one generic representation. That aspect of Conjecture B seems rather challenging.

In the remainder of the introduction we discuss how we proved Theorems D and E.

Reduction from p-adic groups to graded Hecke algebras

Firstly, one can reduce from $\mathrm{Rep}(G)^{\mathfrak{s}}$ to the module category of some kind of affine Hecke algebra. This has been achieved in full generality in [Sol4], but in the most general case some technical difficulties remain, which entail that one does not exactly obtain the module category of a (twisted) affine Hecke algebra. In Section 6 we check that this procedure is still good enough to transfer parts (a),(b),(c) of Conjecture C to statements about modules of (twisted) affine Hecke algebras. When the σ from $\mathfrak{s}$ is simply generic, $\mathrm{Rep}(G)^{\mathfrak{s}}$ is really equivalent to the module category of an extended affine Hecke algebra $\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}}$, and the equivalence of categories preserves genericity [OpSo, Theorem E].

The next step is reduction from a twisted affine Hecke algebra $\mathcal{H}_{\mathfrak{s}} \rtimes \mathbb{C}[\Gamma_{\mathfrak{s}}, \natural_{\mathfrak{s}}]$ to a twisted graded Hecke algebra $\mathbb{H}_\sigma \rtimes \mathbb{C}[\Gamma_\sigma, \natural_\sigma]$, as discussed in Section 5. The procedure for that is known in general from [Lus3, Sol2, AMS3], and preserves all the relevant properties of representations. This translates Conjecture C to statements about twisted graded Hecke algebras. In fact can also reduce directly from $\mathrm{Rep}(G)^{\mathfrak{s}}$ to twisted graded Hecke algebras, skipping the slightly messy step with affine Hecke algebras, that is done in [Sol4].

To proceed, we need the graded Hecke algebra $\mathbb{H}_\sigma$ to be of geometric type, by which we mean that it arises from a cuspidal local system on a nilpotent orbit as in [Lus2, Lus4, AMS2]. This puts a condition on the deformation parameters k_α of $\mathbb{H}_\sigma$, which can be retraced to a condition on the q-parameters of $\mathcal{H}_{\mathfrak{s}}$. That condition is implied by Lusztig's conjecture [Lus7] on the q-parameters of $\mathcal{H}_{\mathfrak{s}}$, but it allows a wider choice of parameters than [Lus7]. In Theorem 6.6 we show that, when $\mathcal{G}(F)$ is quasi-split and ω is generic, all the ensuing extended graded Hecke algebras $\mathbb{H}_\sigma \rtimes \Gamma_\sigma$ have equal parameters, and in particular are of geometric type.

Representation theory of graded Hecke algebras of geometric type Finally, we come to the topic of the largest part of the paper: the standard submodule conjecture for twisted graded Hecke algebras. In Sections 1–4 the setup is quite different from above. We start with a complex reductive group G (not related to $\mathcal{G}$). In the body of the paper G may be disconnected, but in this introduction we slightly simplify the presentation by assuming that G is connected. Let M be a Levi subgroup of G and let $\mathcal{E}$ be a M-equivariant cuspidal local system on a nilpotent orbit $\mathcal{C}_v^M$ in $\mathfrak{m}$. To these data Lusztig [Lus2, Lus4] associated a graded Hecke algebra $\mathbb{H}(G, M, \mathcal{E})$. As a vector space it is a tensor product of three subalgebras:

$$\mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}] \otimes \mathbb{C}[W_{\mathcal{E}}], \text{ where } \mathfrak{t} = \mathrm{Lie}(T), T = Z(M), W_{\mathcal{E}} = N_G(M)/M.$$

In the algebra $\mathbb{H}(G, M, \mathcal{E})$, $\mathbf{r}$ is central and the cross relations between $\mathcal{O}(\mathfrak{t})$ and $\mathbb{C}[W_{\mathcal{E}}]$ are determined by parameters k_α for $\alpha \in R(G, T)$. The graded Hecke algebras $\mathbb{H}_\sigma$ discussed above arise from $\mathbb{H}(G, M, \mathcal{E})$ by specializing $\mathbf{r}$ at some $r \in \mathbb{R}_{>0}$.

The irreducible representations of $\mathbb{H}(G, M, \mathcal{E})$ are naturally parametrized by G-conjugacy classes of "enhanced L-parameters for $\mathbb{H}(G, M, \mathcal{E})$". These are quadruples (y, σ, r, ρ) where $r \in \mathbb{C}$, $y \in \mathfrak{g}$ is nilpotent and $\sigma \in \mathfrak{g}$ is semisimple such that $[\sigma, y] = 2ry$. Further ρ is an irreducible representation of $\pi_0(Z_G(y, \sigma))$, subject to a certain cuspidal support condition. We fix $v \in \mathcal{C}_v^M$ and we extend it to an $\mathfrak{sl}_2$-triple in $\mathfrak{m}$, with semisimple element σ_v. The above conditions force $\sigma \in \mathrm{Ad}(G)(\mathfrak{t} + r\sigma_v)$, so we may assume that $\sigma \in \mathfrak{t} + r\sigma_v$.

For every parameter (y, σ, r, ρ) there is a "geometric standard" module $E_{y,\sigma,r,\rho}$, constructed using equivariant perverse sheaves. It has an irreducible quotient $M_{y,\sigma,r,\rho}$, which is unique if $r \neq 0$. In that sense the Langlands classification holds for $\mathbb{H}(G, M, \mathcal{E})$. It is preferable to pull back $E_{y,\sigma,r,\rho}$ along the sign automorphism of $\mathbb{H}(G, M, \mathcal{E})$, given by

$$\mathrm{sgn}|_{\mathcal{O}(\mathfrak{t})} = \mathrm{id},\ \mathrm{sgn}(\mathbf{r}) = -\mathbf{r},\ \mathrm{sgn}(s_\alpha) = -s_\alpha \text{ for every root } \alpha.$$

When $\Re(r) < 0$, the modules $\mathrm{sgn}^* E_{y,\sigma,r,\rho}$ are precisely the "analytic standard" modules of $\mathbb{H}(G, M, \mathcal{E})/(\mathbf{r} + r)$ in the sense of Langlands (Proposition 3.7).

The centre of $\mathbb{H}(G, M, \mathcal{E})$ is $\mathcal{O}(\mathfrak{t})^{W_\mathcal{E}} \otimes \mathbb{C}[\mathbf{r}]$, so the space of central characters is $\mathfrak{t}/W_\mathcal{E} \times \mathbb{C}$. Then $Z(\mathbb{H}(G, M, \mathcal{E}))$ acts on $\mathrm{sgn}^* E_{y,\sigma,r,\rho}$ with $\sigma \in \mathfrak{t} + r\sigma_v$ by the character $(W_\mathcal{E}(\sigma - r\sigma_v), -r)$. Moreover, every irreducible $\mathbb{H}(G, M, \mathcal{E})$-module with this central character is of the form $\mathrm{sgn}^* M_{y',\sigma,r,\rho'}$ for suitable (y', ρ'). This allows us to focus on a fixed pair $(\sigma, r) \in \mathfrak{t} \oplus \mathbb{C}(\sigma_v, 1)$ in the remainder of the introduction. The nilpotent parameter y lies in

$$\mathfrak{g}_N^{\sigma,r} = \{X \in \mathfrak{g} \text{ nilpotent} : [\sigma, X] = 2rX\}.$$

Analogous to (1), we say that (y, σ, r) is open if $\mathrm{Ad}(Z_G(\sigma))y$ is the unique open orbit in $\mathfrak{g}_N^{\sigma,r}$.

Let us consider the category $\mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G, M, \mathcal{E}))$ of finite dimensional $\mathbb{H}(G, M, \mathcal{E})$-modules all whose irreducible constituents admit the central character $(W_\mathcal{E}(\sigma + r\sigma_v), r)$. According to [Lus4, Sol8], this category is canonically equivalent to

$$\mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}_{\mathcal{D}^b_{Z_G(\sigma)}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})\big),$$

where $K_{N,\sigma,r}$ is a certain $Z_G(\sigma)$-equivariant perverse sheaf on $\mathfrak{g}_N^{\sigma,r}$. With these notations

$$\begin{array}{lll} E_{y,\sigma,r} & = & H^*(\{y\}, i_y^! K_{N,\sigma,r}), \\ E_{y,\sigma,r,\rho} & = & \mathrm{Hom}_{\pi_0(Z_G(\sigma,y))}\big(\rho, E_{y,\sigma,r}\big). \end{array}$$

In this setting we deduce the crucial geometric step in our chain of arguments:

Proposition F. (see Propositions 2.3 and 2.4)
Let (y, σ, r, ρ) be an enhanced L-parameter for $\mathbb{H}(G, M, \mathcal{E})$ and let $\mathcal{L}_\rho$ be the local system on $\mathcal{O}_y = \mathrm{Ad}(Z_G(\sigma))y$ induced by ρ. For another parameter (y', σ, r, ρ'):

$$\mathrm{Hom}_{\mathbb{H}(G,M,\mathcal{E})}(\mathrm{sgn}^* E_{y',\sigma,r,\rho'}, \mathrm{sgn}^* E_{y,\sigma,r,\rho}) = \mathrm{Hom}_{\mathbb{H}(G,M,\mathcal{E})}(E_{y',\sigma,r,\rho'}, E_{y,\sigma,r,\rho})$$

is nonzero if and only if $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$ and $\mathcal{L}_\rho$ appears in $\mathrm{IC}(\mathfrak{g}_N^{\sigma,r}, \mathcal{L}_{\rho'})|_{\mathcal{O}_y}$.

Proposition F quickly implies that the irreducible modules $\mathrm{sgn}^* M_{y',\sigma,r,\rho'}$ with (y', σ, r) open occur as submodules of standard modules $\mathrm{sgn}^* E_{y,\sigma,r,\rho}$ (Theorem 3.2). In Lemma 3.4 and Theorem 3.5 (which comes from [AMS2]), we check that irreducible $\mathbb{H}(G, M, \mathcal{E})$-modules which are tempered or essentially discrete series have open parameters. That proves the larger part of the standard submodule conjecture for graded Hecke algebras of geometric type.

The condition for genericity of $\mathbb{H}(G, M, \mathcal{E})$-module is derived from [OpSo, §6]:

$$V \in \mathrm{Mod}(\mathbb{H}(G, M, \mathcal{E})) \text{ is generic if } \mathrm{Res}^{\mathbb{H}(G,M,\mathcal{E})}_{\mathbb{C}[W_\mathcal{E}]} V \text{ contains the sign representation.}$$

In Proposition 4.1 we show that every standard module $\mathrm{sgn}^* E_{y,\sigma,r,\rho}$ has at most one irreducible generic subquotient, like for standard representations of quasi-split reductive p-adic groups. Hence generic for Hecke algebras corresponds to simply generic for reductive p-adic groups. By reduction to the case $r = 0$, we prove:

Theorem G. (see Theorem 4.5)

(*a*) *For fixed $(\sigma, r) \in \mathfrak{t} \oplus \mathbb{C}(\sigma_v, 1)$, there is a unique (up to conjugacy) pair (y_g, ρ_g) such that $\mathrm{sgn}^* M_{y_g,\sigma,r,\rho_g}$ is generic. Here (y_g, σ, r) is an open parameter.*

(*b*) *Suppose that $(G, M, \mathcal{E})$ has equal parameters, e.g. it arises from a generic Bernstein block for a quasi-split reductive p-adic group. Then $\mathcal{C}_v^M = \{0\}$, $\mathcal{E}$ is the trivial local system and $\rho_g = \mathrm{triv}$.*

Altogether the above results prove the version of Conjecture C for (twisted) graded Hecke algebras of geometric type. Theorem D applies that in the cases where the reduction from $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ to twisted graded Hecke algebras works well. Similarly, Theorem E uses that when a nice LLC via graded Hecke algebras of geometric type is available.

1. Geometric construction of twisted graded Hecke algebras

All the groups in Sections 1–5 will be complex linear algebraic groups. We mainly work in the equivariant bounded derived categories of constructible sheaves from [BeLu]. For a group H acting on a space X, this category will be denoted $\mathcal{D}^b_H(X)$.

Let G be a complex reductive group, possibly disconnected. To construct a graded Hecke algebra geometrically, we need a cuspidal quasi-support $(M, \mathcal{C}_v^M, q\mathcal{E})$ for G [AMS1]. This consists of:

- a quasi-Levi subgroup M of G, which means that M° is a Levi subgroup of G° and $M = Z_G(Z(M^\circ)^\circ)$,
- $\mathcal{C}_v^M$ is a $\mathrm{Ad}(M)$-orbit in the nilpotent variety $\mathfrak{m}_N$ in the Lie algebra $\mathfrak{m}$ of M,
- $q\mathcal{E}$ is a M-equivariant cuspidal local system on $\mathcal{C}_v^M$.

We write $T = Z(M)^\circ = Z(M^\circ)^\circ$, $\mathfrak{t} = \mathrm{Lie}(T)$ and

$$W_{q\mathcal{E}} = \mathrm{Stab}_{N_G(M)}(q\mathcal{E})/M = N_G(M, q\mathcal{E})/M,$$

which is a finite group. Let $\mathcal{E}$ be an irreducible M°-equivariant local system on $\mathcal{C}_v^{M^\circ}$ contained in $q\mathcal{E}|_{\mathcal{C}_v^{M^\circ}}$. Then

$$W_{q\mathcal{E}} = W_{\mathcal{E}} \rtimes \Gamma_{q\mathcal{E}},$$

where $W_{\mathcal{E}}$ is the Weyl group of a root system and $\mathcal{R}_{q\mathcal{E}}$ is the $W_{q\mathcal{E}}$-stabilizer of the set of positive roots. To these data one associates a twisted graded Hecke algebra

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}}), \tag{1.1}$$

see [Sol9, §2.1]. As vector space it is the tensor product of

- a polynomial algebra $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C}) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[\mathbf{r}]$,
- a twisted group algebra $\mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$,

and there are nontrivial cross relations between these two subalgebras. The most important cross relation involves a simple root α. It comes with a simple reflection $s_\alpha \in W_{q\mathcal{E}}^\circ$, a basis element N_{s_α} of $\mathbb{C}[W_{\mathcal{E}}] \subset \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$ and a parameter $k_\alpha \in \mathbb{C}$. For $\xi \in \mathcal{O}(\mathfrak{t})$:

$$N_{s_\alpha}\xi - (\xi \circ s_\alpha) N_{s_\alpha} = k_\alpha \mathbf{r} (\xi - \xi \circ s_\alpha)/\alpha.$$

For elements $\gamma \in \Gamma_{q\mathcal{E}}$ there is a simpler cross relation:

$$N_\gamma \xi = (\xi \circ \gamma^{-1}) N_\gamma.$$

Let $\mathfrak{g}_N$ be the nilpotent variety in the Lie algebra $\mathfrak{g}$ of G. The algebra (1.1) can be realized in terms of suitable equivariant sheaves on $\mathfrak{g}$ or $\mathfrak{g}_N$. We let $\mathbb{C}^\times$ act on $\mathfrak{g}$ and $\mathfrak{g}_N$ by $\lambda \cdot X = \lambda^{-2} X$. Then every M-equivariant local system on $\mathcal{C}_v^M$, and in particular $q\mathcal{E}$, is automatically $M \times \mathbb{C}^\times$-equivariant.

Let $P^\circ = M^\circ U$ be the parabolic subgroup of G° with Levi factor M° and unipotent radical U matching the aforementioned choice of positive roots. Then $P = MU$

is a "quasi-parabolic" subgroup of G. Consider the varieties

$$\dot{\mathfrak{g}} = \{(X, gP) \in \mathfrak{g} \times G/P : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\},$$
$$\dot{\mathfrak{g}}_N = \dot{\mathfrak{g}} \cap (\mathfrak{g}_N \times G/P).$$

We let $G \times \mathbb{C}^\times$ act on these varieties by

$$(g_1, \lambda) \cdot (X, gP) = (\lambda^{-2} \mathrm{Ad}(g_1)X, g_1 gP). \tag{1.2}$$

By [Lus2, Proposition 4.2] there are natural isomorphisms of graded algebras

$$H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}) \cong H^*_{G \times \mathbb{C}^\times}(\dot{\mathfrak{g}}_N) \cong \mathcal{O}(\mathfrak{t}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{r}]. \tag{1.3}$$

Consider the maps

$$\begin{array}{ll} \mathcal{C}_v^M \xleftarrow{f_1} \{(X, g) \in \mathfrak{g} \times G : \mathrm{Ad}(g^{-1})X \in \mathcal{C}_v^M \oplus \mathfrak{t} \oplus \mathfrak{u}\} \xrightarrow{f_2} \dot{\mathfrak{g}}, & \\ f_1(X, g) = \mathrm{pr}_{\mathcal{C}_v^M}(\mathrm{Ad}(g^{-1})X), & f_2(X, g) = (X, gP). \end{array} \tag{1.4}$$

Let $\dot{q\mathcal{E}}$ be the unique $G \times \mathbb{C}^\times$-equivariant local system on $\dot{\mathfrak{g}}$ such that $f_2^* \dot{q\mathcal{E}} = f_1^* q\mathcal{E}$. Let $\mathrm{pr}_1 : \dot{\mathfrak{g}} \to \mathfrak{g}$ be the projection on the first coordinate and define

$$K := \mathrm{pr}_{1,!} \dot{q\mathcal{E}} \qquad \in \mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}).$$

Let $\dot{q\mathcal{E}}_N$ be the pullback of $\dot{q\mathcal{E}}$ to $\dot{\mathfrak{g}}_N$ and put

$$K_N := \mathrm{pr}_{1,N,!} \dot{q\mathcal{E}}_N \qquad \in \mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}_N).$$

It is shown in [Sol9, §2.2] that K_N is a semisimple complex isomorphic to the pullback of K to $\mathfrak{g}_N$, and that K_N can be regarded as the parabolic induction of $\mathrm{IC}_{M \times \mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E})$.

From [Sol9, Theorem 2.2], based on [Lus2, Lus4, AMS2], we recall:

Theorem 1.1. *There exist natural isomorphisms of graded algebras*

$$\mathbb{H}(G, M, q\mathcal{E}) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g})}(K) \longrightarrow \mathrm{End}^*_{\mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}_N)}(K_N).$$

The irreducible modules and the standard modules of $\mathbb{H}(G, M, q\mathcal{E})$ have been constructed and parametrized in [Lus2, Lus4, AMS2]. The parameters consist of:

- a semisimple element $\sigma \in \mathfrak{g}$,
- $r \in \mathbb{C}$,
- a nilpotent element $y \in \mathfrak{g}$ such that $[\sigma, y] = 2ry$,
- an irreducible representation ρ of $\pi_0(Z_{G \times \mathbb{C}^\times}(y))$, such that the quasi-cuspidal support of (σ, y, ρ) is G-conjugate to $(M, \mathcal{C}_v^M, q\mathcal{E})$.

We call (y, σ, r) an L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$ and (y, σ, r, ρ) an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. The relation with Langlands parameters for reductive p-adic groups is explained in [AMS2, §1].

Theorem 1.2. [AMS2, Theorem 4.6]
To each enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$ there is associated a standard module $E_{y,\sigma,r,\rho}$, which has a distinguished (unique if $r \neq 0$) irreducible quotient $M_{y,\sigma,r,\rho}$.

This yields a bijection between $\mathrm{Irr}(\mathbb{H}(G, M, q\mathcal{E}))$ and G-association classes of enhanced L-parameters for $\mathbb{H}(G, M, q\mathcal{E})$.

