The integer case of the plane Jacobian conjecture as a problem on integer points in plane curve

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Given $F = (P, Q) \in \mathbb{Z}[x, y]^2$, a polynomial map with integer coefficients. The mysterious Jacobian conjecture (JC), posed first by Keller in 1939 asserts that such a map F is invertible and has a polynomial inverse with integer coefficients if Jacobian $JF := P_x Q_y - P_y Q_x) \equiv 1$. It was observed in [Ann. Polon. Math. 88, Vol.1(2006), 53-58.] that *If $JF \equiv 1$ and if the complex plane curve $P = 0$ has infinitely many integer points, then such map F has a polynomial inverse with integer coefficients.* This reduces the integer case of (JC) to a question on the number of integer points in a plane curve

Question 1 (Integer case of (JC)): *Whether the Jacobian condition $JF \equiv 1$ ensures that the curve $P = 0$ has infinite many integer points ?*

On the other words, the integer case of (JC) may be regards as a problem of the Algebra-Arithmetic Geometry. In view of Siegel's Theorem [Abh. Deutsch. Akad. Wiss. Berlin Kl. Phys.-Mat. 1929, no. 1], such a curve $P = 0$ with infinite many integer points must be a rational curve.

Question 2 (Rational-Curve case of (JC)): *Whether a polynomial map $f = (p, q) \in \mathbb{C}[x, y]^2$ with $Jf \equiv c \in \mathbb{C}^*$ is invertible if the curve $p = 0$ is a rational curve ?.*