Triangular monomial derivations on $k[X_1, X_2, X_3, X_4]$ have kernel generated by at most four elements

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Abstract

It is shown that any triangular derivation on $k[X_1, X_2, X_3, X_4]$ sending X_i to a monomial has kernel generated by at most four elements, hence is finitely generated. An explicit formula for the generators is given.

1 Introduction

Derivations and the study of their kernels play a crucial role in many problems. (For an excellent account the reader is referred to [Nowicki, 1994]). An important question is if a certain derivation has finitely generated kernel. This question is closely related to Hilbert's 14th problem, stated in 1900:

> Let k be a field and L a subfield of the field of rational functions $k(X_1, \ldots, X_n)$ containing k. Is $L \cap k[X_1, \ldots, X_n]$ a finitely generated k-algebra?

If one has a derivation whose kernel is not finitely generated then one has a counterexample to Hilbert 14 by taking $L = Q(ker(D))$, the quotient field of $ker(D)$.

The first counterexample to Hilbert 14 was found in 1958 by Nagata in dimension 32 [Nagata, 1958]. A counterexample to Hilbert 14 in dimension 7 was given by Roberts in 1990 [Roberts, 1990]. Deveny and Finston showed that this counterexample could be derived from the derivation $D := x^3\partial_S + y^3\partial_T + z^3\partial_U + x^2y^2z^2\partial_V$ whose kernel is not finitely generated [Deveney,Finston, 1994]. Furthermore, Derksen showed in [Derksen, 1993] that any counterexample to Hilbert 14 could be derived as the kernel of a derivation.

It was proved by Zariski in [Zariski, 1954] that Hilbert 14 is true if $trdeg_k(L) \leq 2$, which was used by Nagata and Nowicki to show in [Nagata, Nowicki, 1988] that the kernel of any derivation on $k[X_1, \ldots, X_n]$ has finitely generated kernel if $n \leq 3$.

Recently, a new counterexample to Hilbert 14 was given by Freudenburg in dimension 6, as the kernel of the derivation $D := x^3\partial_s + y^3s\partial_t + y^3t\partial_u + x^2y^2d_v$ [Freudenburg,1998]. This was an important new breakthrough, which leaves Hilbert 14 open in dimensions 4 and 5 only.

It was conjectured by Nowicki that derivations of the form $X_{n-1}^{a_{n-1}}\partial_{X_n} + \ldots + X_0^{a_0}\partial_{X_1}$ could have infinitely generated kernel for $n \geq 4$ if the a_i are chosen wisely. Also one could try to find infinitely generated kernels in dimension 4 or 5 by taking a derivation of the simple form as the Freudenburg or Deveney-Finston derivations. Indeed, all derivations discussed above are of triangular monomial form (see below for definition).

In this article it will be proved that in dimension 4 there are no such easy counterexamples to Hilbert 14 similar to the Freudenburg derivation. As a side result it is proved that Nowicki's conjecture does not hold in dimension 4. The main theorem states that the class of monomial triangular derivations in dimension four has at most four generators, and these generators will be given explicitly .

2 Preliminaries

Throughout this paper we will use the following notations: k is a field of characteristic zero, $k[X] := k[X_1, \dots, X_n]$ the polynomial ring in n variables and D is a k-derivation on $k[X]$ (a map $k[X] \longrightarrow k[X]$ satisfying $D(ab) = aD(b) + D(a)b$, $D(a+b) = D(a) + D(b)$ and which is zero on k). It can be proved that the set of all k-derivations on $k[X]$ is the set of all maps of the form $D := a_1(X)\partial_{X_1} + \dots + a_n(X)\partial_{X_n}$ where $a_i(X) \in k[X]$.
In the proof below an algorithm of van den Essen [Essen, 1993] to calculate the kernel of a given locally nilpotent derivation is used. We will briefly describe the steps of the algorithm, without proofs.