The condition on ρ in enhanced L-parameters for $\mathbb{H}(G, M, q\mathcal{E})$ is rather subtle and restrictive. Some instances can be made more explicit:

(1.5) the quasi-cuspidal support of $(\sigma, y, \mathrm{triv})$ is always of the form $(L, \{0\}, \mathrm{triv})$,

where L is a minimal quasi-Levi subgroup of G, that is, the G-centralizer of a maximal torus in G°. The reason is that quasi-cuspidal supports are unique up to G-conjugation and that (y, triv) already appears in the Springer correspondence for $Z_G(\sigma_0)$, which is based on the quasi-cuspidal support $(L, \{0\}, \mathrm{triv})$. In particular $\rho = \mathrm{triv}$ can only appear in an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$ if $q\mathcal{E}$ is the trivial equivariant local system on $\{0\}$.

The centre of $\mathbb{H}(G, M, q\mathcal{E})$ contains

$$\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}} = \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C}). \tag{1.6}$$

Usually this is the entire centre, and therefore we will just call a character of (1.6), ie. an element $(W_{q\mathcal{E}}\sigma_0, r) \in \mathfrak{t}/W_{q\mathcal{E}} \times \mathbb{C}$, a central character of $\mathbb{H}(G, M, q\mathcal{E})$.

Not every (σ, r) can be extended to an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$, the existence of (y, ρ) already imposes conditions. We pick an algebraic homomorphism

$$\gamma_v : SL_2(\mathbb{C}) \to M \text{ with } \mathrm{d}\gamma_v \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = v$$

and we put $\sigma_v = \mathrm{d}\gamma_v \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{m}$. According to [Sol8, Lemma 2.1], in this setting $\mathrm{Ad}(G)\sigma - r\sigma_v$ intersects $\mathfrak{t}$ in a unique $W_{q\mathcal{E}}$-orbit. Therefore we may, and often will, assume that

(1.7) the semisimple element (σ, r) lies in $\mathfrak{t} \oplus \mathbb{C}(\sigma_v, 1) \subset \mathfrak{m} \oplus \mathbb{C}$.

In this way (σ, r) determines a central character of $\mathbb{H}(G, M, q\mathcal{E})$. We denote the completion of $Z(\mathbb{H}(G, M, q\mathcal{E}))$ with the respect to the powers the ideal

$$\ker \left(\mathrm{ev}_{(\sigma,r)} : \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}} \to \mathbb{C}\right)$$

by $\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma,r}$.

The geometric counterpart of (1.6) is the commutative graded algebra

$$H^*_{G \times \mathbb{C}^\times}(\mathrm{pt}) \cong \mathcal{O}(\mathfrak{g} \oplus \mathbb{C})^G, \tag{1.8}$$

which acts naturally on $\mathrm{End}^*_{\mathcal{D}^b_{G \times \mathbb{C}^\times}(\mathfrak{g}_N)}(K_N)$. The completion of (1.8) with respect to the maximal ideal determined by $(\mathrm{Ad}(G)\sigma, r)$ is denoted $\hat{H}^*_{G \times \mathbb{C}^\times}(\mathrm{pt})_{\sigma,r}$.

Fix (σ, r) as above and write

$$\mathfrak{g}_N^{\sigma,r} = \{X \in \mathfrak{g} : [\sigma, X] = 2rX, X \text{ is nilpotent}\}.$$

When $r \neq 0$, the nilpotency is already guaranteed by the first condition, and $\mathfrak{g}_N^{\sigma,r}$ is a vector space. On the other hand, when $r = 0$, $\mathfrak{g}_N^{\sigma,r}$ is the nilpotent cone in $Z_{\mathfrak{g}}(\sigma)$. In any case $\mathfrak{g}_N^{\sigma,r}$ is an irreducible variety. The group

$$C := Z_{G \times \mathbb{C}^\times}(\sigma) = Z_G(\sigma) \times \mathbb{C}^\times$$

acts on $\mathfrak{g}_N^{\sigma,r}$ like in (1.2):

$$(g, \lambda) \cdot X = \lambda^{-2} \mathrm{Ad}(g) X. \tag{1.9}$$

This action has only finitely many orbits [KaLu2, §5.4], so by the aforementioned irreducibility of $\mathfrak{g}_N^{\sigma,r}$

(1.10) there is a unique open C-orbit in $\mathfrak{g}_N^{\sigma,r}$.

We record the projection and inclusion maps

$$\mathfrak{g}_N^{\sigma,r} \xleftarrow{\mathrm{pr}_{1,N}} \dot{\mathfrak{g}}_N^{\sigma,r} = \mathrm{pr}_1^{-1}(\mathfrak{g}_N^{\sigma,r}) \xrightarrow{j_{N,\sigma,r}} \dot{\mathfrak{g}}_N .$$

With these we define

$$K_{N,\sigma,r} = (\mathrm{pr}_{1,N})_! j_{N,\sigma,r}^* (q\dot{\mathcal{E}}_N) \in \mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r}).$$

It was checked in [Lus4, §5.3 and §8.12] and [Sol9, Lemma 2.8] that this is a semisimple complex. The commutative graded algebra $H^*_C(\mathrm{pt})$ acts naturally on $\mathrm{End}_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$, by the product in equivariant cohomology.

Theorem 1.3. [Sol8, Theorem 2.4]
There are natural algebra isomorphisms

$$\begin{aligned}
&\hat{Z}(\mathbb{H}(G,M,q\mathcal{E}))_{\sigma,r} \underset{Z(\mathbb{H}(G,M,q\mathcal{E}))}{\otimes} \mathbb{H}(G,M,q\mathcal{E}) \xrightarrow{\sim} \\
&\hat{H}^*_{G\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{G\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N) \xrightarrow{\sim} \\
&\hat{H}^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})_{\sigma,r} \underset{H^*_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathrm{pt})}{\otimes} \mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r}).
\end{aligned}$$

These induce equivalences of categories

$$\begin{aligned}
\mathrm{Mod}_{\mathrm{fl},\sigma,r}(\mathbb{H}(G,M,q\mathcal{E})) &\cong \mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}^*_{\mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)}(K_N)\big) \\
&\cong \mathrm{Mod}_{\mathrm{fl},\sigma,r}\big(\mathrm{End}^*_{\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})\big),
\end{aligned}$$

where fl, σ, r stands for finite length modules all whose irreducible subquotients admit the central character given by (σ, r).

In particular Theorem 1.3 can be used to study all irreducible or standard $\mathbb{H}(G,M,q\mathcal{E})$-modules with central character given by (σ, r).

Since the semisimple complexes K_N and $K_{N,\sigma,r}$ are so important in this paper, we provide alternative descriptions. Consider the spaces and maps

$$\mathcal{C}_v^M \to \mathfrak{m}_N \xleftarrow{\mathrm{pr}_{\mathfrak{m}_N}} \mathfrak{m}_N \oplus \mathfrak{u} \to G \times^P (\mathfrak{m}_N \oplus \mathfrak{u}) \xrightarrow{\mu_N} \mathfrak{g}_N,$$

where $\mu_N(g,X) = (g, \mathrm{Ad}(g)X)$. By [BeLu, §2.6.3], for any $P\times\mathbb{C}^\times$-variety Y there is an equivalence of categories

$$\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} : \mathcal{D}^b_{P\times\mathbb{C}^\times}(Y) \to \mathcal{D}^b_{G\times\mathbb{C}^\times}(G\times^P Y),$$

sometimes called equivariant induction. The functor

$$\mu_{N,!}\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}\mathrm{pr}^*_{\mathfrak{m}_N} : \mathcal{D}^b_{P\times\mathbb{C}^\times}(\mathfrak{m}_N) \to \mathcal{D}^b_{G\times\mathbb{C}^\times}(\mathfrak{g}_N)$$

can be regarded as a version of parabolic induction for equivariant constructible sheaves. Notice that we may view $q\mathcal{E}$ as a $P\times\mathbb{C}^\times$-equivariant sheaf on which U acts trivially. By [Sol9, (2.30)–(2.33)] there is an isomorphism

$$K_N \cong \mu_{N,!}\mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times}\mathrm{pr}^*_{\mathfrak{m}_N}\mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}). \tag{1.11}$$

Let $T_{\sigma,r}$ be the smallest algebraic torus in $G^\circ\times\mathbb{C}^\times$ whose Lie algebra contains (σ, r). It was checked in [Sol8, (2.9)] that

$$\mathfrak{g}_N^{T_{\sigma,r}} = \mathfrak{g}_N^{\sigma,r} \quad\text{and}\quad \dot{\mathfrak{g}}_N^{\sigma,r} = (\dot{\mathfrak{g}}_N)^{T_{\sigma,r}} = \dot{\mathfrak{g}} \cap \big(\mathfrak{g}_N^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}\big). \tag{1.12}$$

We let $T_{\sigma,r}$ act on $G \times^P (\mathfrak{m}_N \oplus \mathfrak{u})$ by $(h, z) \cdot (g, X) = (hg, z^{-2}X)$, and we let

$$i_{N,\sigma,r} : (G \times^P \mathfrak{m}_N \oplus \mathfrak{u})^{T_{\sigma,r}} \to G \times^P (\mathfrak{m}_N \oplus \mathfrak{u})$$

be the inclusion.

Lemma 1.4. *In $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\mathfrak{g}_N^{\sigma,r})$ we have isomorphisms*

$$\begin{aligned} K_{N,\sigma,r} &\cong \mathrm{pr}_{1,!}\mathrm{IC}_{Z_G(\sigma)\times\mathbb{C}^\times}\big(\mathfrak{g}_N^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, j^*_{N,\sigma,r} q\dot{\mathcal{E}}_N\big) \\ &\cong \mu_{N,!} i^*_{N,\sigma,r} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathfrak{m}_N} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}). \end{aligned}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} & (G \times^P \mathfrak{m}_N \oplus \mathfrak{u})^{T_{\sigma,r}} \arrow[r, "i_{N,\sigma,r}"] \arrow[dl, "\mu_N"'] & G \times^P (\mathfrak{m}_N \oplus \mathfrak{u}) \arrow[dr, "\mu_N"] & \\ \mathfrak{g}_N^{\sigma,r} = \mathfrak{g}_N^{T_{\sigma,r}} \arrow[r, "\mathrm{pr}_1"] & \dot{\mathfrak{g}}_N^{\sigma,r} = (\dot{\mathfrak{g}}_N)^{T_{\sigma,r}} \arrow[u, "j_{\mathfrak{m}_N}"'] \arrow[r, "j_{N,\sigma,r}"] & \dot{\mathfrak{g}}_N \arrow[u, "j_{\mathfrak{m}_N}"] \arrow[r, "\mathrm{pr}_1"] & \mathfrak{g}_N \end{tikzcd} \tag{1.13}$$

where $j_{\mathfrak{m}_N}(X, gP) = (g, \mathrm{Ad}(g)X)$. With the left half of the diagram we rewrite

$$K_{N,\sigma,r} = \mathrm{pr}_{1,!} j^*_{N,\sigma,r} q\dot{\mathcal{E}}_N = \mu_{N,!} j_{\mathfrak{m}_N,!} j^*_{N,\sigma,r} q\dot{\mathcal{E}}_N. \tag{1.14}$$

Like in (1.12), one checks that

$$\begin{aligned} (G \times^P \mathfrak{m}_N \oplus \mathfrak{u})^{T_{\sigma,r}} &\cong \{(X, gP) \in \mathfrak{g}_N \times G/P : \mathrm{Ad}(g^{-1})X \in \mathfrak{m}_N \oplus \mathfrak{u}\}^{T_{\sigma,r}} \\ &\cong \mathfrak{g}_N^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}. \end{aligned}$$

Via this isomorphism $j_{\mathfrak{m}_N}$ becomes $\dot{\mathfrak{g}}_N^{\sigma,r} \to \mathfrak{g}_N^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}$ and μ_N becomes pr_1. As $j^*_{N,\sigma,r} q\dot{\mathcal{E}}_N$ lies in $\mathcal{D}^b_{Z_G(\sigma)\times\mathbb{C}^\times}(\dot{\mathfrak{g}}_N^{\sigma,r})$, this turns the right hand side of (1.14) into

$$\mathrm{pr}_{1,!}\mathrm{IC}_{Z_G(\sigma)\times\mathbb{C}^\times}\big(\mathfrak{g}_N^{\sigma,r} \times (G/P)^{\exp(\mathbb{C}\sigma)}, j^*_{N,\sigma,r} q\dot{\mathcal{E}}_N\big),$$

which proves the first isomorphism in the statement.

The square in (1.13) is Cartesian, so by base change [BeLu, Theorem 3.4.3], (1.14) becomes $\mu_{N,!} i^*_{N,\sigma,r} j_{\mathfrak{m}_N,!} q\dot{\mathcal{E}}_N$. By [Sol9, (2.22)], this is isomorphic to

$$\mu_{N,!} i^*_{N,\sigma,r} j_{\mathfrak{m}_N,!} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathfrak{m}_N} q\mathcal{E} \cong \mu_{N,!} i^*_{N,\sigma,r} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} (j_{\mathfrak{m}_N}|_{\mathcal{C}_v^M \oplus \mathfrak{u}})_! \mathrm{pr}^*_{\mathfrak{m}_N} q\mathcal{E} \cong$$

$$\mu_{N,!} i^*_{N,\sigma,r} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathfrak{m}_N} (j_{\mathfrak{m}_N}|_{\mathcal{C}_v^M})_! q\mathcal{E} \cong \mu_{N,!} i^*_{N,\sigma,r} \mathrm{ind}_{P\times\mathbb{C}^\times}^{G\times\mathbb{C}^\times} \mathrm{pr}^*_{\mathfrak{m}_N} \mathrm{IC}_{P\times\mathbb{C}^\times}(\mathfrak{m}_N, q\mathcal{E}). \quad \square$$

Comparing (1.11) and Lemma 1.4, we see that $K_{N,\sigma,r}$ can be regarded as a restricted parabolic induction of $\mathrm{IC}(\mathfrak{m}_N, q\mathcal{E})$.

2. The internal structure of standard modules

From Theorem 1.3 one sees that all irreducible or standard $\mathbb{H}(G, M, q\mathcal{E})$-modules with central character $(W_{q\mathcal{E}}\sigma - r\sigma_v, r)$ arise in some way from the semisimple complex $K_{N,\sigma,r}$ on $\mathfrak{g}_N^{\sigma,r}$. For $y \in \mathfrak{g}_N^{\sigma,r}$ we write

$$C_y = Z_C(y) = (Z_G(\sigma) \times \mathbb{C}^\times) \cap Z_{G\times\mathbb{C}^\times}(y),$$

where $G \times \mathbb{C}^\times$ acts as in (1.9). Let

$$\mathcal{O}_y = \mathrm{Ad}(C)y \subset \mathfrak{g}_N^{\sigma,r}$$

be the C-orbit of y. The equivalence of categories

$$\mathrm{ind}_{C_y}^C : \mathcal{D}^b_{C_y}(\{y\}) \to \mathcal{D}^b_C(\mathcal{O}_y) \tag{2.1}$$

transforms any representation ρ of $\pi_0(C_y)$ into a C-equivariant local system on $\mathcal{O}_y$. We form the (equivariant) intersection cohomology complex $\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho))$, a C-equivariant perverse sheaf on $\mathfrak{g}_N^{\sigma,r}$. In the literature this object is often denoted $\mathrm{IC}_C(\mathcal{O}_y, \mathrm{ind}_{C_y}^C(\rho))$, but we prefer a notation that specifies the variety on which it is defined.

Theorem 2.1. [Sol8, Theorem 4.2]
Every simple direct summand of $K_{N,\sigma,r}$ is (up to a degree shift) isomorphic to $\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho))$. for (y,ρ) such that (y,σ,r,ρ) is an enhanced L-parameter for $\mathbb{H}(G,M,q\mathcal{E})$. Conversely, for every such (y,σ,r,ρ), $\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho))$ is (up to a degree shift) a direct summand of $K_{N,\sigma,r}$.

With Theorems 1.3 and 2.1 and techniques from [Lus4, AMS2], we provide a description of the irreducible $\mathbb{H}(G,M,q\mathcal{E})$-modules.

Lemma 2.2. *The irreducible modules of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$ are*

$$M_{y,\sigma,r,\rho} = \mathrm{Hom}^0_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}\big(\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho))', K_{N,\sigma,r}\big),$$

where (y,σ,r,ρ) is an enhanced L-parameter for $\mathbb{H}(G,M,q\mathcal{E})$. Here the prime indicates a suitable degree shift, so that $\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho))'$ becomes a direct summand of $K_{N,\sigma,r}$.

Proof. We use a modified version $K'_{N,\sigma,r}$ of $K_{N,\sigma,r}$. The only change is that for every simple direct summand of $K_{N,\sigma,r}$ the degrees are shifted, so that it becomes an actual perverse sheaf. Thus $K'_{N,\sigma,r}$ is a direct sum of simple perverse sheaves.

Then $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})$ is naturally isomorphic to $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$ as algebras, only the gradings are different. Recall that

$$\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r}) = \mathrm{End}^*_{\mathcal{D}^b_{C^\circ}(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})^{C/C^\circ}. \tag{2.2}$$

From [Lus4, §5] and (2.2) we see that:

- $\mathrm{End}^n_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r}) = 0$ for $n < 0$,
- $\mathbb{C}_{\sigma,r} \otimes_{H^*_{C^\circ}(\mathrm{pt})} \bigoplus_{n \in \mathbb{Z}_{>0}} \mathrm{End}^n_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})$ is the nilpotent radical of $\mathbb{C}_{\sigma,r} \otimes_{H^*_{C^\circ}(\mathrm{pt})} \mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})$,
- $\mathrm{End}^0_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})$ is finite dimensional and semisimple.

We conclude that the irreducible modules of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$ can be identified with the irreducible modules of $\mathrm{End}^0_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K'_{N,\sigma,r})$. By Theorem 2.1, those are the

$$\mathrm{Hom}^0_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}\big(\mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho)), K'_{N,\sigma,r}\big),$$

where (y,σ,r,ρ) is an enhanced L-parameter for $\mathbb{H}(G,M,q\mathcal{E})$. □

With Lemma 4.2 we can decompose

$$K_{N,\sigma,r} \cong \bigoplus\nolimits_{y,\rho} \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_y}^C(\rho)) \otimes M_{y,\sigma,r,\rho}. \tag{2.3}$$

We warn that, while this is a direct sum of constructible sheaves, there may be nonzero morphisms (in higher degrees) between different intersection cohomology complexes in the sum.

Let $i_y : \{y\} \to \mathfrak{g}_N^{\sigma,r}$ and $i_{\mathcal{O}_y} : \mathcal{O}_y \to \mathfrak{g}_N^{\sigma,r}$ be the inclusions. From [Lus4, §10] and [Sol8, §3.2] we see that one way to define the standard $\mathbb{H}(G, M, q\mathcal{E})$-modules is:

$$\begin{aligned} E_{y,\sigma,r} &= H^*(\{y\}, i_y^! K_{N,\sigma,r}), \\ E_{y,\sigma,r,\rho} &= \mathrm{Hom}_{\pi_0(C_y)}\big(\rho, E_{y,\sigma,r}\big). \end{aligned}$$

The action of $\mathbb{H}(G, M, q\mathcal{E})$ comes from Theorem 1.3 and $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$, which acts via the natural homomorphism to $\mathrm{End}^*_{\mathcal{D}^b_{C_y}(\{y\})}(i_y^! K_{N,\sigma,r})$.

