DO THERE EXIST ODD POLYNOMIAL AUTOMORPHISMS OVER $\mathbb{F}_4, F_8, \ldots$?

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Write $\mathbb{F}_q$ for the field with $q = p^m$ elements. Given a polynomial automorphism F of $\mathbb{F}_q^{[n]}$, we get a bijection $B_F : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$. Note that, contrary to infinite fields, an endomorphism of $\mathbb{F}_q^{[n]}$ can be non-invertible but induce a bijective map $B_F : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$ (like the map X^3 on $\mathbb{F}^{[1]}$).
What was done in the paper [1] is compute which bijections of $\mathbb{F}_q^n$ can be made by tame automorphisms of $\mathbb{F}_q^{[n]}$. It turned out that

- if q is odd, or if $q = 2$, one can make any bijection.
- If $q = 2^m$ where $m \geq 2$, then one can only make half of the bijection: any tame automorphism of $\mathbb{F}_q^{[n]}$, seen as a bijection $\mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$ will induce an even permutation of the symmetric group with q^n elements.

The question is thus:

Conjecture: If $q = 2^m$ where $m \geq 2$, then any polynomial automorphism of $\mathbb{F}_q^{[n]}$ induces an even bijection $\mathbb{F}_1^n \longrightarrow \mathbb{F}_q^n$.

Note that answering this question in the negative would imply that one has found a non-tame automorphism, with trivial proof that it is non-tame.

References

[1] [Mau] S. Maubach,, *Polynomial automorphisms over finite fields.* Serdica Math. J. 27 (2001), no. 4, 343–350.