Find $p \in k[X]$ such that $D(p) \neq 0$, $D^2(p) = 0$. Choose $q \in k[X]$ such that $D(p) = uq^l$ for some $u \in k^*$ and some integer $l > 0$. Let $s := p/q$ in $k[X, q^{-1}]$. Now define

$$r_i := q^{e_i} exp(-sD)(X_i)$$

where $e_i \in \mathbb{N}$ is chosen such that $r_i \in k[X]$, q does not divide r_i. Define

$$R_0 := k[r_1, \dots, r_n, q].$$

Notice that $R_0 \subset k[X]$. Now we define inductively R_m for $m \in \mathbb{N}$. If $R_m = k[F_1, \dots, F_t]$ and $I := \{P \in k[Y_1, \dots, Y_t] \mid P(F_1, \dots, F_t) \in k[X] \cdot q\}$ is generated by $P_1(Y), \dots, P_s(Y)$ then $R_{m+1} = k[F_1, \dots, F_t, f_1, \dots, f_s]$ where $f_i = q^{-1} \cdot P_i(F_1, \dots, F_t)$. It is proved that R_{m+1} is a finitely generated k-algebra. Now, if ever $R_m = R_{m+1}$ for some m then $ker(D) = R_m$.

3 Main theorem

Definition 3.1. A derivation $D := a_1(X)\partial_{X_1} + \dots + a_n(X)\partial_{X_n}$ is called

1. *monomial* if each $a_i(X)$ is a monomial.
2. *triangular* if $a_i(X) \in k[X_{i+1}, \dots, X_n]$ if $1 \leq i \leq n-1$ and $a_n \in k$.

In the theorem below we use the following notations: $D := \lambda_1 X_2^a X_3^b X_4^c \partial_{X_1} + \lambda_2 X_3^d X_4^e \partial_{X_2} + \lambda_3 X_4^f \partial_{X_3} + \lambda_4 \partial_4$ where $a, b, c, d, e, f \in \mathbb{N}$ and $\lambda_i \in k$. This is the general triangular monomial k-derivation. Furthermore we write

$$\begin{aligned} r_1 &:= X_4^F (X_1 - \textstyle\sum_{i=0}^a \mu_i X_2^{a-i} X_3^{b+1+i(d+1)} X_4^{i(e-f)+c-f}) \\ r_2 &:= X_4^G (X_2 - \tfrac{1}{d+1} \tfrac{\lambda_2}{\lambda_3} X_4^{e-f} X_3^{d+1}) \\ r_5 &:= X_4^{-l} (\tfrac{1}{d+1} \tfrac{\lambda_2}{\lambda_3} r_1^\alpha - \mu_a r_2^\beta) \end{aligned}$$

where

- $G = max\{0, f-e\}$, $F = max\{0, fa + f - ae - c\}$
- $\mu_i = \prod_{j=1}^i \left((\frac{a-j+1}{b+1+j(d+1)}) (\frac{-\lambda_2}{\lambda_3})^i \right) \frac{\lambda_1}{(b+1)\lambda_3}$
- $\alpha := \frac{1}{E}(b+1+a(d+1)), \beta = \frac{1}{E}(d+1)$ in which $E = gcd(b+1+a(d+1), d+1)$
- l is some integer.

The only new part of the following theorem is the case $\lambda_4 = 0$, $\lambda_1\lambda_2\lambda_3 \neq 0$. For completeness sake the generators of the kernel of D for this case have been written down exactly.

Theorem 3.2. *Let $A := k[X_1, X_2, X_3, X_4]$ and let D be a monomial triangular k-derivation on A.*

1. *If $\lambda_4 \neq 0$ then $ker(D) = k[exp(-sD)(X_1), exp(-sD)(X_2), exp(-sD)(X_3)]$ where $s = \lambda_4^{-1} X_4$;*
2. *If $\lambda_4 = 0$ and $\lambda_1\lambda_2\lambda_3 = 0$ then $ker(D) = k[F_1, F_2, F_3]$ for some F_i;*
3. *If $\lambda_4 = 0$,$\lambda_1\lambda_2\lambda_3 \neq 0$, $ae + c - fa - f < 0$ and $e - f < 0$, then $ker(D) = k[X_4, r_1, r_2, r_5]$ where r_i as above;*
4. *If $\lambda_4 = 0$,$\lambda_1\lambda_2\lambda_3 \neq 0$, $ae + c - fa - f \geq 0$ or $e - f \geq 0$ then $ker(D) = k[X_4, r_1, r_2]$ where r_i as above.*