Let us denote equality in the Grothendieck of finite length $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$-modules by $\doteq$. Like in [Lus4, 10.3.(c)] we deduce from (2.3) that

$$E_{y,\sigma,r} \doteq \bigoplus\nolimits_{y',\rho'} H^*\big(i_y^! \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'))\big) \otimes M_{y',\sigma,r,\rho'}, \tag{2.4}$$

where the sum runs over all enhanced L-parameters (y', σ, r, ρ') for $\mathbb{H}(G, M, q\mathcal{E})$. Here and later $H^*(?)$ is an abbreviation of $\bigoplus_{n\in\mathbb{Z}} H^n(?)$. The action of $\pi_0(C_y)$ on $E_{y,\sigma,r}$ corresponds to the natural action of $\pi_0(C_y)$ on the first tensor factors on the right hand side of (2.4).

Recall that $i_y^! = D i_y^* D$ where D denotes the Verdier duality operator. For $\rho \in \mathrm{Irr}(\pi_0(C_y)) \subset \mathcal{D}^b_{C_y}(\{y\})$ we have $D\rho = \rho^\vee$, the contragredient representation. The analogue of (2.4) for $E_{y,\sigma,r,\rho}$ involves the space

$$\begin{aligned} &\mathrm{Hom}_{\pi_0(C_y)}\big(\rho, H^*\big(i_y^! \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'))\big)\big) \cong \\ &\mathrm{Hom}_{\pi_0(C_y)}\big(H^*\big(i_y^* D \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'))\big), \rho^\vee\big) \cong \\ &\mathrm{Hom}_{\pi_0(C_y)}\big(H^*\big(i_y^* \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'^\vee))\big), \rho^\vee\big). \end{aligned} \tag{2.5}$$

By definition the dimension of the last line of (2.5) is the multiplicity of $\rho^\vee$ in $i_y^* \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'^\vee))$. Equivalently the dimension of (2.5) equals

$$\text{the multiplicity of } \mathrm{ind}_{C_y}^C(\rho^\vee) \text{ in } i_{\mathcal{O}_y}^* \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho'^\vee)). \tag{2.6}$$

By [Lus4, §10.6], or by applying complex conjugation to both sides, that equals

$$\text{the multiplicity } \mu(y, \rho, y', \rho') \text{ of } \mathrm{ind}_{C_y}^C(\rho) \text{ in } i_{\mathcal{O}_y}^* \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}_{C_{y'}}^C(\rho')). \tag{2.7}$$

Notice that (2.6) and (2.7) can only be nonzero if $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$. As in [Lus4, Corollary 10.7] and [Sol8, Proposition 5.1], (2.4) and (2.7) determine the semisimplification of standard modules, namely

$$E_{y,\sigma,r,\rho} \doteq \text{ direct sum of the } M_{y',\sigma,r,\rho'} \text{ with multiplicities (2.7).} \tag{2.8}$$

To arrive at (2.4) and (2.8), Lusztig uses a filtration of $E_{y,\sigma,r}$ by submodules defined in terms of the cohomological grading from $K'_{N,\sigma,r}$ [Lus4, §10.2]. That provides some information about which constituents of $E_{y,\sigma,r,\rho}$ appear as submodules or quotients, but it is not yet explicit.

Proposition 2.3. *Let $y, y' \in \mathfrak{g}_N^{\sigma,r}$.*

(*a*) *If $r \neq 0$ and $\mathcal{O}_y \not\subset \overline{\mathcal{O}_{y'}}$, then* $\mathrm{Hom}_{\mathbb{H}(G,M,q\mathcal{E})}(E_{y',\sigma,r}, E_{y,\sigma,r}) = 0$.

(*b*) *Suppose that $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$. There is a homomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules*

$$J_{y',y} : E_{y',\sigma,r} \to E_{y,\sigma,r},$$

canonical up to the action of $\pi_0(C_{y'})$.

Proof. (a) As $r \neq 0$, by [AMS2, Theorems 3.20 and 4.6], every irreducible quotient of $E_{y',\sigma,r}$ is isomorphic to $M_{y',\sigma,r,\rho'}$ for some enhancement ρ'. Let

$$\phi \in \mathrm{Hom}_{\mathbb{H}(G,M,q\mathcal{E})}(E_{y',\sigma,r}, E_{y,\sigma,r})$$

and suppose that $\phi \neq 0$. Then $\ker(\phi)$ is a proper submodule, so $E_{y',\sigma,r}/\ker(\phi)$ has at least one quotient of the form $M_{y',\sigma,r,\rho'}$. Now ϕ induces an injection

$$E_{y',\sigma,r}/\ker(\phi) \to E_{y,\sigma,r},$$

which in particular maps the quotient $M_{y',\sigma,r,\rho'}$ injectively to a subquotient of $E_{y,\sigma,r}$. However, by (2.8), (2.7) and the assumption, $E_{y,\sigma,r}$ does not have any subquotients isomorphic to $M_{y',\sigma,r}$. This contradiction shows that $\phi \neq 0$ is impossible.
(b) Let $K^\vee_{N,\sigma,r}$ be the version of $K_{N,\sigma,r}$ obtained from $q\mathcal{E}^\vee$ instead of $q\mathcal{E}$. We need to construct a homomorphism of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K_{N,\sigma,r})$-modules

$$J_{y',y} : H^*(\{y'\}, i^!_{y'} K_{N,\sigma,r}) \to H^*(\{y\}, i^!_y K_{N,\sigma,r})$$

Via Verdier duality, this equivalent to the construction of a homomorphism of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K^\vee_{N,\sigma,r})$-modules

$$DJ_{y',y} : H^*(\{y\}, i^*_y K^\vee_{N,\sigma,r}) \to H^*(\{y'\}, i^*_{y'} K^\vee_{N,\sigma,r}). \tag{2.9}$$

Recall that $K^\vee_{N,\sigma,r}$ is a bounded complex of C-equivariant constructible sheaves with finite dimensional stalks on $\mathfrak{g}_N^{\sigma,r}$. Since there are only finitely many C-orbits in $\mathfrak{g}_N^{\sigma,r}$ [KaLu2, §5.4], there exists an open neighborhood U_y of y in $\mathfrak{g}_N^{\sigma,r}$ such that every section of $K^\vee_{N,\sigma,r}$ over U_y is completely determined by its stalk at y. From $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$ we see that $U_y \cap \mathcal{O}_{y'}$ is nonempty. Pick $y_1 \in U_y \cap \mathcal{O}_{y'}$. Every element of the stalk $i^*_y K^\vee_{N,\sigma,r}$ comes from a unique section over U_y, so it determines an element of the stalk $i^*_{y_1} K^\vee_{N,\sigma,r}$. That yield canonical maps

$$i^*_y K^\vee_{N,\sigma,r} \to i^*_{y_1} K^\vee_{N,\sigma,r} \quad \text{and} \quad H^*(\{y\}, i^*_y K^\vee_{N,\sigma,r}) \to H^*(\{y_1\}, i^*_{y_1} K^\vee_{N,\sigma,r}). \tag{2.10}$$

Pick $g_1 \in C$ with $g_1 \cdot y_1 = y'$. The action of g_1 provides an isomorphism of $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K^\vee_{N,\sigma,r})$-modules

$$g_1 : H^*(\{y_1\}, i^*_{y_1} K^\vee_{N,\sigma,r}) \to H^*(\{y'\}, i^*_{y'} K^\vee_{N,\sigma,r}). \tag{2.11}$$

The composition of (2.10) and (2.11) is the desired map (2.9).

It remains to analyse the dependence on the choices of U_y, y_1 and g_1. Consider different $y_2 \in U_y$ and $g_2 \in C$ with $g_2 \cdot y_2 = y'$. Then

$$g_2^{-1} g_1 : H^*(\{y_1\}, i^*_{y_1} K^\vee_{N,\sigma,r}) \to H^*(\{y_2\}, i^*_{y_2} K^\vee_{N,\sigma,r}) \tag{2.12}$$

is an isomorphism, canonical up to multiplying g_2 on the right by elements of C_{y_2}. The isomorphism

$$g_2 : H^*(\{y_2\}, i^*_{y_2} K^\vee_{N,\sigma,r}) \to H^*(\{y'\}, i^*_{y'} K^\vee_{N,\sigma,r}) \tag{2.13}$$

is the canonical in the same sense. Equivalently, (2.12) and (2.13) are canonical up to multiplying g_2 on the left by elements of $C_{y'}$. The constructibility of the involved sheaves entails that this action of $C_{y'}$ factors through $\pi_0(C_{y'})$.

From this we deduce that the choice of U_y was inessential. For any alternative $\tilde{U}_y$, the intersection $U_y \cap \tilde{U}_y$ contains an open neighborhood of y with the same property. We can take y_2 in that smaller neighborhood, and by (2.12) and (2.13) that is just as good as y_1. Altogether the only non-canonicity of (2.9) comes from the $\mathrm{End}^*_{\mathcal{D}^b_C(\mathfrak{g}_N^{\sigma,r})}(K^\vee_{N,\sigma,r})$-linear action of $\pi_0(C_{y'})$ on $H^*(\{y'\}, i^*_{y'} K^\vee_{N,\sigma,r})$. $\square$

We note that at this point it is still possible that $E_{y,\sigma,r} = 0$ or $E_{y',\sigma,r} = 0$, because it may be impossible to extend (σ, y) or (σ, y') to an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. To improve on that, we bring in enhanced L-parameters (y, σ, r, ρ) and (y', σ, r, ρ'). We denote the vector space underlying ρ by V_ρ.

Proposition 2.4. (*a*) *If the multiplicities* (2.6) *and* (2.7) *are zero and* $r \neq 0$*, then*

$$\mathrm{Hom}_{\mathbb{H}(G,M,q\mathcal{E})}(E_{y',\sigma,r,\rho'}, E_{y,\sigma,r,\rho}) = 0.$$

(*b*) *Suppose that the multiplicities* $\mu(y, \rho, y', \rho')$ *and* (2.6) *are nonzero, so in particular* $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$*. Then* $J_{y',y}$ *induces a nonzero* $\mathbb{H}(G, M, q\mathcal{E})$*-module homomorphism from* $V_{\rho'} \otimes E_{y',\sigma,r,\rho'}$ *to* $V_\rho \otimes E_{y,\sigma,r,\rho}$ *(with* $\mathbb{H}(G, M, q\mathcal{E})$*-action only on the second tensor factors).*

(*c*) $J_{y',y}$ *gives rise to* $\mu(y, \rho, y', \rho')$ *linearly independent* $\mathbb{H}(G, M, q\mathcal{E})$*-homomorphisms from* $E_{y',\sigma,r,\rho'}$ *to* $E_{y,\sigma,r,\rho}$*.*

Proof. (a) This can be shown in the same way as Proposition 2.3.a.
(b) The map $DJ_{y',y}$ from (2.9) sends the linear subspace

$$H^*\big(\{y\}, i^*_y \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}^C_{C_{y'}}(\rho'^\vee))\big) \subset H^*(\{y\}, i^*_y K^\vee_{N,\sigma,r}) \tag{2.14}$$

to the linear subspace

$$H^*\big(\{y'\}, i^*_{y'} \mathrm{IC}_C(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}^C_{C_{y'}}(\rho'^\vee))\big) \cong \rho'^\vee. \tag{2.15}$$

The map $DJ_{y',y}$ from (2.14) to (2.15) is injective because

$$\mathrm{IC}_C\big(\mathfrak{g}_N^{\sigma,r}, \mathrm{ind}^C_{C_{y'}}(\rho'^\vee)\big) \cong \mathrm{IC}\big(\overline{\mathcal{O}_{y'}}, \mathrm{ind}^C_{C_{y'}}(\rho'^\vee)\big)$$

has no subsheaves supported on $\overline{\mathcal{O}_{y'}} \setminus \mathcal{O}_{y'}$ [Ach, Lemma 3.3.3]. By assumption (2.14) contains a copy of $\rho^\vee$, and $DJ_{y',y}$ sends that nontrivially to $\rho'^\vee$. In other words, $J_{y',y}$ sends $\rho' \subset E_{y',\sigma,r}$ nontrivially to a copy of ρ in $E_{y,\sigma,r}$. Now we split $E_{y',\sigma,r}$, resp. $E_{y,\sigma,r}$, into isotypic component for the action of $\pi_0(C_{y'})$, resp. of $\pi_0(C_y)$. We obtain that via these splittings $J_{y',y}$ restricts to a nonzero homomorphism from the ρ'-isotypic component $V_{\rho'} \otimes E_{y',\sigma,r,\rho'}$ of $E_{y',\sigma,r}$ to the ρ-isotypic component $V_\rho \otimes E_{y,\sigma,r,\rho}$ of $E_{y,\sigma,r}$.
(c) By the equality of (2.6) and (2.7), the $\pi_0(C_y)$-representation (2.14) contains a direct sum of $\mu(y, \rho, y', \rho')$ copies of $\rho^\vee$. As we observed in the proof of part (b), $DJ_{y',y}$ injects that into ${\rho'}^\vee$. By duality, there are $\mu(y, \rho, y', \rho')$ linearly independent surjections of $\rho' \subset E_{y',\sigma,r}$ onto a copy of ρ in $E_{y,\sigma,r}$. To each of these we can apply the argument from part (b), and that provides $\mu(y, \rho, y', \rho')$ linearly independent elements of $\mathrm{Hom}_{\mathbb{H}(G,M,q\mathcal{E})}(E_{y',\sigma,r,\rho'}, E_{y,\sigma,r,\rho})$. $\square$

3. Open parameters for twisted graded Hecke algebras

We say that an L-parameter (y, σ, r) or (y, σ, r, ρ) for $\mathbb{H}(G, M, q\mathcal{E})$ is open if the $Z_{G\times\mathbb{C}^\times}(\sigma)$-orbit of y is open in $\mathfrak{g}_N^{\sigma,r}$. We may also use $Z_G(\sigma)$ instead of $C = Z_{G\times\mathbb{C}^\times}(\sigma) = Z_G(\sigma) \times \mathbb{C}^\times$, for they have the same nilpotent orbits. Since there is a unique open orbit in $\mathfrak{g}_N^{\sigma,r}$ (1.10), we could equivalently require that the C-orbit of y is dense in $\mathfrak{g}_N^{\sigma,r}$.

For $r = 0$ we can reformulate the above condition in easier terms:

$$(y, \sigma, 0) \text{ is open if and only if } y \text{ is regular nilpotent in } Z_{\mathfrak{g}}(\sigma). \tag{3.1}$$

Lemma 3.1. *Let (y, σ, r, ρ) be an open enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. Then $E_{y,\sigma,r,\rho}$ is irreducible and equals $M_{y,\sigma,r\rho}$.*

Proof. For connected G this is [Lus4, Corollary 10.9.c]. We spell out that argument in general. Any constituent of $E_{y,\sigma,r,\rho}$ is of the form $M_{y',\sigma,r,\rho'}$, where $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$. The "open" property of y forces $\mathcal{O}_y = \mathcal{O}_{y'}$. Next (2.8) shows that $\mathrm{ind}_{C_y}^{C}(\rho)$ must equal $\mathrm{ind}_{C_{y'}}^{C}(\rho')$, so that $M_{y',\sigma,r,\rho'} \cong M_{y,\sigma,r,\rho}$. From (2.8) and (2.7) we see that $M_{y,\sigma,r,\rho}$ appears with multiplicity one in $E_{y,\sigma,r,\rho}$. We conclude that the two modules coincide. □

The main result of this section is a quick consequence of the insights collected so far.

Theorem 3.2. *Let (y, σ, r, ρ) be an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. Suppose that (y', σ, r, ρ') is an open L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$, such that $M_{y',\sigma,r,\rho'}$ occurs as a subquotient of $E_{y,\sigma,r,\rho}$.*

(a) $M_{y',\sigma,r,\rho'}$ is isomorphic to a submodule of $E_{y,\sigma,r,\rho}$.
(b) Every irreducible subquotient of $E_{y,\sigma,r,\rho}$ isomorphic to $M_{y',\sigma,r,\rho'}$ is a submodule of $E_{y,\sigma,r,\rho}$.

Proof. (a) From (2.8) we see that the multiplicity (2.8) is nonzero and that $\mathcal{O}_y \subset \overline{\mathcal{O}_{y'}}$. By Proposition 2.4.b there exists a nonzero $\mathbb{H}(G, M, q\mathcal{E})$-module homomorphism from $E_{y',\sigma,r,\rho'}$ to $E_{y,\sigma,r,\rho}$. By Lemma 3.1 $E_{y',\sigma,r,\rho'} = M_{y',\sigma,r,\rho'}$ is irreducible, so this homomorphism is injective.
(b) Proposition 2.4 enables us to apply the proof of part (a) in $\mu(y, \rho, y', \rho')$ linearly independent ways. That produces a direct sum of $\mu(y, \rho, y', \rho')$ copies of $M_{y',\sigma,r,\rho'}$ as a submodule of $E_{y,\sigma,r,\rho}$. By (2.8), this exhausts all occurences of $M_{y',\sigma,r,\rho'}$ as a subquotient of $E_{y,\sigma,r,\rho}$. □

The parameters (y, σ, r, ρ) can also be presented in another way. By [KaLu2, §2.4] we can find a homomorphism of algebraic groups

$$\gamma_y : SL_2(\mathbb{C}) \to G^\circ \text{ such that:} \tag{3.2}$$
- $\mathrm{d}\gamma_y(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = y$,
- $\sigma_0 := \sigma - \mathrm{d}\gamma_y(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right))$ commutes with the image of $\mathrm{d}\gamma_y$.

Moreover the G°-conjugacy class of (y, σ, r) determines the G°-conjugacy class of (y, σ_0, r) and conversely. We recall from [AMS2, Lemma 3.6] that there is a natural isomorphism $\pi_0(C_y) \cong \pi_0(Z_G(\sigma_0, y))$, and that the data (y, σ, r, ρ) up to G-association carry exactly the same information as (y, σ_0, r, ρ) up to G-association.

Lemma 3.3. *Let (y, σ, r) be an L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$, with $r \neq 0$. Equivalent are:*

(i) *(y, σ, r) is open,*
(ii) *$\mathfrak{g}_N^{\sigma,r}$ equals $\{X \in Z_{\mathfrak{g}}(\sigma_0) : [\sigma - \sigma_0, X] = 2rX\}$*

Proof. Since $\mathfrak{g}_N^{\sigma,r} = \mathfrak{g}_N^{r^{-1}\sigma,1}$ and $Z_{G^\circ}(\sigma) = Z_{G^\circ}(r^{-1}\sigma)$, we may replace (y, σ, r) by $(y, r^{-1}\sigma, 1)$ and assume that $r = 1$.

(i) $\Rightarrow$ (ii) Pick a maximal toral subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$ containing σ, and let R be the root system of $(\mathfrak{g}, \mathfrak{t}')$. Then

$$\mathfrak{g}_N^{\sigma,1} = \bigoplus\nolimits_{\alpha \in R : \alpha(\sigma) = 2} \mathfrak{g}_\alpha .$$

The open $Z_{G^\circ}(\sigma)$-orbit in $\mathfrak{g}_N^{\sigma,1}$ contains an element with nonzero parts in every root subspace $\mathfrak{g}_\alpha$ with $\alpha(\sigma) = 2$. Conjugating y by an element of $Z_{G^\circ}(\sigma)$ if necessary, we may assume that y has this property. Let R' be the minimal parabolic root subsystem of R containing $\{\alpha \in R : \alpha(\sigma) = 2\}$. It gives a Levi subalgebra

$$\mathfrak{g}' = \mathfrak{t} \oplus \bigoplus\nolimits_{\alpha \in R'} \mathfrak{g}_\alpha ,$$

which contains y as a distinguished nilpotent element. Let G' be the algebraic Lie subgroup of G° with Lie algebra $\mathfrak{g}'$, and let $\gamma_y : SL_2(\mathbb{C}) \to G'$ be as in (3.2). As $Z_{\mathfrak{g}'}(\sigma_0)$ is a Levi subalgebra of $\mathfrak{g}'$ containing y,

(3.3) the distinguishedness of y forces $\sigma_0 \in Z(\mathfrak{g}') \subset \mathfrak{t}'$.

