

A problem on polynomial maps over finite fields

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Recently I noticed that a conjecture of mine got quite some interest of people working outside of my field, especially people who work in discrete mathematics, number theory, code theory, combinatorics, even cryptographers. Due to that interest, I thought it might be a nice idea to write some background etc. on the problem, to make it as easy as possible to understand what is going on, and to explain its importance. Well, first let me state the question without explaining the notations.

Problem: Do there exist odd polynomial automorphisms over the finite fields $\mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_{16}, \mathbb{F}_{32}, \dots$?

Below I will explain all terms used in this problem, in rather elaborate terms. You might want to skip some parts.

1 Polynomial automorphisms

For simplicity, I will explain everything in dimension 2. Let \mathbb{K} be a field. If $F_1, F_2 \in \mathbb{K}[X, Y]$ then we write $F := (F_1(X, Y), F_2(X, Y))$ is a polynomial endomorphism. One can see such a map in several ways:

1. As an element of $(\mathbb{K}[X, Y])^2$, this is how I explained it above.
2. As a map $\mathbb{K}^2 \longrightarrow \mathbb{K}^2$, induced by polynomials. The map sending $(a, b) \in \mathbb{K}^2$ to $(F_1(a, b), F_2(a, b)) \in \mathbb{K}^2$.
3. As a map $\mathbb{K}[X, Y] \longrightarrow \mathbb{K}[X, Y]$, sending $p(X, Y)$ to $p(F_1(X, Y), F_2(X, Y))$.

Let's stick to point 1 for the moment. We say that $I := (X, Y)$ is the identity map. If F, G are polynomial maps, we can make their composition $F \circ G := (F_1(G_1, G_2), F_2(G_1, G_2))$. If $F \circ G = I$, then we say that G is the inverse of F .

(Automatically, $G \circ F = I$.) We say that F is a *polynomial automorphism*.

Note that polynomial automorphisms include all invertible linear maps (or “non-singular matrices” if you like that more). Just to give you one example of a “non-trivial” polynomial automorphism: $(X + Y^2, Y)$ has as inverse $(X - Y^2, Y)$.

Remark 1.1. Obviously, if F is a polynomial automorphism, then F induces a bijection $\mathbb{K}^2 \rightarrow \mathbb{K}^2$. However, the converse is only true if \mathbb{K} is infinite!! Since we are talking exactly about the finite field case, we do NOT have this situation here! For example, $(X^p, Y^p) \in (\mathbb{F}_p[X, Y])^2$ is not a polynomial automorphism, but it does induce a bijection $\mathbb{F}_p^2 \rightarrow \mathbb{F}_p^2$.

2 What does “odd” mean?

From now on, $\mathbb{K} = \mathbb{F}_q$, a field with q elements. If you have a polynomial automorphism F , then it induces a map $\mathcal{E}(F) : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q^2$. It’s not just a map, it is a bijection of \mathbb{F}_q^2 . Since these are q^2 elements, one can see $\mathcal{E}(F)$ as an element of the symmetric group with q^2 elements. In general, in dimension n :

$$\mathcal{E}(F) \in \text{Sym}(q^n).$$

Obviously, you can get any endomorphism of \mathbb{F}_q^2 induced by an element of $(\mathbb{F}_q[X, Y])^2$ (even in dimension 1!) and hence any element of $\text{Sym}(q^n)$ can be given by an element of $(\mathbb{F}_q[X, Y])^2$. But, what about which bijections are induced by polynomial automorphisms? This question is answered in [1], and the answer is quite surprising:

Theorem 2.1. *Let $n \geq 2$.*

- 1. If q is odd or $q = 2$ then any element of $\text{Sym}(q^n)$ can be obtained as the image of a polynomial automorphism.*
- 2. If $q = 2^m$ where $m \geq 2$ then any element of the subgroup $\text{Alt}(q^n)$ can be obtained as the image of a polynomial automorphism.*

The group $\text{Alt}(q^n)$ is the alternating subgroup of elements which are even. Obviously, the above theorem is screaming the question: do there exist elements of $\text{Sym}(q^n) \setminus \text{Alt}(q^n)$ which can be obtained as the image of a polynomial automorphism over $\mathbb{F}_4, \mathbb{F}_8$ etc.? So let us repeat the problem:

Problem: Do there exist odd polynomial automorphisms over the finite fields $\mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_{16}, \mathbb{F}_{32}, \dots$?

3 Why is it important?

If one can find an odd polynomial automorphism, then it has many consequences!! Quite a few conjectures are killed, then, immediately. But before I can explain why, I have to throw some theory at you (which is easy enough to read before going to bed, or while having a headache, or both).

Let us denote the polynomial automorphism group in dimension n over \mathbb{K} by $\text{GA}_n(\mathbb{K})$. One of the big problems in my field (affine algebraic geometry) is trying to understand the group $\text{GA}_n(\mathbb{K})$. There are quite a few problems that we can only address when we have understood the automorphism sufficiently enough.

At the moment, we don't even know if we have a generating set for the polynomial automorphism group in dimension 3 and higher!!

We have a reasonable description in dimension 2, which then has as a side effect that many results are only known in dimension 2 and not beyond!!

