PROBLEM 2
EMBEDDING OF $\mathbb{R}$ IN $\mathbb{C}$.

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It is known that we have the following relations

$$\mathbb{R} \subset \mathbb{C},$$
$$\mathbb{R} \neq \mathbb{C},$$
$$\mathbb{R}[i] = \mathbb{C}.$$

Problem 1. *Is $\mathbb{R}$ the unique subfield of $\mathbb{C}$ having the above properties?*

Answer. No.

This answer was found during the Hanoi Conference in collaboration with Furter and M. Koras.

An explicit example.

Consider the splitting field $K \subset \mathbb{C}$ of the polynomial $x^3 - 2$ defined over $\mathbb{Q}$ (it is $K = \mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$). Since $x^3 - 2$ is irreducible over $\mathbb{Q}$ there exists an automorphism A of K over $\mathbb{Q}$ which sends the root $\sqrt[3]{2}$ of $x^3 - 2$ into another $\sqrt[3]{2}e^{2\pi i/3}$ i.e. $A(\sqrt[3]{2}) = \sqrt[3]{2}e^{2\pi i/3}$. Next we extend the automorphism A to an automorphism $\mathbb{A}$ of the whole field $\mathbb{C}$. We define

$$\tilde{\mathbb{R}} := \mathbb{A}(\mathbb{R}).$$

Then

$$\tilde{\mathbb{R}} \neq \mathbb{R},$$
$$\tilde{\mathbb{R}} \subset \mathbb{C},$$
$$\tilde{\mathbb{R}} \neq \mathbb{C},$$
$$\tilde{\mathbb{R}}[i] = \mathbb{C}.$$

Remark 1. *Notice the following Artin Theorem, which explains the structure of finite extensions $\mathbb{C} : K$*

Theorem 1. *If $\mathbb{K}$ is algebraically closed field and $K \subset \mathbb{K}$ is a finite extension (i.e. $[\mathbb{K} : K] < \infty$) then* $\operatorname{char} \mathbb{K} = 0$ *and* $\mathbb{K} = K[i]$ *(i is a root of $x^2 + 1 = 0$ in $\mathbb{K}$).*

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