In particular any root α with $\alpha(\sigma) = 2$ satisfies $\alpha(\sigma_0) = 0$ and $\alpha(\sigma - \sigma_0) = 2$. Hence (ii) holds for this specific σ_0. Since σ_0 is determined by (y, σ, r) up to G°-conjugacy, (ii) holds for all possible σ_0.
(ii) $\Rightarrow$ (i) Notice that $y, \mathrm{d}\gamma_y(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)), \mathrm{d}\gamma_y(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ is an $\mathfrak{sl}_2$-triple in $Z_{\mathfrak{g}}(\sigma_0)$. It is known from [Kos, Lemma 4.2.c] or [ChGi, Lemma 3.7.24] that the orbit of $Z_{G^\circ}(\sigma_0) \cap Z_{G^\circ}\big(\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big)$ through y is open in

$$\{X \in Z_{\mathfrak{g}}(\sigma_0) : [\mathrm{d}\gamma_y(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)), X] = 2X\} = \{X \in Z_{\mathfrak{g}}(\sigma_0) : [\sigma - \sigma_0, X] = 2X\} = \mathfrak{g}_N^{\sigma,1}.$$

The group $Z_{G^\circ}(\sigma)$ contains $Z_{G^\circ}(\sigma_0) \cap Z_{G^\circ}\big(\mathrm{d}\gamma_y \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big)$, so its orbit through y is also open in $\mathfrak{g}_N^{\sigma,1}$. $\square$

An L-parameter (y, σ, r) or (y, σ_0, r) (or with ρ included) is called bounded if a G-conjugate of σ_0 lies in $i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$. We recall that by [Sol8, Lemma 2.2] $\mathrm{Ad}(G)\sigma$ intersects $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} X_*(T)$ whenever there exists an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$ with this σ. To explain the terminology, we note that $\exp(i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T))$ is the maximal compact subgroup of T. Thus a parameter is bounded if and only if $\exp(\sigma_0)$ lies in a bounded closed subgroup of G. More generally we say that (y, σ, r) is essentially bounded if a G-conjugate of σ_0 lies in $Z(\mathfrak{g}) \oplus i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T)$.

The next result is a variation on a property of Langlands parameters for reductive groups, announced in [CFZ, §0.6].

Lemma 3.4. *Let (y, σ, r) be an L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$.*
(*a*) *If y is distinguished in $\mathfrak{g}$, then (y, σ, r) is essentially bounded.*
(*b*) *If $\Re(r) \neq 0$ and (y, σ, r) is essentially bounded, then it is an open parameter.*

Proof. (a) By the same reasons as for (3.3), σ_0 lies in $Z(\mathfrak{g})$.
(b) By [AMS3, Proposition 1.4] we may assume that $\sigma_0, \sigma - r\sigma_v \in \mathfrak{t}$. Let T' be a maximal torus of $Z_G(\sigma_0)$ whose Lie algebra $\mathfrak{t}'$ contains $\mathfrak{t} + \mathbb{C}\sigma = \mathfrak{t} + \mathbb{C}\sigma_v$. Then

$$\sigma = \sigma_0 + \mathrm{d}\gamma_y(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)) \text{ with } \sigma_0 \in Z(\mathfrak{g}) \oplus i\mathbb{R} \otimes_{\mathbb{Z}} X_*(T) \text{ and } \mathrm{d}\gamma_y(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)) \in \mathbb{R} \otimes_{\mathbb{Z}} X_*(T').$$

In particular σ_0 takes imaginary values on all roots of $(\mathfrak{g}, \mathfrak{t}')$, while $\sigma - \sigma_0 = \mathrm{d}\gamma_y(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right))$ takes values in $r\mathbb{R}$ on all roots. As $r\mathbb{R} \cap i\mathbb{R} = \{0\}$, (ii) from Lemma 3.3 holds, and we conclude by applying Lemma 3.3. $\square$

We warn that Lemmas 3.3 and 3.4.b are false for $r = 0$. Just take $\sigma = 0$ and note that $(y, 0, 0)$ is a bounded parameter satisfying (ii), for any nilpotent element y.

Bounded parameters are related to $\mathbb{H}(G, M, q\mathcal{E})$-modules which are tempered in the sense of [AMS2, Definition 3.24]. The definition says that $\mathcal{O}(\mathfrak{t})$-weights of a module must lie in a certain negative cone. In particular any subquotient of a tempered module is again tempered.

Besides tempered representations, in an important role in harmonic analysis is played by (essentially) discrete series representations. For graded Hecke algebras they are also defined in [AMS2, Definition 3.24], it says that their weights must belong to the interior of a suitable negative cone.

To see the connection between tempered representations and bounded parameters best, we involve the sign automorphism of $\mathbb{H}(G, M, q\mathcal{E})$. Let $\det : W_{q\mathcal{E}} \to \{\pm 1\}$ be the determinant of the action of $W_{q\mathcal{E}}$ on $X^*(T)$, an extension of the sign character of the Weyl group $W_{q\mathcal{E}}^\circ$. Then $\mathrm{sgn} : \mathbb{H}(G, M, q\mathcal{E}) \to \mathbb{H}(G, M, q\mathcal{E})$ is defined by

$$\mathrm{sgn}(N_w) = \det(w) N_w,\ \mathrm{sgn}(\mathbf{r}) = -\mathbf{r},\ \mathrm{sgn}(\xi) = \xi \qquad w \in W_{q\mathcal{E}}, \xi \in \mathcal{O}(\mathfrak{t} \oplus \mathbb{C}).$$

Theorem 3.5. [AMS2, §3.5 and §4] and [Sol8, Theorem 3.4]

(*a*) *Let (y, σ, r, ρ) be an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. The following are equivalent when $\Re(r) \leq 0$:*
- *the parameter (σ, y, r) is bounded,*
- *$E_{y,\sigma,r,\rho}$ is tempered,*
- *$M_{y,\sigma,r,\rho}$ is tempered.*

When $\Re(r) < 0$, $M_{y,\sigma,r,\rho}$ is essentially discrete series if and only if y is distinguished nilpotent in $\mathfrak{g}$.

(*b*) *Let $(y, \sigma, -r, \rho)$ be an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$. The following are equivalent when $\Re(r) \geq 0$:*
- *the parameter $(\sigma, y, -r)$ is bounded,*
- *$\mathrm{sgn}^* E_{y,\sigma,-r,\rho}$ is tempered,*
- *$\mathrm{sgn}^* M_{y,\sigma,-r,\rho}$ is tempered.*

When $\Re(r) > 0$, $\mathrm{sgn}^* M_{y,\sigma,-r,\rho}$ is essentially discrete series if and only if y is distinguished nilpotent in $\mathfrak{g}$.

(*c*) *When $\Re(r) = 0$, $\mathbb{H}(G, M, q\mathcal{E})$ does not have nonzero essentially discrete modules on which $\mathbf{r}$ acts as r.*

We will refer to $\mathrm{sgn}^* E_{y,\sigma,-r,\rho}$ as an analytic standard module. These are useful because often $\mathbf{r}$ has to be specialized to a positive real number r, and then Theorem 3.5.b yields tempered or essentially discrete $\mathbb{H}(G, M, q\mathcal{E})$-modules on which $\mathbf{r}$ acts as r. To emphasize the contrast, we will sometime call $E_{y,\sigma,r,\rho}$ a geometric standard module.

Theorem 3.6. *Let E be a $\mathbb{H}(G, M, q\mathcal{E})$-module on which $\mathbf{r}$ acts as $r \in \mathbb{C}$. Assume one of the following:*
- *$\Re(r) < 0$ and E is geometric standard,*
- *$\Re(r) > 0$ and E is analytic standard.*

Let V be an irreducible subquotient of E, which is tempered or essentially discrete series. Then V is a submodule of E.

Proof. First we suppose that $\Re(r) > 0$, and we write $E = \mathrm{sgn}^* E_{y,\sigma,-r,\rho}$. In view of (2.8), $V \cong \mathrm{sgn}^* M_{y',\sigma,-r,\rho'}$ for some y', ρ'. Theorem 3.5.b says that $(y', \sigma, -r)$ is bounded or that y' is distinguished nilpotent in $\mathfrak{g}$. By Lemma 3.4, $(y', \sigma, -r)$ is an open parameter. Now Theorem 3.2 tells us that $M_{y',\sigma,-r,\rho'}$ is a submodule $E_{y,\sigma,-r,\rho}$, so V is a submodule E. The case $\Re(r) < 0$ is entirely analogous, only now using part (a) of Theorem 3.5. $\square$

There are yet other standard modules, namely those appearing in the Langlands classification for graded Hecke algebras [Eve]. To construct those one starts with an irreducible tempered representation τ of a parabolic subalgebra $\mathbb{H}^P$ of $\mathbb{H}$ (determined by a set of simple roots P). One twists τ by a character t in positive position with respect to the set of simple roots outside P, and then induces to $\mathbb{H}$.

This works when $\Gamma_{q\mathcal{E}}$ is trivial, when we include a nontrivial $\Gamma_{q\mathcal{E}}$ or $\mathbb{C}[\Gamma_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$ we have to be careful because it might mess up the uniqueness of irreducible quotients in the Langlands classification. A solution is provided by [Sol2, Corollary 2.2.5] and [Sol4, (8.11)]: one must choose the subgroup of $\Gamma_{q\mathcal{E}}$ that occurs in a parabolic subalgebra of

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})$$

depending on the data τ, t. Namely, one takes the largest subgroup $\Gamma_{P,t}$ of $\Gamma_{q\mathcal{E}}$ that stabilizes P and t. Next one replaces τ by an irreducible representation τ' of $\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}})$ whose restriction to $\mathbb{H}^P$ contains τ, or equivalently an irreducible direct summand of $\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P, k, \mathbf{r})}^{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}})} \tau$. We call modules of the form

$$\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}})}^{\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})} (\tau' \otimes t) \tag{3.4}$$

Langlands standard modules. With Clifford theory, in the version of [Sol1, §11] and [AMS3, §1], we can write

$$\tau' = \mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t,\tau}, k, \mathbf{r}, \natural_{q\mathcal{E}})}^{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}})} (\rho \otimes \tau)$$

for a suitable projective representation ρ of $\Gamma_{P,t,\tau}$. In that notation, (3.4) becomes

$$\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t,\tau}, k, \mathbf{r}, \natural_{q\mathcal{E}})}^{\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})} (\rho \otimes \tau \otimes t) = \mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_{\mathcal{E}} \Gamma_{P,t,\tau}, k, \mathbf{r}, \natural_{q\mathcal{E}})}^{\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})} \big(\rho \otimes \mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P, k, \mathbf{r})}^{\mathbb{H}(\mathfrak{t}, W_{\mathcal{E}}, k, \mathbf{r})} (\tau \otimes t) \big).$$

This shows that every Langlands standard module of $\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})$ is an indecomposable direct summand of

$$\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_{\mathcal{E}}, k, \mathbf{r})}^{\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})} V \tag{3.5}$$

for some Langlands standard $\mathbb{H}(\mathfrak{t}, W_{\mathcal{E}}, k, \mathbf{r})$-module V, namely $\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P, k, \mathbf{r})}^{\mathbb{H}(\mathfrak{t}, W_{\mathcal{E}}, k, \mathbf{r})} (\tau \otimes t)$ above. If $Q \subset G$ is the standard parabolic subgroup such that $W_{q\mathcal{E}}^Q = W_P \mathcal{R}_{P,t}$, then

$$\mathbb{H}(Q, M, q\mathcal{E}) = \mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}}),$$

and the Langlands standard module (3.4) depends only on the data (Q, τ', t) up to G-conjugacy. The relations between the various kinds of standard $\mathbb{H}(G, M, q\mathcal{E})$-modules are as follows.

Proposition 3.7. *Let $r \in \mathbb{C}^\times$.*

(*a*) *When $\Re(r) > 0$, the Langlands standard modules of $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r)$ are precisely the analytic standard modules.*

(*b*) *When $\Re(r) < 0$, the Langlands standard modules of $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r} - r)$ are precisely the geometric standard modules.*

(c) *The Langlands standard modules of* $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r})$ *are precisely its irreducible modules.*

Proof. (a) This is shown in [Sol8, Proposition B.4].
(b) This can be shown in the same way as [Sol8, Proposition B.4], just apply sgn^* to all the modules in the proof of parts (c) and (d).
(c) By Clifford theory in the form [Sol4, Lemma 8.4], the set of irreducible representations of

$$\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r}) = \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$$

with central character $W_{q\mathcal{E}}c$ is naturally in bijection with $\mathrm{Irr}(\mathbb{C}[(W_{q\mathcal{E}})_c, \natural_{q\mathcal{E}}])$, via $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[(W_{q\mathcal{E}})_c, \natural_{q\mathcal{E}}]}^{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[(W_{q\mathcal{E}}), \natural_{q\mathcal{E}}]}$. Consider a Langlands standard module

$$V = \mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_P \Gamma_{P,t}, k, \mathbf{r}, \natural_{q\mathcal{E}})}^{\mathbb{H}(\mathfrak{t}, W_{q\mathcal{E}}, k, \mathbf{r}, \natural_{q\mathcal{E}})} (\tau' \otimes t) = \mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_P \Gamma_{P,t}, \natural_{q\mathcal{E}}]}^{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]} (\tau' \otimes t).$$

Since $\tau' \in \mathrm{Irr}(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_P \Gamma_{P,t}, \natural_{q\mathcal{E}}])$ is tempered, the real part of its central character $W_P \Gamma_{P,t} c$ is 0. Hence the real part of the central character $W_P \Gamma_{P,t} ct$ of $\tau' \otimes t$ is t. Furthermore t is positive with respect to the simple roots outside P, so

$$(W_{q\mathcal{E}})_t \subset \mathrm{Stab}_{W_{q\mathcal{E}}}(\mathbb{Z}P, t) \subset W_P \mathrm{Stab}_{\Gamma_{q\mathcal{E}}}(\mathbb{Z}P, t) = W_P \Gamma_{P,t}.$$

Now [Sol4, Lemma 8.4] says that $\mathrm{Irr}(\mathbb{C}[(W_{q\mathcal{E}})_{ct}, \natural_{q\mathcal{E}}])$ parametrizes the irreducible modules with central character represented by ct of both $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]$ and $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_P \Gamma_{P,t}, \natural_{q\mathcal{E}}]$. It follows that $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_P \Gamma_{P,t}, \natural_{q\mathcal{E}}]}^{\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W_{q\mathcal{E}}, \natural_{q\mathcal{E}}]}$ preserves the irreducibility of $\tau' \otimes t$, so V is irreducible.

By the Langlands classification from [Eve], in the form [Sol4, Proposition 8.5], every irreducible $\mathbb{H}(G, M, q\mathcal{E})/(\mathbf{r})$-module V' occurs as a quotient of some Langlands standard module V. By the irreducibility of V we have $V' = V$. $\square$

4. Generic representations of graded Hecke algebras

In this section we assume that the 2-cocycle $\natural_{q\mathcal{E}}$ involved in $\mathbb{H}(G, M, q\mathcal{E})$ is trivial. Then (1.1) simplifies to

$$\mathbb{H}(G, M, q\mathcal{E}) = \mathbb{H}(G^\circ, M^\circ, \mathcal{E}) \rtimes \Gamma_{q\mathcal{E}}, \tag{4.1}$$

see [AMS2, §4]. In other words, $\mathbb{H}(G, M, q\mathcal{E})$ is a graded Hecke algebra extended with a finite group. The triviality of $\natural_{q\mathcal{E}}$ is known when $\mathbb{H}(G, M, q\mathcal{E})$ arises:

- (i) from an extended affine Hecke algebra $\mathcal{H} \rtimes \Gamma$ via localization, as in [Lus3, Sol2, AMS3],
- (ii) from a classical p-adic group [Hei, AMS4],
- (iii) from a Bernstein component of a reductive p-adic group, such that the underlying supercuspidal representations are simply generic [OpSo, Theorem A.1].

In the references to (ii) and (iii) this is shown for the relevant extended affine Hecke algebras, and then one can apply (i).

Recall that $\det : W_{q\mathcal{E}} \to \{\pm 1\}$ denotes the determinant of the action of $W_{q\mathcal{E}}$ on $X^*(T)$. It can also be regarded as a onedimensional representation of $\mathbb{C}[W_{q\mathcal{E}}]$. We say that

(4.2) a $\mathbb{H}(G, M, q\mathcal{E})$-module V is generic if $\mathrm{Res}_{\mathbb{C}[W_{q\mathcal{E}}]}^{\mathbb{H}(G,M,q\mathcal{E})} V$ contains det.

This compares well with the definition of genericity for representations of extended affine Hecke algebras, see Theorem 5.1. Like for quasi-split reductive groups [Rod, Shal], there are multiplicity one properties for generic representations of extended graded Hecke algebras.

Proposition 4.1. *Let $Q \subset G$ be a quasi-Levi subgroup containing M, so that $\mathbb{H}(Q, M, q\mathcal{E})$ is a parabolic subalgebra of $\mathbb{H}(G, M, q\mathcal{E})$. Let (π, V) be a $\mathbb{H}(Q, M, q\mathcal{E})$-module.*

(*a*) *The multiplicity of* $\det$ *in* $\mathrm{Res}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{C}[W_{q\mathcal{E}}}\big(\mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} V\big)$ *equals the multiplicity of* $\det$ *in V, as representations of the version $W^Q_{q\mathcal{E}}$ of $W_{q\mathcal{E}}$ for $(Q, M, q\mathcal{E})$. In particular V is generic if and only if* $\mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} V$ *is generic.*

(*b*) *Suppose that (π, V) is irreducible and generic. Then*

$$\dim \mathrm{Hom}_{\mathbb{C}[W_{q\mathcal{E}}]}\big(\mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} V, \det\big) = 1$$

and $\mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} V$ *has a unique generic irreducible subquotient, appearing with multiplicity one.*

(*c*) $\dim \mathrm{Hom}_{\mathbb{C}[W_{q\mathcal{E}}]}(\mathrm{sgn}^*(E_{y,\sigma,r,\rho}), \det) \leq 1$ *for every enhanced L-parameter (y, σ, r, ρ) for $\mathbb{H}(G, M, q\mathcal{E})$.*

Proof. (a) and (b) These can be shown in the same way as for extended affine Hecke algebras, see [Sol10, Lemma 3.5] and [OpSo, Lemma 7.2]. Alternatively, one can apply [Sol10, Theorems 6.1 and 6.2] to [Sol10, Lemma 3.5].