First, let us make some polynomial automorphisms. As we already noticed, all invertible linear maps are polynomial automorphisms. So:

$$\text{GL}_n(\mathbb{K}) \subseteq \text{GA}_n(\mathbb{K}).$$

Then we have a class of very simple polynomial automorphisms. The *elementary* ones. They look like this:

$$(X_1 + f(X_2, \dots, X_n), X_2, \dots, X_n)$$

where $f \in \mathbb{K}[X_2, \dots, X_n]$ arbitrarily. Its inverse is the elementary map where you replace f by $-f$. If you compose a few of them, you can make any *triangular* map, I will show it to you in dimension three, and then you will get the idea. Let $f \in \mathbb{K}[Y, Z], g \in \mathbb{K}[Z], h \in \mathbb{K}, a, b, c \in \mathbb{K}$. Then

$$(X, Y, cZ+h) \circ (X, bY+g(Z), Z) \circ (aX+f(Y, Z), Y, Z) = (aX+f(Y, Z), bY+g(Z), cZ+h)$$

and thus any of such maps is invertible. The set of these maps has many names: two are Jonqui re group and Triangular group. It is denoted by

$$\text{BA}_n(\mathbb{K}) \subseteq \text{GA}_n(\mathbb{K})$$

since it resembles the Borel subgroup of the linear maps (the triangular linear maps). (In fact, $\text{BA}_n(\mathbb{K}) \cap \text{GL}_n(\mathbb{K}) = B_n(\mathbb{K})$, the Borel group.) Now you can make the group generated by $\text{BA}_n(\mathbb{K})$ and $\text{GL}_n(\mathbb{K})$, which is called the *tame automorphism group*:

$$\text{TA}_n(\mathbb{K}) := \langle \text{BA}_n(\mathbb{K}), \text{GL}_n(\mathbb{K}) \rangle .$$

We call automorphisms tame if they are elements of $\text{TA}_n(\mathbb{K})$ and wild otherwise.

In dimension 2 there is the famous Jung-van der Kulk-theorem, which states among others (for any field \mathbb{K}):

$$\mathrm{TA}_2(\mathbb{K}) = \mathrm{GA}_2(\mathbb{K}).$$

This is the reason why we can do so much more in dimension 2: we have a set of generators of the automorphism group, and even quite a nice description of it.

Now in dimension three or higher there were some candidate examples of which no one knew if they are tame or not. The most famous is *Nagata's map*:

$$(X - 2Y\Delta - Z\Delta^2, Y + Z\Delta, Z)$$

where $\Delta = XZ + Y^2$. (The inverse is $(X + 2Y\Delta - Z\Delta^2, Y - Z\Delta, Z)$, by the way.)

The biggest breakthrough in the last twenty years in our field was the proof in 2004 of Shestakov-Umirbaev [2, 3], which showed that Nagata's map (and others) were not tame. That was and is an amazing result, which surprised our community. For these papers, they got the AMS 2007 Moore Prize (a "best paper award" given every three years [4]). It's quite a nontrivial paper, and has lots of technicalities. For now, it only works in characteristic zero.

Now, in [1] the above theorem 2.1 was actually the following:

Theorem 3.1. *Let $n \geq 2$.*

1. *If q is odd or $q = 2$ then $\mathcal{E}(\mathrm{TA}_n(\mathbb{F}_q)) = \mathrm{Sym}(q^n)$.*
2. *If $q = 2^m$ where $m \geq 2$ then $\mathcal{E}(\mathrm{TA}_n(\mathbb{F}_q)) = \mathrm{Alt}(q^n)$.*

In other words, if you find an "odd" polynomial automorphism over for example \mathbb{F}_4 , then it cannot be tame! Checking that a concrete polynomial automorphism is odd, can be easily done by computer or by hand. So, that would be a very, very easy proof of an automorphism being non-tame, giving you a great result in a one-page paper.

As you can guess, all examples I and some others have tried so far are even. But, I do not know what to believe! There are nowadays several ways to make automorphisms, and it is very unclear if not one of these could be odd.

On the other hand, one could try to prove that some classes of polynomial automorphism over $\mathbb{F}_4, \mathbb{F}_8, \dots$ are even. These are nice, concrete questions which you can work with.

If you are ambitious, you could try to do this for *all* polynomial automorphisms, solving the problem. (If you don't believe there are odd polynomial automorphisms.) But it is hard to use the input that you have a polynomial automorphism in this particular problem! My guess is that one should consider a larger class of polynomial endomorphisms, and prove something about them. This approach could lead to something, I think. If there are no odd polynomial automorphisms, that is...

4 More consequences of an odd automorphism

You can stop reading here, this is only a little extra stuff for the interested.

There are several conjectures on generating sets of automorphism groups. Some of them would be killed by an odd automorphism.

Conjecture 4.1. $\text{GA}_3(\mathbb{K})$ is generated by all automorphisms fixing one variable.

If $\mathbb{K} = \mathbb{F}_4, \mathbb{F}_8, \dots$ then you can show that any automorphism fixing one variable is even. For, let $(F_1(X, Y, Z), F_2(X, Y, Z), Z)$ be such an automorphism, then for any value $Z = a \in \mathbb{K}$, $(F_1(X, Y, a), F_2(X, Y, a))$ must be an automorphism in dimension 2. Here we have the Jung-van der Kulk theorem, so this map is even. So, on each hypersurface $\mathbb{K}^2 \times \{a\}$ the map is even. So the total map is even too.

Conjecture 4.2. $\text{GA}_n(\mathbb{K})$ is generated by all linearizable automorphisms.

An automorphism F is linearizable if there exists another automorphism φ such that $\varphi^{-1}F\varphi$ is linear. Since all linear maps over $\mathbb{F}_4, \mathbb{F}_8, \dots$ are even, this conjecture would imply that each map is even, as being even or odd is invariant under conjugation.

References

- [1] S. Maubach, *Polynomial automorphisms over finite fields*, Serdica Math. J. 27 (2001) no.4. 343-350.
- [2] I. Shestakov and U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. 17 (2004), no. 1, 197–227
- [3] I. Shestakov and U. Umirbaev, *Poisson brackets and two-generated subalgebras of rings of polynomials*, J. Amer. Math. Soc. 17 (2004), no. 1, 181–196
- [4] <http://www.ams.org/ams/press/moore2007.html>