(c) Conjugating the parameters by a suitable element of $N_{G^\circ}(T)$, we may assume that $\Re(\sigma)$ lies in the closed positive cone in $\mathfrak{t}_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} X_*(T)$. Alternatively, we can maneuver $\Re(-\sigma)$ to the closed positive cone. Therefore we can arrange that we are in one of the situations where [Sol8, Lemma B.3] applies, with $Q = Z_G(\sigma_0)$. It says that $\epsilon(\sigma, r) \neq 0$, which is needed to use [Sol8, Theorem B.2]. That result tells us

$$E_{y,\sigma,r,\rho} = \mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} E^Q_{y,\sigma,r,\rho}.$$

This remains valid upon applying sgn^* on both sides, by the isomorphism

$$\begin{array}{ccc} \mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})}(\mathrm{sgn}^* E^Q_{y,\sigma,r,\rho}) & \to & \mathrm{sgn}^*\big(\mathrm{ind}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{H}(Q,M,q\mathcal{E})} E^Q_{y,\sigma,r,\rho}\big) \\ h \otimes v & \mapsto & \mathrm{sgn}(h) \otimes v \end{array}.$$

Now part (a) shows it suffices to prove that

$$\dim \mathrm{Hom}_{\mathbb{C}[W^Q_{q\mathcal{E}}]}\big(\mathrm{sgn}^*(E^Q_{y,\sigma,r,\rho}), \det\big) \leq 1. \tag{4.3}$$

Notice that $\sigma_0 \in Z(\mathfrak{q})$, so that (y, σ, r) is an essentially bounded parameter for $\mathbb{H}(Q, M, q\mathcal{E})$. Suppose for the moment that $r \neq 0$. Then Lemma 3.4.b says that (y, σ, r) is an open parameter, and by Lemma 3.1 $E^Q_{y,\sigma,r,\rho}$ is irreducible. In this case part (b) proves (4.3).

Recall from [AMS2, Lemma 3.6] that $\pi_0(Z_G(\sigma_r, y)) = \pi_0(Z_G(\sigma_0, y))$ does not depend on r, where $\sigma_r = \sigma_0 + \mathrm{d}\gamma_y\big(\big(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\big)\big)$. Hence there is a family of $\mathbb{H}(Q, M, q\mathcal{E})$-modules $\mathrm{sgn}^* E_{y,\sigma_r,r,\rho}$, parametrized by $r \in \mathbb{C}$. It follows from [Sol8, Theorem 3.2.b] that the underlying family of $\mathbb{C}[W^Q_{q\mathcal{E}}]$-modules is constant. We already showed that for $r \neq 0$ it contains $\det$ at most one time, so the same holds when $r = 0$. □

It is known, for unipotent representations of (adjoint) p-adic groups from [Ree] and for principal series representations of quasi-split p-adic groups from [Sol10], that the Langlands parameters of generic representations are precisely the open parameters, with the trivial representation of a component group as enhancement. We intend to prove an analogous statement for extended graded Hecke algebras.

Lemma 4.2. *Fix* $(\sigma, r) = (\sigma_0 + r\sigma_v, r) \in \mathfrak{t} \oplus \mathbb{C}(\sigma_v, 1)$.

(*a*) *Every irreducible* $\mathbb{H}(G, M, q\mathcal{E})$*-module with central character* $(W_{q\mathcal{E}}\sigma_0, r)$ *is a subquotient of* $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0,r}$.

(*b*) *Up to* $Z_G(\sigma)$*-conjugacy, there exists precisely one enhanced L-parameter* (y, σ, r, ρ) *for* $\mathbb{H}(G, M, q\mathcal{E})$ *such that* $\mathrm{sgn}^*(M_{y,\sigma,r,\rho})$ *is generic.*

Proof. (a) Any irreducible $\mathbb{H}(G, M, q\mathcal{E})$-module V with central character $(W_{q\mathcal{E}}\sigma_0, r)$ has an $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$-weight (σ_0', r) with $\sigma' \in W_{q\mathcal{E}}\sigma_0$. Then

$$\mathrm{Hom}_{\mathbb{H}(G,M,q\mathcal{E})}\big(\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0',r}, V\big) \cong \mathrm{Hom}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}(\mathbb{C}_{\sigma_0',r}, V) \neq 0,$$

so V is a quotient of $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0',r}$. On the other hand, $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0',r}$ and $\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0,r}$ have the same irreducible subquotients, with the same multiplicities [Sol1, Lemma 9.1.a].

(b) The group $W_{q\mathcal{E}}^M$ for $\mathbb{H}(M, M, q\mathcal{E}) \cong \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$ is trivial, so

$$\mathrm{Hom}_{\mathbb{C}[W_{q\mathcal{E}}^M]}(\mathbb{C}_{\sigma_0,r}, \det) \cong \mathbb{C}.$$

By Proposition 4.1.a also

$$\mathrm{Hom}_{\mathbb{C}[W_{q\mathcal{E}}]}\big(\mathrm{ind}_{\mathcal{O}(\mathfrak{t}\oplus\mathbb{C})}^{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{C}_{\sigma_0,r}, \det\big) \cong \mathbb{C}. \tag{4.4}$$

By [AMS2, Theorem 4.6] every irreducible $\mathbb{H}(G, M, q\mathcal{E})$-module with central character $(W_{q\mathcal{E}}\sigma_0, -r)$ is of the form $\mathrm{sgn}^*(M_{y,\sigma,r,\rho})$. By part (a) and (4.4), exactly one of these modules is generic. That corresponds to a unique G-conjugacy class of (y, σ, ρ), and since (σ, r) was fixed we find that (y, ρ) is unique up to $Z_G(\sigma)$. $\square$

The $\mathbb{C}[W_{q\mathcal{E}}]$-module structure of $M_{y,\sigma,r,\rho}$ can be studied more easily in the case $r = 0$, so we consider that first.

Proposition 4.3. *Let* $(y, \sigma_0, 0)$ *be an open L-parameter for* $\mathbb{H}(G, M, q\mathcal{E})$*, so* y *is regular in* $Z_{\mathfrak{g}}(\sigma_0)$.

(*a*) *There exists a unique enhancement* $\rho_0 \in \mathrm{Irr}\big(\pi_0(Z_G(\sigma_0, y))\big)$ *such that* $\mathrm{sgn}^*(M_{y,\sigma_0,0,\rho_0})$ *is generic.*

(*b*) *Suppose that* $v = 0$ *and that* $q\mathcal{E}$ *is the trivial equivariant local system on* $\mathcal{C}_v^M = \{0\}$*. Then* $\rho_0 = triv$ *in part (a).*

Proof. (a) The condition to be checked is equivalent with: $\mathrm{Res}_{\mathbb{C}[W_{q\mathcal{E}}]}^{\mathbb{H}(G,M,q\mathcal{E})} M_{y,\sigma_0,0,\rho}$ contains $\mathrm{triv}_{W_{q\mathcal{E}}}$. First we consider the analogous question for $\mathbb{H}(G^\circ, M^\circ, \mathcal{E})$. To avoid confusion, we endow modules for this algebra with a superscript $\circ$. Write $Q = Z_G(\sigma_0)$ and $\mathfrak{q} = \mathrm{Lie}(Q)$. By [AMS2, (34)], for $\rho^\circ \in \mathrm{Irr}(Z_{G^\circ}(\sigma_0, y))$:

$$M_{y,\sigma_0,0,\rho^\circ}^\circ = \mathrm{ind}_{\mathbb{H}(Q^\circ,M^\circ,\mathcal{E})}^{\mathbb{H}(G^\circ,M^\circ,\mathcal{E})} M_{y,\sigma_0,0,\rho^\circ}^{Q^\circ}. \tag{4.5}$$

The $\mathbb{C}[W_\mathcal{E}]$-module structure of (4.5) follows from [AMS2, (33)]:

$$\mathrm{Res}_{\mathbb{C}[W_\mathcal{E}]}^{\mathbb{H}(G^\circ,M^\circ,\mathcal{E})} M_{y,\sigma_0,0,\rho^\circ}^\circ = \mathrm{ind}_{\mathbb{C}[W_\mathcal{E}^{Q^\circ}]}^{\mathbb{C}[W_\mathcal{E}]} M_{y,\rho^\circ}, \tag{4.6}$$

where M_{y,ρ° comes from the generalized Springer correspondence [Lus1] for $(Q^\circ, M^\circ, \mathcal{E})$. By Frobenius reciprocity (4.6) contains $\mathrm{triv}_{W_\mathcal{E}}$ if and only if M_{y,ρ° contains $\mathrm{triv}_{W_\mathcal{E}^{Q^\circ}}$. By [Lus1, Theorem 9.2], the latter happens if and only if

$$\mathrm{Ad}(Q^\circ)y \cap (\mathcal{C}_v^M \oplus \mathfrak{u} \cap \mathfrak{q}) \text{ is dense in } \mathcal{C}_v^M \oplus \mathfrak{u} \cap \mathfrak{q}.$$

This holds in our setting because (by assumption) $\mathrm{Ad}(Q^\circ)y$ is the regular nilpotent orbit in $\mathfrak{q} = Z_\mathfrak{g}(\sigma_0)$ and $\mathcal{C}_v^M \oplus (\mathfrak{u} \cap \mathfrak{q}) \subset \mathfrak{q}$. From [Lus1, Theorem 9.2] we also obtain that ρ° is unique.

Consider $\rho \in \mathrm{Irr}\big(\pi_0(Z_Q(y))\big)$ whose restriction to $\pi_0(Z_{Q^\circ}(y))$ contains ρ°. In the notation from [AMS2, Lemma 3.18] we have $\rho = \rho^\circ \rtimes \tau^\vee$ where $\tau^\vee$ where $\tau^\vee$ is an irreducible representation of the stabilizer S_{ρ° of ρ° in $Z_Q(y)/Z_{Q^\circ}(y)$. Then [AMS2, (67)] says that

$$M_{y,\sigma_0,0,\rho} \cong \tau \ltimes M^\circ_{y,\sigma_0,0,\rho^\circ}, \tag{4.7}$$

where the latter module is described explicitly in [AMS2, Lemma 3.16]. This description simplifies a bit in our setup, because the 2-cocycles in [AMS2, §3] are by assumption trivial. Namely, the structure of (4.7) as $\mathbb{C}[W_{q\mathcal{E}}]$-module is

$$\mathrm{ind}_{\mathbb{C}[W'_{q\mathcal{E}}]}^{\mathbb{C}[W_{q\mathcal{E}}]}(\tau \otimes J) \tag{4.8}$$

for some extension J of $\mathrm{triv}_{W_\mathcal{E}}$ of $W'_{q\mathcal{E}}$. Here $W_\mathcal{E} \subset W'_{q\mathcal{E}} \subset W_{q\mathcal{E}}$ such that $W'_{q\mathcal{E}}/W_\mathcal{E}$ is naturally isomorphic to S_{ρ°. This J comes from [AMS2, Proposition 3.15], and it is only unique up a characters. As the underlying vector space of J is that of $\mathrm{triv}_{W_\mathcal{E}}$, we may renormalize the operators $J(\gamma)$ with $\gamma \in W'_{q\mathcal{E}}$, and arrange that $J =$ triv. Now it is clear that, if we take $\tau = \mathrm{triv}_{S_{\rho^\circ}}$, then (4.8) contains $\mathrm{triv}_{W_{q\mathcal{E}}}$. Thus $\rho_0 := \rho^\circ \rtimes \mathrm{triv}$ fulfills the requirements. By Lemma 4.2.a it is unique.
(b) Under these assumptions, the generalized Springer correspondence for $(Q^\circ, M^\circ, \mathcal{E})$, encountered in the proof of part (a), becomes the classical Springer correspondence for Q°. Then (4.6) contains $\mathrm{triv}_{W_\mathcal{E}}$ if and only y is regular nilpotent in $\mathfrak{q}$ and $\rho^\circ = \mathrm{triv}$. (Since these constructions are in the end based on [Lus1], we have to use the normalization of the Springer correspondence from there.) Then ρ_0 in part (a) reduces to $\mathrm{triv}_{\pi_0(Z_{Q^\circ}(y))} \rtimes \mathrm{triv} = \mathrm{triv}_{\pi_0(Z_Q(y))}$. □

With geometric arguments we will deduce a property of generic representations for $r \neq 0$, which will enable us to rule out that their L-parameters are not open.

Lemma 4.4. *Let (y, σ, r, ρ) be an enhanced L-parameter for $\mathbb{H}(G, M, q\mathcal{E})$, with $r \neq 0$. Let (y', σ, r) be an open parameter with the same (σ, r). If $\mathrm{sgn}^* M_{y,\sigma,r,\rho}$ is generic, then $\mathrm{sgn}^* E_{y',\sigma,r,\rho'}$ is generic for some enhancement ρ'.*

Proof. As $\mathrm{sgn}^* E_{y,\sigma,r,\rho}$ is generic,

$$\mathrm{Res}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{C}[W_{q\mathcal{E}}]} E_{y,\sigma,r} \text{ contains } \mathrm{triv}_{W_{q\mathcal{E}}}.$$

Recall from Section 3 that $\sigma = \sigma_r = \sigma_0 + \mathrm{d}\gamma_y\left(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\right)$. We can vary r in $\mathbb{C}$ and obtain a family of $\mathbb{H}(G, M, q\mathcal{E})$-modules $E_{y,\sigma_r,r}$.

From Lemma 3.3 we know that $\sigma'_0 = \sigma - \mathrm{d}\gamma_{y'}\left(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\right)$ commutes with $y \in \mathfrak{g}_N^{\sigma,r}$. That enables us to regard $E_{y,\sigma_0,0}$ as a member of a family of $\mathbb{H}(G, M, q\mathcal{E})$-modules $E_{y,t\sigma_0+(1-t)\sigma'_0,0}$ parametrized by $t \in \mathbb{C}$.

By [Sol8, Theorem 3.2.b], which is based on [Lus2], the two underlying families of $\mathbb{C}[W_{q\mathcal{E}}]$-modules are constant. All those modules, in particular $E_{y,\sigma_0,0}$ and $E_{y,\sigma'_0,0}$, contain $\mathrm{triv}_{W_{q\mathcal{E}}}$. From Lemma 4.2.b and Proposition 4.3.a we deduce that this is only possible if $E_{y,\sigma'_0,0}$ has a constituent $M_{y_g,\sigma'_0,0,\rho_0}$, where y_g is regular nilpotent in $Z_{\mathfrak{g}}(\sigma'_0)$. From Theorem 3.2 and Proposition 2.3.b we see that

$$M_{y,\sigma'_0,0,\rho_0} = E_{y,\sigma'_0,0,\rho_0} \text{ embeds in } E_{y,\sigma'_0,0} \text{ via } J_{y_g,y}. \tag{4.9}$$

The proof of (ii) $\Rightarrow$ (i) in Lemma 3.3 shows that not only $\overline{\mathrm{Ad}(Z_G(\sigma))y'}$, but also $\mathrm{Ad}(Z_{G^\circ}(\sigma'_0,\sigma-\sigma'_0))y'$ is dense in $\mathfrak{g}_N^{\sigma,r}$. In particular $\overline{\mathrm{Ad}(Z_{G^\circ}(\sigma'_0))y'}$ contains y, which shows that

$$\overline{\mathrm{Ad}(Z_{G^\circ}(\sigma'_0))y} \;\subset\; \overline{\mathrm{Ad}(Z_{G^\circ}(\sigma'_0))y'} \;\subset\; \overline{\mathrm{Ad}(Z_{G^\circ}(\sigma'_0))y_g} \;\subset\; \mathfrak{g}_N^{\sigma'_0,0}.$$

From Proposition 2.3.b we see that $J_{y_g,y}$ factors as

$$E_{y_g,\sigma'_0,0} \xrightarrow{J_{y_g,y'}} E_{y',\sigma'_0,0} \xrightarrow{J_{y',y}} E_{y,\sigma'_0,0}. \tag{4.10}$$

With (4.9) we conclude that the module $M_{y_g,\sigma'_0,0,\rho_0}$ is a constituent of $E_{y',\sigma'_0,0}$, so both contain $\mathrm{triv}_{W_{q\mathcal{E}}}$.

Consider the family of $\mathbb{H}(G,M,q\mathcal{E})$-modules $E_{y',\sigma'_r,r}$, where $\sigma'_r = \sigma'_0 + \mathrm{d}\gamma_{y'}(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right))$ for $r \in \mathbb{C}$. Again by [Sol8, Theorem 3.2.b], as $\mathbb{C}[W_{q\mathcal{E}}]$-modules they form a constant family. In particular all members contain $\mathrm{triv}_{W_{q\mathcal{E}}}$. We return to our initial $r \in \mathbb{C}$, so that $\sigma'_r = \sigma$. As $E_{y',\sigma,r}$ is a direct sum of modules $E_{y',\sigma,0,\rho'}$ (with multiplicities), there exists a ρ' such that $\mathrm{Res}^{\mathbb{H}(G,M,q\mathcal{E})}_{\mathbb{C}[W_{q\mathcal{E}}]} E_{y',\sigma,r,\rho'}$ contains $\mathrm{triv}_{W_{q\mathcal{E}}}$. Now $\mathrm{sgn}^* E_{y',\sigma,r,\rho'}$ is generic. $\square$

We are ready to complete the analysis of the L-parameters of generic irreducible $\mathbb{H}(G,M,q\mathcal{E})$-modules.

Theorem 4.5. *Consider an enhanced L-parameter (y,σ,r,ρ) for $\mathbb{H}(G,M,q\mathcal{E})$.*

(a) If (y,σ,r) is not open, then $\mathrm{sgn}^*(M_{y,\sigma,r,\rho})$ is not generic.

(b) If (y,σ,r) is open, then $\mathrm{sgn}^*(M_{y,\sigma,r,\rho})$ is generic for a unique enhancement ρ, say ρ_g.

(c) If $q\mathcal{E}$ is the trivial equivariant local system on $\mathcal{C}_v^M = \{0\}$, then $\rho_g = \mathrm{triv}$.

Proof. (a) Suppose that $\mathrm{sgn}^*(M_{y,\sigma,r,\rho})$ is generic. By Lemma 4.4 (or Proposition 4.3.a if $r=0$) $E_{y',\sigma,r,\rho'}$ is generic for some open parameter (y',σ,r,ρ'). From Lemma 3.1 we know that $M_{y',\sigma,r,\rho'} = E_{y',\sigma,r,\rho'}$, so $\mathrm{sgn}^*(M_{y',\sigma,r,\rho'})$ is generic for an enhanced L-parameter with the same (σ,r) as before. This contradicts Lemma 4.2.b.
(b) This follows from Lemma 4.2.b and part (a).
(c) Under the current assumptions M is a minimal quasi-Levi subgroup L of G, see (1.5). Let (y,σ,r,ρ_g) be as in part (b) and write $\sigma_r = \sigma_0 + \mathrm{d}\gamma_y(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right))$. Recall from [AMS2, Lemma 3.6] that $\pi_0(Z_G(\sigma_r,y)) = \pi_0(Z_G(\sigma_0,y))$ does not depend on r.

Arguing with families of $\mathbb{H}(G,M,q\mathcal{E})$-modules as in the proof of Lemma 4.4, we deduce that

$$\mathrm{sgn}^*(E_{y,\sigma_r,r,\rho_g}) \text{ is generic for every } r \in \mathbb{C}, \tag{4.11}$$

and in particular for $r=0$. We can get some useful information from the proof of Lemma 4.4, with $y' = y$. There we encountered the enhancement ρ_0 of $(y_g,\sigma_0,0)$. Proposition 4.3.b says that ρ_0 equals the trivial representation of $\pi_0(Z_G(\sigma_0,y_g))$. In view of (4.10) and (4.11), we need to identify the unique enhancement ρ_g of $(y,\sigma_0,0)$ such that $J_{y_g,y}$ induces a nonzero homomorphism

$$M_{y_g,\sigma_0,0,\mathrm{triv}} = V_{\mathrm{triv}} \otimes E_{y_g,\sigma_0,0,\mathrm{triv}} \longrightarrow V_{\rho_g} \otimes E_{y,\sigma_0,0,\rho_g}.$$

Put $Q = Z_G(\sigma_0)$ and notice that $\pi_0(Z_G(\sigma_0, y)) = \pi_0(Z_Q(y))$. Standard modules are compatible with parabolic induction, under a condition $\epsilon(\sigma, r) \neq 0$ [Sol8, Theorem B.1]. Here $\mathrm{ad}(\sigma_0)$ is invertible on $\mathrm{Lie}(G)/\mathrm{Lie}(Q)$, so $\epsilon(\sigma_0, 0) \neq 0$. We obtain

$$E_{y,\sigma_0,0,\rho_g} = \mathrm{ind}_{\mathbb{H}(Q,L,\mathrm{triv})}^{\mathbb{H}(G,L,\mathrm{triv})} E^Q_{y,\sigma_0,0,\rho_g},$$

and similarly for $E_{y_g,\sigma_0,0,\mathrm{triv}}$. Hence it suffices to identify the ρ_g so that $J_{y_g,y}$ induces a nonzero homomorphism of $\mathbb{H}(Q, L, \mathrm{triv})$-modules

$$E^Q_{y_g,\sigma_0,0,\rho_g} \to E^Q_{y,\sigma_0,0,\rho_g}. \tag{4.12}$$

We consider the $\mathbb{C}[W_{q\mathcal{E}}]$-modules underlying (4.12). These are standard modules as appearing in the Springer correspondence for (possibly disconnected) reductive group Q and its (possibly extended) Weyl group $W_{q\mathcal{E}}$.

Since $q\mathcal{E}$ is trivial, so is the local system $\dot{q\mathcal{E}}_N$ used to construct K_N. With that in mind, [Sol8, Proposition 3.6] says that

$$E^Q_{y,\sigma_0,0} = H_*(\mathcal{P}^{Q,\sigma_0}_y), \qquad \mathcal{P}^{Q,\sigma_0}_y = \mathcal{P}^Q_y = \{gL \in Q/L : \mathrm{Ad}(g^{-1})y \in \mathfrak{u} \cap \mathfrak{q}\}.$$

The action of $\pi_0(Z_Q(y))$ on $E^Q_{y,\sigma_0,0}$ is induced by the left multiplication action of $Z_Q(y)$ on $\mathcal{P}^Q_y$. The same holds with y_q instead of y. By the regularity of y_g in $\mathfrak{q}$:

$$\mathcal{P}^Q_{y_g} \cap Q^\circ L/L = L/L \quad \text{and} \quad \mathcal{P}^Q_{y_g} \cong Z_Q(y_g)/Z_{Q^\circ}(y_g) \cong Q/Q^\circ.$$

We obtain

$$E^Q_{y_g,\sigma_0,0,\mathrm{triv}} = H_*(\mathcal{P}^Q_{y_g})^{\pi_0(Z_Q(y_g))} \cong H_0(Z_Q(y_g)/Z_{Q^\circ}(y_g))^{Z_Q(y_g)} \cong H_0(Q/Q^\circ)^Q, \tag{4.13}$$

which has dimension one. The image of (4.13) by $J_{y_g,y}$ is a subspace of $H_0(\mathcal{P}^Q_y)$ fixed by $Z_Q(y)$, so it only appears in $V_{\rho_g} \otimes E^Q_{y,\sigma_0,0,\rho_g}$ when $\rho_g = \mathrm{triv}$. $\square$

We conclude this section with a proof of the generalized injectivity conjecture for geometric graded Hecke algebras.

Corollary 4.6. *Let E be an analytic standard $\mathbb{H}(G, M, q\mathcal{E})$-module and let M be a generic irreducible subquotient of E. Then M is a submodule of E.*

Proof. Write $E = \mathrm{sgn}^* E_{y,\sigma,r,\rho}$. Since M has the same central character as E, it equals $\mathrm{sgn}^* M_{y',\sigma,r,\rho'}$ for some y', ρ'. By Theorem 4.5 (y', σ, r) is open. Now $\mathrm{sgn}^* M = M_{y',\sigma,r,\rho'}$ is a subqotient of $\mathrm{sgn}^* E = E_{y,\sigma,r,\rho}$, and Theorem 3.2.a says that it is isomorphic to a submodule of $\mathrm{sgn}^* E$. We apply sgn^* again to conclude that M is isomorphic to a submodule of E. As E has a unique generic irreducible subquotient (Proposition 4.1.c), that must be a submodule and equal to M. $\square$

5. Transfer to affine Hecke algebras

We will show how the representation theoretic results from the previous sections can be translated to suitable affine Hecke algebras. This section is largely based on [Lus3, Sol2, AMS3].

Let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be a based root datum and let W be the Weyl group of R. Let $\lambda, \lambda^* : R \to \mathbb{Z}_{\geq 0}$ be W-invariant functions such that $\lambda^*(\alpha) = \lambda(\alpha)$ whenever $\alpha^\vee \notin 2Y$. Let $\mathbf{q}$ be an invertible indeterminate. To these data one can associate an affine Hecke algebra

$$\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}),$$

as for instance in [Lus3, Sol3]. The underlying vector space is $\mathbb{C}[X] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$ and the quadratic relation for a simple reflection s_α is

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - \mathbf{q}^{2\lambda(\alpha)}) = 0.$$

Let Γ be a finite group acting on $\mathcal{R}$ and on $T = \mathrm{Hom}(X, \mathbb{C}^\times)$. Assume that λ and λ^* are Γ-invariant and that $\alpha(\gamma(1_T)) = 1$ for all $\gamma \in \Gamma, \alpha \in R$. For any 2-cocycle $\natural : \Gamma^2 \to \mathbb{C}^\times$ we can build the twisted affine Hecke algebra

$$\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural] = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}) \rtimes \mathbb{C}[\Gamma, \natural],$$

see [AMS3, Proposition 2.2]. As a vector space it is the tensor product of its subalgebras $\mathcal{H}$ and $\mathbb{C}[\Gamma, \natural]$, and for a standard basis element T_γ of $\mathbb{C}[\Gamma, \natural]$ we have the cross relations

$$T_\gamma T_w \theta_x T_\gamma^{-1} = T_{\gamma w \gamma^{-1}} \gamma(\theta_x) \qquad w \in W, x \in X.$$

We can specialize $\mathbf{q}$ to any $q \in \mathbb{C}^\times$, and then we obtain Hecke algebras denoted

$$\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) \quad \text{and} \quad \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) \rtimes \mathbb{C}[\Gamma, \natural].$$

In practice we will only specialize to $q \in \mathbb{R}_{>0}$.

When $\natural$ is trivial, [OpSo, §6 and (8.9)] provide a good notion of genericity for $\mathcal{H} \rtimes \Gamma$-modules, as follows. The elements $T_{w\gamma}$ with $w \in W$ and $\gamma \in \Gamma$ form a $\mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}]$-basis of a subalgebra $\mathcal{H}(W, q^\lambda) \rtimes \Gamma$. The Steinberg representation of $\mathcal{H}(W, \mathbf{q}^\lambda) \rtimes \Gamma$ (with $\mathbf{q}$ specialized to some chosen $q \in \mathbb{C}^\times$) has dimension one and is defined by

$$\mathrm{St}(T_{w\gamma}) = \det(w\gamma). \tag{5.1}$$

Here det denotes the determinant of the action of $W\Gamma$ on X. We say that a $\mathcal{H} \rtimes \Gamma$-module V is generic if $\mathbf{q}$ acts as multiplication by some $q \in \mathbb{C}^\times$ and $\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}(W, \mathbf{q}^\lambda) \rtimes \Gamma} V$ contains St.

The centre of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ contains

$$\mathcal{O}(T)^{W\Gamma} \otimes \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}] \cong \mathcal{O}(T/W\Gamma \times \mathbb{C}^\times). \tag{5.2}$$

Often we will analyse representations of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ via localization to suitable subsets of $T/W\Gamma \times \mathbb{C}^\times$. That involves decomposing representations along their weights for (5.2), which works well for finite length representations but does not always apply to infinite dimensional representations. Therefore we will usually restrict our attention to the category $\mathrm{Mod}_{\mathrm{fl}}(\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural])$ of finite length (or equivalently finite dimensional) modules.

There is a two-step reduction procedure which assigns to $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ a twisted graded Hecke algebra that governs a well-defined part of its representation theory. A suitable family of such twisted graded Hecke algebras covers the entire category $\mathrm{Mod}_{\mathrm{fl}}(\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural])$.

We write $\mathfrak{t}_\mathbb{R} = R \otimes_\mathbb{Z} X_*(T)$ and $T_\mathbb{R} = \exp(\mathfrak{t}_\mathbb{R})$. We fix a unitary element $u \in \mathrm{Hom}(X, S^1) \subset T$, and we want to study representations whose $\mathcal{O}(T)^{W\Gamma}$-weights are close to $W\Gamma u T_\mathbb{R}$ in $T/W\Gamma$. There is a subroot system $R_u = \{\alpha \in R : s_\alpha(u) = u\}$, with a basis Δ_u determined by Δ. These fit into a based root datum

$$\mathcal{R}_u = (X, R_u, Y, R_u^\vee, \Delta_u).$$

The group $(W\Gamma)_u$ decomposes as

$$(W\Gamma)_u = W(R_u) \rtimes \Gamma_u, \qquad \Gamma_u = \{\gamma \in (W\Gamma)_u : \gamma(\Delta_u) = \Delta_u\}.$$

Let λ_u, λ_u^* be the restrictions of λ, λ^* to R_u and let $\natural_u$ be the restriction of $\natural : (W\Gamma)^2 \to \Gamma^2 \to \mathbb{C}^\times$ to Γ_u^2. Altogether these objects yield a new twisted affine Hecke algebra

$$\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u] = \mathcal{H}(\mathcal{R}_u, \lambda_u, \lambda_u^*, \mathbf{q}) \rtimes \mathbb{C}[\Gamma_u, \natural_u],$$

a subalgebra of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. An advantage is that u is fixed by $W(R_u)$, so $\alpha(u) \in \{\pm 1\}$ for all $\alpha \in R_u$. There is a $(W\Gamma)_u$-equivariant map

$$\exp_u : \mathfrak{t} \to T, \exp_u(\sigma) = u \exp(\sigma).$$

It is a local diffeomorphism around $\mathfrak{t}_\mathbb{R}$ and restricts to a diffeomorphism $\mathfrak{t}_\mathbb{R} \to uT_\mathbb{R}$. Via this map we can pass from $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$ to a twisted graded Hecke

$$\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u] = \mathbb{H}(\mathfrak{t}, (W\Gamma)_u, k_u, \mathbf{r}, \natural_u).$$

Here the parameter function $k_u : R_u \to \mathbb{Z}$ is given by

$$k_{u,\alpha} = (\lambda(\alpha) + \alpha(u)\lambda^*(\alpha))/2. \tag{5.3}$$

The next theorem was proven in [AMS3, Theorems 2.5, 2.11 and Proposition 2.7], based on similar results in [Lus3, §8–9] and [Sol2, §2.1]. The part about genericity was checked in [Sol10, Theorems 6.1 and 6.2].

Theorem 5.1. *The following three categories are canonically equivalent:*

- *finite dimensional $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-modules, all whose $\mathcal{O}(T/W\Gamma \times \mathbb{C}^\times)$-weights belong to $W\Gamma uT_\mathbb{R} \times \mathbb{R}_{>0}$,*
- *finite dimensional $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$-modules, all whose $\mathcal{O}(T/(W\Gamma_u \times \mathbb{C}^\times)$-weights belong to $uT_\mathbb{R} \times \mathbb{R}_{>0}$,*
- *finite dimensional $\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$-modules, all whose $\mathcal{O}(\mathfrak{t}/(W\Gamma)_u \times \mathbb{C})$-weights belong to $\mathfrak{t}_\mathbb{R} \times \mathbb{R}$.*

The equivalences have the following features:

(i) They are compatible with parabolic induction and parabolic restriction.
(ii) They respect temperedness.
(iii) They respect essentially discrete series when $\mathrm{rk}(R_u) = \mathrm{rk}(R)$, and otherwise the involved category of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-modules does not contain essentially discrete series representations.
(iv) They respect genericity whenever $\natural$ is trivial.
(v) Any $\mathcal{O}(\mathfrak{t} \oplus \mathbb{C})$-weight of a $\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$-module is transformed into a $\mathcal{O}(T \times \mathbb{C}^\times)$-weight $(\exp_u(\sigma), \exp(r))$ for $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$ and into a collection of $\mathcal{O}(T \rtimes \mathbb{C}^\times)$-weights $(w \exp_u(\sigma), \exp(r))$ for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$, where w runs through a certain set of representatives for $W\Gamma/(W\Gamma)_u$.

Langlands standard modules for twisted affine Hecke algebras can be defined like for twisted graded Hecke algebras, see (3.4). This provides satisfactory collections of standard modules in each of the three categories in Theorem 5.1. In each case they are in bijection with the irreducible modules in that category, via taking irreducible quotients of standard modules.

Lemma 5.2. *The equivalences of categories in Theorem 5.1 restrict to bijections between the three sets of Langlands standard modules.*

Proof. Theorem 5.1 respects almost all the operations and properties involved in (Langlands) standard modules, the only potential issue being the weights in part (v). Between $\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$ and $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$, Theorem 5.1 induces a bijection on weights, so the equivalence of categories provides a bijection between the respective sets of standard modules.

It may seem that Theorem 5.1 does not necessarily match those two sets with standard modules for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. The problems lies in part (v) at the level of parabolic subalgebras (associated to a set of simple roots P), which entails that a positive character t for $\mathcal{H}_u^P \rtimes \mathbb{C}[\gamma_{P,u}, \natural_u]$ may be moved by Γ_P even if it is fixed by $\Gamma_{P,t}$. In such a situation the essentially tempered irreducible representation $\tau \otimes t$ of is sent by Theorem 5.1 to

$$\mathrm{ind}_{\mathbb{H}^P \rtimes \mathbb{C}[\Gamma_{P,t},\natural]}^{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]} (\tau' \otimes t),$$

where τ' is the image of τ via Theorem 5.1 for the appropriate subalgebras. The standard $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-module associated to (P, τ', t) is

$$\mathrm{ind}_{\mathbb{H}^P \rtimes \mathbb{C}[\Gamma_{P,t},\natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]} (\tau' \otimes t).$$

In Theorem 5.1 this is matched with the standard $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$-module

$$\mathrm{ind}_{\mathbb{H}_u^P \rtimes \mathbb{C}[\Gamma_{P,u},\natural_u]}^{\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} (\tau \otimes t).$$

Thus Theorem 5.1 sends standard modules for $\mathcal{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$ to standard modules for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. Since we have an equivalence of categories and on both sides the standard modules are canonically in bijection with the irreducible modules, the equivalence is also bijective on standard modules. $\square$

There are always classifications of irreducible $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-modules, see [Sol3], but in general these do not involve parameters like Langlands parameters for reductive p-adic groups. To get the geometry from Sections 1–4 into play, we need fairly specific parameter functions λ, λ^*, and the 2-cocycle $\natural$ cannot be arbitrary either. Some twisted affine Hecke algebras that can be analysed geometrically feature in [AMS3, §2], they are based on reductive complex groups and cuspidal local systems like our graded Hecke algebras.

But the class of twisted affine Hecke algebras to which Sections 1–3 can be applied is larger, we only need that for every fixed u Theorem 5.1 yields a geometric graded Hecke algebra. If we only want to transfer the results about submodules of standard modules, the subalgebra $\mathbb{C}[\Gamma, \natural]$ does not cause additional complications, and a slightly more relaxed condition suffices:

Condition 5.3. The twisted affine Hecke algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \mathbf{q}) \rtimes \mathbb{C}[\Gamma, \natural]$ is such that, for each twisted graded Hecke algebra

$$\mathbb{H}(\mathfrak{t}, (W\Gamma)_u, k_u, \mathbf{r}, \natural_u) = \mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$$

involved in Theorem 5.1 for some unitary element $u \in \mathrm{Hom}(X, S^1) \subset T$, there are data $(G_u, M_u, q\mathcal{E}_u, \mathcal{E}_u)$ like in Section 1 and a Lie group isomorphism $\mathfrak{t} \rtimes W_{q\mathcal{E}_u} \cong \mathrm{Lie}(T_u) \rtimes (W\Gamma)_u$ which induces an algebra isomorphism $\mathbb{H}_u \cong \mathbb{H}(G_u^\circ, M_u^\circ, \mathcal{E}_u)$.

Notice that Condition 5.3 puts no restrictions on $\natural$, which is good because often it is difficult to make $\natural$ explicit. That renders it largely irrelevant how Γ_u arises from $G_u \supset G_u^\circ$. In Condition 5.3 we could simpy take G_u of the form $G_u^\circ \rtimes \Gamma_u$ where the action of Γ_u on G_u° preserves a pinning.

Theorem 5.4. *Let $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ be a twisted affine Hecke algebra satisfying Condition 5.3. Let E be a Langlands standard $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-module on which $\mathbf{q}$ acts as multiplication by $q \in \mathbb{R}_{>0}$ and let V be an irreducible subquotient of E.*

(*a*) *If V is tempered or essentially discrete series, then it is a submodule of E.*
(*b*) *Suppose $q > 1$, $\natural$ is trivial and V is generic. Then V is a submodule of E.*

Proof. Since E is standard, it admits a central character, say $(W\Gamma t, q)$. Put $u = t|t|^{-1} \in \mathrm{Hom}(X, S^1)$, so that $t \in uT_{\mathbb{R}}$. By Theorem 5.1, the category $\mathrm{Mod}_{\mathrm{fl},W\Gamma t,q}(\mathcal{H})$ is equivalent with the category

$$\mathrm{Mod}_{\mathrm{fl},(W\Gamma)_u|t|,\log q}(\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]),$$

where $\mathbb{H}_u = \mathbb{H}(G_u, M_u, q\mathcal{E}_u)$ by Condition 5.3. By Lemma 5.2 E corresponds to a Langlands standard module E_u of $\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u, \natural_u]$, on which $\mathbf{r}$ acts as $\log q \in \mathbb{R}$. By (3.5), E_u is a direct summand of $\mathrm{ind}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} E_u^\circ$ for some Langlands standard $\mathbb{H}_u$-module E_u°. More concretely, the steps from (3.4) to (3.5) show that

$$E_u = \mathrm{ind}_{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma'_u,\natural_u]}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} (\rho_u \otimes E_u^\circ), \tag{5.4}$$

where Γ'_u is the stabilizer of E_u° in Γ_u and ρ_u is a projective representation of Γ'_u. By Proposition 3.7

- E_u° is analytic standard if $q > 1$,
- E_u° is geometric standard if $q < 1$,
- E_u° is irreducible if $q = 1$.

In the case $q = 1$, we deduce from (5.4) that E_u inherits the complete reducibility of E_u°, and then all its subquotients are submodules. Therefore we may assume from now on that $q \neq 1$.

Via Theorem 5.1, V corresponds to an irreducible subquotient V_u of E_u. By Clifford theory, see for instance [RaRa, Appendix], [Sol1, §11] and [AMS3, §1], $\mathrm{Res}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} V_u$ is completely reducible, and all its irreducible summands are in one Γ_u-orbit. Via a composition series of $\rho_u \otimes E_u^\circ$ we see that V_u arises from a subquotient of that, unique up to Γ_u. It follows that V_u contains an irreducible subquotient of E_u°, say V_u°, and is generated by V_u° as $\mathbb{C}[\Gamma_u, \natural_u]$-module.

Clifford theory tells us that V_u is a direct summand of $\mathrm{ind}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} V_u^\circ$. The action of Γ_u on R_u stabilizes Δ_u, so preserves positivity and negativity of $\mathcal{O}(\mathfrak{t})$-weights. Hence V_u° is tempered if and only if $\mathrm{ind}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} V_u^\circ$ is tempered, if and only if V_u is tempered. The same holds with essentially discrete series instead of tempered. If V_u is generic and $\natural_u$ is trivial, then [OpSo, (8.13)] says that $V_u = \det \ltimes V_u^\circ$ and V_u° is generic. Thus, in both cases (a) and (b), V_u° has the same property as supposed for V_u. Now Theorem 3.6 in case (a) and Corollary 4.6 in case (b) prove the theorem for the subquotient V_u° of E_u°.

Let $\mathrm{soc}(E_u)$ denote the socle of E_u, that is, the sum of all irreducible submodules. Since $\mathrm{ind}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]}$ preserves completely reducibility [Sol1, Theorem 11.2] and E_u is a direct summand of $\mathrm{ind}_{\mathbb{H}_u}^{\mathbb{H}_u \rtimes \mathbb{C}[\Gamma_u,\natural_u]} E_u^\circ$:

$$\mathrm{soc}(E_u) = \mathbb{C}[\Gamma_u, \natural_u] \cdot \mathrm{soc}(E_u^\circ).$$

We already saw that V_u° is an irreducible submodule of E_u° which generates V_u, so $V_u \subset \mathbb{C}[\Gamma_u, \natural_u] \cdot \mathrm{soc}(E_u^\circ)$. Thus $V_u \subset \mathrm{soc}(E_u)$, which means that it is a submodule. We can go back to $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$-modules via Theorem 5.1, from which we conclude that V is a submodule of E. $\square$

6. Transfer to reductive p-adic groups

Let F be a non-archimedean local field and let $\mathcal{G}$ a connected reductive F-group. We will call $\mathcal{G}(F)$ a reductive p-adic group, although $\mathrm{char}(F) > 0$ is allowed. We warn that $\mathcal{G}$ is not related to G from Section 1.

We are interested in smooth complex representations of $\mathcal{G}(F)$, which form a category $\mathrm{Rep}(\mathcal{G}(F))$. Let $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ be a Bernstein block in there, coming from a unitary supercuspidal representation ω of a Levi subgroup $\mathcal{M}(F) \subset \mathcal{G}(F)$.

It is well-known that in many cases $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ is closely related to the module category of a (twisted) affine Hecke algebra. At the same time, it is known from [Sol4] that one can increase the generality of such comparison results by using graded instead of affine Hecke algebras.

Let $X_{\mathrm{nr}}(\mathcal{M}(F))$ be the group of unramified characters of $\mathcal{M}(F)$, and let $X_{\mathrm{nr}}^+(\mathcal{M}(F))$ be the subgroup $\mathrm{Hom}(\mathcal{M}(F), \mathbb{R}_{>0})$. Recall that $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ consists of all smooth $\mathcal{G}(F)$-representations π such that every irreducible subquotient of π has cuspidal support in $(\mathcal{M}(F), X_{\mathrm{nr}}(\mathcal{M}(F))\omega)$ up to $\mathcal{G}(F)$-conjugacy. Let $W_{\mathfrak{s}}$ be the finite group associated to $\mathfrak{s} = [\mathcal{M}(F), \omega]$ by Bernstein, and let $W_{\mathfrak{s},\omega}$ be the subgroup that stabilizes ω. Let $\mathfrak{t}$ be the Lie algebra of $X_{\mathrm{nr}}(\mathcal{M}(F))$, identified with the tangent space to $X_{\mathrm{nr}}(\mathcal{M}(F))\omega$ at ω. Since $W_{\mathfrak{s}}$ operates faithfully on $X_{\mathrm{nr}}(\mathcal{M}(F))\omega$ [BeDe, §2.16], $W_{\mathfrak{s},\omega}$ acts faithfully on $\mathfrak{t}$.

We define a root system R_ω as in [Sol4, §6.1], where it is called $\Sigma_{\sigma \otimes u}$. Parameters k^ω and a 2-cocycle $\natural_\omega$ of $\Gamma_\omega \cong W_{\mathfrak{s},\omega}/W(R_\omega)$ (denoted $\natural_u^{-1}$ in [Sol4]) are constructed in [Sol4, §7].

Theorem 6.1. [Sol4, Theorems B and C and (8.2)]
There exists an equivalence between the following categories:

- *finite length smooth $\mathcal{G}(F)$-representations π, such that all irreducible subquotients of π have cuspidal support in $(\mathcal{M}(F), X_{\mathrm{nr}}^+(\mathcal{M}(F))\omega)$ up to $\mathcal{G}(F)$-conjugacy,*
- *finite dimensional modules of the twisted graded Hecke algebra*
$$\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega) = \mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \mathbb{C}[\Gamma_\omega, \natural_\omega],$$
all whose $\mathcal{O}(\mathfrak{t})$-weights belong to $\mathfrak{t}_{\mathbb{R}} = \mathrm{Lie}\big(X_{\mathrm{nr}}^+(\mathcal{M}(F))\big)$.

This equivalence is canonical up to the choice of the 2-cocycle $\natural_\omega$, and it has the following properties:

(i) compatibility with normalized parabolic induction and restriction,
(ii) respects temperedness,
(iii) sends essentially square-integrable $\mathcal{G}(F)$-representations to essentially discrete series $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega)$-modules (but not always conversely),
(iv) compatibility for twisting $\mathcal{G}(F)$-representations by elements of $X_{\mathrm{nr}}^+(\mathcal{M}(F))$ and $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega)$-modules by elements of $\mathrm{Lie}\big(X_{\mathrm{nr}}^+(\mathcal{G}(F))\big) \subset \mathfrak{t}_{\mathbb{R}}$.

Notice that there is no $\mathbf{r}$ in the graded Hecke algebras in Theorem 6.1. They relate to Sections 1–4 by specializing $\mathbf{r}$ at some $r > 0$.

Let $\mathcal{L}(F) \subset \mathcal{G}(F)$ be a Levi subgroup containing $\mathcal{M}(F)$ and let $\tau \in \mathrm{Irr}(\mathcal{L}(F))$ be a tempered representation with cuspidal support in $(\mathcal{M}(F), X_{\mathrm{nr}}^+(\mathcal{M}(F))\omega)$. Let $\chi \in X_{\mathrm{nr}}^+(\mathcal{L}(F))$ be in positive position with respect to a parabolic subgroup $\mathcal{P}(F) \subset \mathcal{G}(F)$ with Levi factor $\mathcal{L}(F)$. Then $I_{\mathcal{P}(F)}^{\mathcal{G}(F)}(\tau \otimes \chi)$ is a standard representation as in the Langlands classification for $\mathcal{G}(F)$. Moreover every standard representation

$\mathcal{G}(F)$-representation with cuspidal support in $(\mathcal{M}(F), X_{\mathrm{nr}}^+(\mathcal{M}(F))\omega)$ (up to $\mathcal{G}(F)$-conjugation) is of this form.

Lemma 6.2. *The equivalence in Theorem 6.1 restricts to a bijection between the sets of Langlands standard representations in both categories.*

Proof. Let $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}^{\mathcal{L}}, k^\omega, \natural_\omega)$ be the parabolic subalgebra of $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega)$ determined by $\mathcal{L}(F)$. The properties in Theorem 6.1 imply that $I_{\mathcal{P}(F)}^{\mathcal{G}(F)}(\tau \otimes \chi)$ is matched with

$$\mathrm{ind}_{\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}^{\mathcal{L}}, k^\omega, \natural_\omega)}^{\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega)} \big(\tau_{\mathbb{H}} \otimes \log(\chi)\big), \tag{6.1}$$

where $\tau_{\mathbb{H}}$ denotes the image of τ under Theorem 6.1 for $\mathcal{L}(F)$ and $\log(\chi)$ is fixed by $W_{\mathfrak{s},\omega}^{\mathcal{L}}$. By assumption χ is positive with respect to all roots of $Z^\circ(\mathcal{M})(F)$ in $\mathrm{Lie}(\mathcal{P}(F)/\mathcal{L}(F))$. The root system R_ω consists of scalar multiples of the roots of $Z^\circ(\mathcal{M})(F)$ in $\mathrm{Lie}(\mathcal{G}(F))$, but some of those roots may be left out depending on ω. As a consequence the condition for a character of $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}^{\mathcal{L}}, k^\omega, \natural_\omega)$ to be in positive position may be weaker than the corresponding condition for $X_{\mathrm{nr}}(\mathcal{L}(F))$. Thus $\log(\chi)$ is in positive position (but one cannot conclude that in opposite direction). This shows that (6.1) is a standard module in the traditional sense, and since $W_{\mathfrak{s},\omega}^{\mathcal{L}}$ fixes $\log(\chi)$ it is also a Langlands standard module as in (3.4).

Thus the equivalence of categories in Theorem 6.1 sends standard representations to Langlands standard modules. These two "standard" sets are canonically in bijection with the irreducible representations in the respective categories. Hence the equivalence of categories is bijective on standard representations. $\square$

When ω is simply generic [BuHe], one can improve on Theorem 6.1. Let $\mathcal{U}$ be the unipotent radical of a minimal parabolic F-subgroup $\mathcal{B}$ of $\mathcal{G}$. For a nondegenerate character ξ of $\mathcal{U}(F)$, the $\mathcal{G}(F)$-orbit of the pair $(\mathcal{U}(F), \xi)$ is called a Whittaker datum for $\mathcal{G}(F)$. By conjugating with a suitable element of $\mathcal{G}$, we may assume that $\mathcal{M}$ contains a Levi factor of $\mathcal{B}$. We recall that an $\mathcal{M}(F)$-representation π is called simply generic if $\mathrm{Hom}_{\mathcal{U}(F)\cap\mathcal{M}(F)}(\pi, \xi)$ has dimension one. Although this depends on the choice of the Whittaker datum for $\mathcal{G}(F)$, we suppress that in our terminology.

Theorem 6.3. [OpSo, Theorem E] and [Sol4, Theorem 10.9]
Assume that the supercuspidal unitary representation $\omega \in \mathrm{Irr}(\mathcal{M}(F))$ is simply generic. There exists an extended affine Hecke algebra $\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}}$ whose module category is canonically equivalent with $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$. This $\mathcal{H}_{\mathfrak{s}}$ is constructed from the following data:

- *the complex torus $X_{\mathrm{nr}}(\mathcal{M}(F))\omega \subset \mathrm{Irr}(\mathcal{M}(F))$,*
- *a root system $R_{\mathfrak{s}}$ such that $W(R_{\mathfrak{s}}) \rtimes \Gamma_{\mathfrak{s}} = W_{\mathfrak{s}}$,*
- *q-parameters in $\mathbb{R}_{\geq 1}$ as in* [Sol4, (3.7)] *and λ, λ^* as in* [Sol4, (9.5)].

The equivalence of categories $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}})$:

(i) is compatible with normalized parabolic induction and restriction,
(ii) respects temperedness,
(iii) sends essentially square-integrable $\mathcal{G}(F)$-representations to essentially discrete series $\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}}$-modules and conversely,
(iv) preserves genericity.
(v) is compatible with twisting by unramified characters of $\mathcal{G}(F)$,

From Theorem 6.3 one can obtain Theorem 6.1 (when ω is simply generic) by applying a variation on Theorem 5.1 to $\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}}$, that is essentially what happens in [Sol4, §6–7]. We need the version of Theorem 5.1 proven in [Sol2, §2.1], with $\mathbf{q}$ specialized to $q \in \mathbb{R}_{>1}$, $\mathbf{r}$ specialized to $r \in \mathbb{R}_{>0}$ and λ, λ^*, k^u real-valued (but not necessarily integral). From Theorem 6.3.iv and Theorem 5.1.iv we deduce:

Corollary 6.4. *If ω is simply generic, then $\natural_\omega = 1$ and the equivalence of categories in Theorem 6.1 preserves genericity.*

Like in Section 5, to apply our results from Sections 1–4 we need the graded Hecke algebras in Theorem 6.1 to be of geometric type, a condition on the parameter functions k^ω. Lusztig [Lus7] has conjectured that it is valid in general.

Condition 6.5. Let $\mathcal{G}(F), \mathcal{M}(F), \omega$ and $\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \mathbb{C}[\Gamma_\omega, \natural_\omega]$ be as in Theorem 6.1. There must exist data $(G_\omega, M_\omega, q\mathcal{E}_\omega, \mathcal{E}_\omega)$ as in Section 1, $r \in \mathbb{R}_{>0}$, and an isomorphism $\mathfrak{t} \rtimes W_{\mathfrak{s},\omega} \cong \mathrm{Lie}(T_\omega) \rtimes W_{q\mathcal{E}_\omega}$, which induce an algebra isomorphism

$$\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \cong \mathbb{H}(G_\omega^\circ, M_\omega^\circ, \mathcal{E}_\omega)/(\mathbf{r} - r).$$

In [Sol7], it is shown that Condition 6.5 holds when all simple factors $\tilde{\mathcal{G}}(F)$ of $\mathcal{G}(F)$ satisfy:

- $\tilde{\mathcal{G}}(F)$ is not of type $E_7, {}^2E_7, E_8$,
- $\tilde{\mathcal{G}}(F)$ is not isogenous to a symplectic or special orthogonal group of quaternionic type.

Theorem 6.6. *Assume that Condition 6.5 holds for a unitary supercuspidal representation ω of a Levi subgroup $\mathcal{M}(F) \subset \mathcal{G}(F)$. Let π_{st} be a standard $\mathcal{G}(F)$-representation with cuspidal support in $X_{\mathrm{nr}}^+(\mathcal{M}(F))$ and let π be an irreducible subquotient of π_{st}.*

(*a*) *If π is tempered or essentially square-integrable, then it is a subrepresentation of π_{st}.*

(*b*) *Suppose that ω is simply generic and π is generic. Then π is a subrepresentation of π_{st}.*

Proof. This can be shown exactly like in Theorem 5.4, using the results in Section 6 instead of those in Section 5. $\square$

For quasi-split groups, we will improve on Theorem 6.6 by verifying Condition 6.5. The next result was already known for principal series representations [Sol10, Lemma 6.4], and anticipated in [OpSo, Appendix].

Theorem 6.7. *Let $\mathcal{G}(F)$ be quasi-split and let ω be a generic unitary supercuspidal representation of $\mathcal{M}(F)$. Then the twisted graded Hecke algebra $\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \mathbb{C}[\Gamma_\omega, \natural_\omega]$ from Theorem 6.1 is isomorphic to an extended graded Hecke algebra with equal parameters.*

Proof. As observed in and before Corollary 6.4, $\natural_\omega = 1$ and $\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \Gamma_\omega$ can be obtained from the extended affine Hecke algebra $\mathcal{H}_{\mathfrak{s}} \rtimes \Gamma_{\mathfrak{s}}$ in Theorem 6.3 by applying the reduction procedure from [Sol2, §2.1] and more generally [AMS3, Theorems 2.5 and 2.11]. The effect on the parameters is given by (5.3). In view of [Sol4, (95)], this works out to

$$k^\omega_\alpha = \log(q_\alpha)/\log(q_F) \quad \text{or} \quad k^\omega_\alpha = \log(q_{\alpha*})/\log(q_F), \tag{6.2}$$

with q_α and $q_{\alpha *}$ as in Harish-Chandra's μ-function [Sol4, (3.7)]. Which of the options from (6.2) depends on ω. We must use q_α if α (as a function on $X_{\mathrm{nr}}(\mathcal{M}(F))\omega$) takes the same value at ω and at the base point of $X_{\mathrm{nr}}(\mathcal{M}(F))\omega$ chosen in [Sol4, §3], and we must use $q_{\alpha *}$ otherwise.

Thus k^ω agrees with the function $k^{\sigma'}$ (for $\sigma' \cong \omega$) from [OpSo, Proposition A.2], except that the domain of $k^{\sigma'}$ is obtained from the domain of k^ω by omitting the α's with $k^\omega_\alpha = 0$. Put

$$R_{\sigma'} = \{\alpha \in R_\omega : k^\omega_\alpha \neq 0\}$$

and let $\Gamma_{\sigma'}$ be the stabilizer in $W(R_\omega)$ of the set of positive roots in $R_{\sigma'}$. As in [Sol10, Lemma 6.3], one checks that

$$\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \Gamma_\omega \cong \mathbb{H}(\mathfrak{t}, W(R_{\sigma'}), k^{\sigma'}) \rtimes (\Gamma_{\sigma'} \rtimes \Gamma_\omega).$$

The setup for $\mathcal{H}$ and $\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega)$ in [Sol4, §3] entails that the roots in our setting are the coroots in [OpSo, Appendix A]. More precisely, from [OpSo, (A.4) and (A.5)] one sees that $\alpha^\vee$ over there corresponds to $h^\vee_\alpha$ from [Sol4], which is just $\alpha \in R_{\mathfrak{s}}$ in Theorem 6.3. Consider an irreducible component R of $R_{\sigma'}$. Let $\alpha \in R$ be long and $\beta \in R$ be short. Then [OpSo, Proposition A.2] says that $\kappa_R := k^{\sigma'}_\alpha / k^{\sigma'}_\beta$ equals either 1 or the square of the ratio of the lengths of α and β (which is 1, 2 or 3). If $\kappa_R \neq 1$, then we can divide all long roots in R by κ_R, and obtain a new root system R' with the same Weyl group. As observed in [Sol3, Example 5.4], this gives rise to an algebra isomorphism

$$\mathbb{H}(\mathfrak{t}, W(R), k^{\sigma'}) \to \mathbb{H}(\mathfrak{t}, W(R'), k'),$$

which is the identity on $\mathcal{O}(\mathfrak{t})$, such that k' takes the value $k^{\sigma'}_\beta \in \mathbb{R}_{>0}$ on all roots in R'. Rescaling all the elements of R' by a factor $2/k^{\sigma'}_\beta$ (still not touching $\mathfrak{t}$), we may further assume that $k' = 2$ on R'. We do this for all irreducible components R of $R^{\sigma'}$, and we obtain an algebra isomorphism

$$\mathbb{H}(\mathfrak{t}, W(R_{\sigma'}), k^{\sigma'}) \to \mathbb{H}(\mathfrak{t}, W(R'_{\sigma'}), k') \tag{6.3}$$

which is the identity on $\mathcal{O}(\mathfrak{t})$, and $k' = 2$ on $R'_{\sigma'}$. In (6.3) each root is scaled by a factor that depends only on $k^{\sigma'}$. Since $k^{\sigma'} = k^\omega|_{R_{\sigma'}}$ is $\Gamma_{\sigma'} \rtimes \Gamma_\omega$-invariant, the isomorphism (6.3) is $\Gamma_{\sigma'} \rtimes \Gamma_\omega$-equivariant. Hence it extends to an algebra isomorphism

$$\begin{aligned} \mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \Gamma_\omega &\cong \mathbb{H}(\mathfrak{t}, W(R_{\sigma'}), k^{\sigma'}) \rtimes (\Gamma_{\sigma'} \rtimes \Gamma_\omega) \\ &\to \mathbb{H}(\mathfrak{t}, W(R'_{\sigma'}), k') \rtimes (\Gamma_{\sigma'} \rtimes \Gamma_\omega). \end{aligned} \qquad \square$$

From Theorem 6.7 we deduce that Condition 6.5 is automatic.

Lemma 6.8. *In the setting of Theorem 6.7, Condition 6.5 holds with* $(M, \mathcal{C}^M_v, q\mathcal{E}) = (T, \{0\}, \mathrm{triv})$.

Proof. Let G° be a connected complex reductive group with a maximal torus T, so that $R(G^\circ, T) \cong R_{\sigma'}$ and

$$\mathfrak{t} \rtimes W(R_{\sigma'}) \cong \mathrm{Lie}(T) \rtimes W(G^\circ, T).$$

By passing to a cover, we may assume that the derived group of G° is simply connected. The action of $\Gamma_{\sigma'} \rtimes \Gamma_\omega$ on $\mathfrak{t} \rtimes W(R_{\sigma'})$ can be transferred to $\mathrm{Lie}(T) \rtimes W(G^\circ, T)$, and then (using the simply connectedness of G°_{der}) lifted to an action on G° that preserves a pinning. In this way we build the complex reductive group $G = G^\circ \rtimes (\Gamma_{\sigma'} \rtimes \Gamma_\omega)$.

Since $W_{\mathfrak{s},\omega}$ acts faithfully on $\mathfrak{t}$, $Z_G(T) = Z_{G^\circ}(T) = T$. In particular T is a quasi-Levi subgroup of G, and it admits a quasi-cuspidal support $(T, \{0\}, \text{triv})$. All the parameters k_α for $(T, \{0\}, \text{triv})$ are equal to 2 [Lus2, §0.3]. Moreover the 2-cocycle $\natural_{\text{triv}}$ is trivial because $\Gamma_{\sigma'} \rtimes \Gamma_\omega$ acts naturally on all the relevant perverse sheaves constructed from $(T, \{0\}, \text{triv})$. We conclude that

$$\mathbb{H}(\mathfrak{t}, W(R_\omega), k^\omega) \rtimes \Gamma_\omega \cong \mathbb{H}(\mathfrak{t}, W(R_{\sigma'}), k^{\sigma'}) \rtimes (\Gamma_{\sigma'} \rtimes \Gamma_\omega) \cong \mathbb{H}(G, T, \text{triv})/(\mathbf{r} - 1). \qquad \square$$

We are ready to prove the generalized injectivity conjecture from [CaSh].

Theorem 6.9. *Let π_{st} be a standard representation of a quasi-split reductive p-adic group $\mathcal{G}(F)$. Let π be a generic irreducible subquotient of π_{st}. Then π is a subrepresentation of π_{st}.*

Proof. Let $(\mathcal{M}(F), \omega')$ be the cuspidal support of π (and hence of π_{st}). The normalized parabolic induction of a nongeneric irreducible representation is not generic [BuHe], so ω' must be generic. More precisely, when we choose a representative $(\mathcal{U}(F), \xi)$ for the Whittaker datum so that $\mathcal{M} \cap \mathcal{U}$ is a maximal unipotent subgroup of $\mathcal{M}$, then ω' is $(\mathcal{U}(F) \cap \mathcal{M}(F), \xi)$-generic. Let $|\chi|$ be the absolute value of the central character of ω'. Then $|\chi| \in X_{\text{nr}}^+(\mathcal{M}(F))$ and $\omega = \omega' \otimes |\chi|^{-1}$ is unitary. Twisting by unramified characters does not perturb the genericity of ω', so we are in the setting of Theorem 6.6. Condition 6.5 holds by Lemma 6.8, so Theorem 6.6.b yields the desired statement. $\square$

7. Relations with the local Langlands correspondence

To prepare for the upcoming arguments, we compare properties of (enhanced) L-parameters for reductive p-adic groups and for twisted graded Hecke algebras.

Let $\mathbf{W}_F$ be the Weil group of F and let ${}^L\mathcal{G} = \mathcal{G}^\vee \rtimes \mathbf{W}_F$ be the Langlands dual group of $\mathcal{G}(F)$. Let $\mathcal{M}(F)$ be a standard Levi subgroup of $\mathcal{G}(F)$ and let $\mathcal{M}^\vee \subset \mathcal{G}^\vee$ be the associated standard Levi subgroup on the dual side. Let $(\phi_b, q\epsilon)$ be a bounded enhanced cuspidal L-parameter for $\mathcal{M}(F)$, in the form

$$\phi_b : \mathbf{W}_F \times SL_2(\mathbb{C}) \to {}^L\mathcal{M}$$

and with $q\epsilon$ an irreducible representation of the appropriate component group S_ϕ. Consider the group

$$G_{\phi_b} = Z^1_{\mathcal{G}^\vee_{\text{sc}}}(\phi_b|_{\mathbf{W}_F})$$

from [AMS1, (75)]. We mainly need its identity component

$$G^\circ_{\phi_b} = Z_{\mathcal{G}^\vee_{\text{sc}}}(\phi_b(\mathbf{W}_F))^\circ.$$

Let M_{ϕ_b} be the quasi-Levi subgroup of G_{ϕ_b} determined by $(\mathcal{M}^\vee, \phi_b, q\epsilon)$ and let $q\mathcal{E}$ be the cuspidal local system (determined by $q\epsilon$) on the nilpotent orbit in $\text{Lie}(M_{\phi_b})$ from $\phi_b|_{SL_2(\mathbb{C})}$.

Recall from [Hai, §3.3.1] that the group of unramified characters $X_{\text{nr}}(G)$ is naturally isomorphic to the complex torus

$$X_{\text{nr}}({}^LG) := (Z(\mathcal{G}^\vee)^{\mathbf{I}_F})^\circ_{\text{Frob}_F},$$

where $\mathbf{I}_F$ denotes the inertia subgroup of $\mathbf{W}_F$. From these data we build

$$\mathbb{H}(\phi_b, q\epsilon) = \mathbb{H}\big(G_{\phi_b} \times X_{\text{nr}}(\mathcal{G}^\vee), M_{\phi_b} \times X_{\text{nr}}({}^L\mathcal{G}), q\mathcal{E}\big).$$

It is almost the same as the algebra

$$\mathbb{H}(\phi_b, v, q\epsilon, \vec{\mathbf{r}}) = \mathbb{H}\big(G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), M_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G}), q\mathcal{E}, \vec{\mathbf{r}}\big)$$

from [AMS3, (3.9)]. The only difference is that we set all the indeterminates $\mathbf{r}_i$ from [AMS3] equal to $\mathbf{r}$, by dividing out the ideal generated by the $\mathbf{r}_i - \mathbf{r}_j$ with $i \neq j$.

Let $X^+_{\mathrm{nr}}({}^L\mathcal{M}) \subset X_{\mathrm{nr}}({}^L\mathcal{M})$ be the subgroup corresponding to $X^+_{\mathrm{nr}}(\mathcal{M}(F)) \subset X_{\mathrm{nr}}(\mathcal{M}(F))$. As

$$Z(M)^\circ \times X_{\mathrm{nr}}(\mathcal{G}^\vee) \to X_{\mathrm{nr}}(\mathcal{M}^\vee)$$

is a finite covering [AMS3, Lemma 3.7], $X^+_{\mathrm{nr}}({}^L\mathcal{M})$ can be identified with a subgroup of $Z(M)^\circ \times X^+_{\mathrm{nr}}({}^L\mathcal{G})$.

Recall that an L-parameter (y, σ, r) for $\mathbb{H}(\phi_b, \rho)$ contains (up to conjugation) precisely the same information as (y, σ_0, r), for σ_0 as in (3.2). As worked out in [AMS3, §3.1], the exponential map for $\mathcal{G}^\vee$ provides a natural bijection from

- the set of enhanced L-parameters (y, σ, r, ρ) or (y, σ_0, r, ρ) for $\mathbb{H}(\phi_b, q\epsilon)$, with $r = -\log(q_F)/2$ and $\sigma_0 \in X^+_{\mathrm{nr}}({}^L\mathcal{M})$ to
- the set of enhanced L-parameters for $\mathcal{G}(F)$ with cuspidal support in $(\mathcal{M}(F), X^+_{\mathrm{nr}}({}^L\mathcal{M})\phi_b, q\epsilon)$.

Here the group $X_{\mathrm{nr}}(\mathcal{M}(F)) \cong X_{\mathrm{nr}}({}^L\mathcal{M})$ acts naturally on L-parameters for $\mathcal{M}(F)$, by adjusting the value $\phi(\mathrm{Frob}_F)$ while keeping $\phi|_{\mathbf{I}_F \times SL_2(\mathbb{C})}$ fixed. We refer to this as twisting Langlands parameters by unramified characters.

We note that in [AMS3, §3.1] the role of σ_0 is played by $\log(\phi(\mathrm{Frob}_F)\phi_b(\mathrm{Frob}_F)^{-1})$. Explicitly, $\exp(y, \sigma, -\log(q_F)/2, \rho) = (\phi, \rho)$ where

$$\phi|_{\mathbf{I}_F} = \phi_b|_{\mathbf{I}_F},\ \phi(\mathrm{Frob}_F) = \exp(\sigma_0)\phi_b(\mathrm{Frob}_F),\ \phi\big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) = \exp(y).$$

If we present ϕ as a Weil–Deligne parameter ϕ', then

$$\phi'(\mathrm{Frob}_F) = \exp(\sigma)\phi_b(\mathrm{Frob}_F) \text{ and the nilpotent operator for } \phi' \text{ is } y.$$

Recall the definition of an open Langlands parameter from (1). We call a Langlands parameter $\psi : \mathbf{W}_F \times SL_2(\mathbb{C}) \to {}^L\mathcal{G}$ for essentially bounded if $\psi(\mathrm{Frob}_F) = (a, \mathrm{Frob}_F)$ with $aZ(\mathcal{G}^\vee)$ in a bounded subgroup of $\mathcal{G}^\vee / Z(\mathcal{G}^\vee)$. By the Langlands classification for L-parameters [SiZi], every discrete L-parameter ψ is essentially bounded. More precisely, ψ can be expressed (uniquely up to conjugation) as a bounded discrete L-parameter twisted by an element of $X^+_{\mathrm{nr}}({}^L\mathcal{G})$.

Lemma 7.1. *Let (y, σ, r, ρ) be an enhanced L-parameter for $\mathbb{H}(\phi, \rho)$, with $r = -\log(q_F)/2$ and $\sigma_0 \in X^+_{\mathrm{nr}}({}^L\mathcal{M})$.*

(a) (y, σ, r, ρ) is bounded if and only if $\exp(y, \sigma, r, \rho)$ is bounded. The same holds with essentially bounded.

(b) (y, σ, r, ρ) is open if and only if $\exp(y, \sigma, r, \rho)$ is open.

Proof. (a) Recall from (3.2) that $\sigma = \sigma_0 + \mathrm{d}\gamma_y\big(\left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)\big)$. The first condition is equivalent with $\sigma_0 = 0$ because $\sigma_0 \in X^+_{\mathrm{nr}}({}^L M)$, and the second condition is equivalent with $\sigma_0 = 0$ because ϕ_b is bounded.

Both for $\mathbb{H}(\phi_b, \rho)$ and for $\mathcal{G}(F)$, essential boundedness is equivalent with $\sigma_0 \in Z(\mathfrak{g}^\vee) \cap \log(X^+_{\mathrm{nr}}({}^L\mathcal{M}))$.

(b) We analyse the vector space in which y lives:

$$\begin{aligned}\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N} &= \{X \in \mathfrak{g}^\vee_{\phi_b} : [\sigma, X] = 2rX\}\\ &= \big\{X \in \mathrm{Lie}\big(Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b(\mathbf{W}_F))\big) : [\sigma, X] = -\log(q_F)X\big\}\\ &= \{X \in \mathfrak{g}^\vee : [\sigma, X] = \log(q_F^{-1})X, \mathrm{Ad}(\phi_b(w))X = X \ \forall w \in \mathbf{W}_F\}.\end{aligned}$$

As ϕ_b is bounded, $\mathrm{Ad}(\phi_b(w))$ only has eigenvalues of absolute value 1. Hence

$$\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N} = \{X \in \mathfrak{g}^\vee : \mathrm{Ad}(\exp(\sigma)\phi_b(\mathrm{Frob}_F))X = q_F^{-1}X, \mathrm{Ad}(\phi_b(w))X = X \ \forall w \in \mathbf{I}_F\},$$

which equals $\mathfrak{g}^\vee_{\phi'}$ for (ϕ', ρ) the Weil–Deligne parameter associated to $\exp(y, \sigma, r, \rho)$. This shows that the nilpotent parts of the L-parameters on both sides can be chosen form the same vector space $\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N} = \mathfrak{g}^\vee_{\phi'}$.

The group whose orbits on $\mathfrak{g}^\vee_{\phi'}$ determine the "open" property is

$$Z_{\mathcal{G}^\vee}(\phi'(\mathbf{W}_F)) = Z_{\mathcal{G}^\vee}(\phi_b(\mathbf{W}_F), \exp(\sigma)) = Z_{\mathcal{G}^\vee_{\mathrm{der}}}(\phi_b(\mathbf{W}_F), \exp(\sigma))Z(\mathcal{G}^\vee)^{\mathbf{W}_F}. \tag{7.1}$$

On the other hand, for $\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N}$ openness comes from orbits for the group $Z_{G_{\phi_b} \times X_{\mathrm{nr}}({}^L\mathcal{G})}(\sigma)$, which has identity component

$$Z_{G_{\phi_b}}(\sigma)^\circ \times X_{\mathrm{nr}}({}^L\mathcal{G}) = Z_{\mathcal{G}^\vee_{\mathrm{sc}}}(\phi_b(\mathbf{W}_F), \exp(\sigma))^\circ \times X_{\mathrm{nr}}({}^L\mathcal{G}). \tag{7.2}$$

On both sides, the unique open orbit in the vector space is already a single orbit for the identity component of the acting group, by [KaLu2, §2.6]. Since $Z(\mathcal{G}^\vee)$ acts trivially, the identity components of the groups (7.1) and (7.2) have exactly the same orbits on $\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N} = \mathfrak{g}^\vee_{\phi'}$. Thus (y, σ, r, ρ) is open if and only if y lies in the unique open orbit of (7.2) in $\mathfrak{g}^{\vee,\sigma,r}_{\phi_b,N}$, which is the same as the open orbit of (7.1) in $\mathfrak{g}^\vee_{\phi'}$. That condition is equivalent to openness of $\exp(y, \sigma, r, \rho)$. □

The following result has been proven in another way in [CDFZ, Proposition 4.1 and §6], although in less generality.

Proposition 7.2. *Every essentially bounded (e.g. bounded or discrete) L-parameter for $\mathcal{G}(F)$ is open.*

Proof. Every enhanced L-parameter for $\mathcal{G}(F)$ has a cuspidal support [AMS1]. That can be written as $(\mathcal{M}(F), \phi, q\epsilon)$ with $(\phi, q\epsilon)$ a cuspidal enhanced L-parameter for a Levi subgroup $\mathcal{M}(F)$ of $\mathcal{G}(F)$. Then ϕ is discrete, so by [SiZi] of the form $z\phi_b$ with ϕ_b bounded discrete and $z \in X^+_{\mathrm{nr}}({}^L\mathcal{M})$. Now Lemma 7.1 allows us to transfer the issue to L-parameters for $\mathbb{H}(\phi_b, q\epsilon)$. In that setting, we proved the desired statement in Lemma 3.4.

Alternatively, one can translate the proofs of Lemmas 3.3 and 3.4 directly to the current setting. □

To get Langlands parameters into play from representations of a reductive p-adic group $\mathcal{G}(F)$, we need to assume some reasonable form of the local Langlands correspondence involving Hecke algebras. As in Section 6 we consider a Bernstein block $\mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}$ determined by a unitary supercuspidal representation ω of $\mathcal{M}(F)$. We write

$$\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}} = \mathrm{Irr}(\mathcal{G}(F)) \cap \mathrm{Rep}(\mathcal{G}(F))^{\mathfrak{s}}.$$

Condition 7.3. A local Langlands correspondence is known for $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ and $\mathrm{Irr}(\mathcal{M}(F))^{\mathfrak{s}}$. Let (ϕ, ρ) be the enhanced L-parameter of ω.

There is an isomorphism between the graded Hecke algebras of geometric type

- $\mathbb{H}(\mathfrak{t}, W_{\mathfrak{s},\omega}, k^\omega, \natural_\omega)$ from Theorem 6.1,
- $\mathbb{H}(\phi,\rho)/(\mathbf{r} - \log(q_F)/2)$ from [AMS3, §3.1]

induced by isomorphisms between $(\mathfrak{t} \rtimes W_{\mathfrak{s},\omega}, R_\omega)$ and the analogous data for $\mathbb{H}(\phi,\rho,\mathbf{r})$.

The same holds if we twist ω by a unitary unramified character of $\mathcal{M}(F)$.

Furthermore, the local Langlands correspondence for $\mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$ can be constructed via Theorem 6.1, the above isomorphisms (for all such twists of ω) and the parametrization of $\mathrm{Irr}\big(\mathbb{H}(\phi,\rho,\mathbf{r})/(\mathbf{r} - \log(q_F)/2)\big)$ from [AMS3, Theorem 3.8].

We point out that Condition 7.3 is stronger than Condition 6.5, and we refer to [AMS1, AMS2, AMS3] for more background. A list of cases in which Condition 7.3 has been verified can be found in [Sol8, Theorem 5.4] and in the introduction before Theorem E. We expect that Condition 7.3 is always fulfilled.

Theorem 7.4. *Assume Condition 7.3 and let* $\pi \in \mathrm{Irr}(\mathcal{G}(F))^{\mathfrak{s}}$.

(*a*) *If* π *is tempered or essentially square-integrable, then its L-parameter is open.*

(*b*) *Suppose that* $\mathfrak{s} = [\mathcal{M}(F), \omega]$ *with* ω *simply generic, and that* π *is generic. Then the L-parameter of* π *is open.*

(*c*) *Suppose that the L-parameter of* π *is open and that* π *is a subquotient of a standard* $\mathcal{G}(F)$*-representation* π_{st}*. Then* π *is a subrepresentation of* π_{st}*.*

Proof. (a) By the known properties of the LLC imposed by Condition 7.3, or by [AMS2, AMS3], the L-parameter of π is bounded or discrete. As mentioned in [CFZ, §0.6] and shown in [CDFZ], such an L-parameter is open. Alternatively, that can also be shown with Lemma 3.4 and the translation from L-parameters for $\mathbb{H}(\phi,\rho,\mathbf{r})$ to L-parameters for $\mathcal{G}(F)$ in [AMS3, §3].

(b) By Theorem 6.1 and Corollary 6.4, the associated $\mathbb{H}(\phi,\rho,\mathbf{r})$-module $\pi_{\mathbb{H}}$ is generic. Then Theorem 4.5 says that the L-parameter of $\pi_{\mathbb{H}}$ is open, and the constructions in [AMS3, §3] entail that the L-parameter of $\pi \in \mathrm{Irr}(\mathcal{G}(F))$ is open.

(c) Condition 7.3 and the comparison/reduction results in Sections 5 and 6 transfer this to a statement about $\mathbb{H}(\phi,\rho,\mathbf{r})$-modules. That statement is proven in Theorem 3.2. $\square$

We note that Theorem 7.4.b proves Conjecture B.a for all the cases listed in [Sol8, Theorem 5.4] or just before Theorem E. Theorem 7.4 also verifies Conjecture C in all those instances.

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