Proof. (1): We use the algorithm described in section 2, and use the same notations. If $\lambda_4 \neq 0$ then take $p = X_4$, $q = \lambda_4$ (and $l = 1$) and $s = \lambda^{-1}q$. Now $R_0 = k[exp(-sD)(X_1), exp(-sD)(X_2), exp(-sD)(X_3), q]$. But since q is invertible in $k[X_1, X_2, X_3, X_4]$ any new step won't introduce any new elements. Hence $R_0 = R_1$ and the kernel is as stated.
(2): For this result we refer to [Daigle,Freudenburg, 1998].
(3): We will apply the algorithm described in section 2 again. Note $D(X_3) = \lambda_3 X_4^f$ and define $q = X_4$ and $s = X_3/D(X_3)$. Now when we want to calculate

$$r_i := q^{e_i} \exp(-sD)(X_i).$$

We know $ae + c - fa - f < 0$ and $e - f < 0$.
Claim: In this case one has

$$\begin{aligned}
&r_1 = X_4^{fa+f-ae-c} X_1 - \textstyle\sum_{i=1}^{a} \mu_i X_2^{a-i} X_3^{b+1+i(d+1)} X_4^{(a-i)(f-e)}.\\
&r_2 = X_4^{f-e} X_2 - \tfrac{1}{d+1}\tfrac{\lambda_2}{\lambda_3} X_3^{d+1}\\
&r_3 = 0\\
&r_4 = X_4
\end{aligned}$$

where μ_i is as in the theorem. The only thing which needs to be proved of this claim is that the formula for r_1 is correct. By the lemma following this proof we are done. Let $R_0 := k[r_1, r_2, r_3, r_4] = k[r_1, r_2, X_4]$. We want to calculate R_1. For such a $g \in R_1$ we must have $X_4^l g = G(r_1, r_2)$ for some $G(U_1, U_2) \in k[U_1, U_2]$, $l \geq 1$. Hence $G(r_1, r_2) = 0 (mod\ X_4)$. So $G(r_1 (mod\ X_4), r_2 (mod\ X_4)) = 0$. Hence $G(\mu_a X_3^{b+1+a(d+1)}, \frac{1}{d+1}\frac{\lambda_2}{\lambda_3} X_3^{d+1}) = 0$. If G is taken of minimal degree then it must be of the form $(c_1 U_1)^\alpha - (c_2 U_2)^\beta$ where $\alpha = \frac{1}{E}(d+1), \beta = \frac{1}{E}(b+1+a(d+1))$ in which $E = gcd(b+1+a(d+1), d+1)$ and $c_1 = \frac{1}{\mu_a}$, $c_2 = (d+1)\frac{\lambda_3}{\lambda_2}$. Hence we can take a maximal $l \in \mathbb{N}$ such that $X_4^{-l} G(r_1, r_2) \in A$. Say $r_5 := X_4^{-l} G(r_1, r_2) = X_4^{-l}(c_1 r_1^\alpha - c_2 r_2^\beta)$. Since l is taken as large as possible we have $r_5 (mod\ X_4) \neq 0$. We now leave it to the reader to verify that $r_5 mod(X_4)$ depends on X_2 (a real detailed proof would be very tedious: as a hint, notice that $r_5 mod(X_4)$ is the lowest degree term with respect to X_4 of $G(r_1, r_2)$). It is easy to see that for any $\widetilde{G} \in k[U_1, U_2]$ satisfying $\widetilde{G}(r_1 (mod\ X_4), r_2 (mod\ X_4)) = 0$ G divides $\widetilde{G}$. Hence $R_1 = k[X_4, r_1, r_2, r_5]$. Now let us attempt to construct R_2. Suppose we have $H \in k[U_1, U_2, U_3]$ such that $H(r_1, r_2, r_5) = X_4(\ldots)$. Then $H(r_1 (mod\ X_4), r_2 (mod\ X_4), r_5 (mod\ X_4)) = 0$. But since $r_5 (mod\ X_4)$ depends on X_2 this means that H is independent of U_3 and that we have a polynomial from our previous step. Hence $R_2 = R_1$ and thus $ker(D) = R_1 = k[X_4, r_3, r_4, r_5]$.
(4): This case (in fact: these 3 cases) can be handled with similar arguments as in (3). For example, $e - f \geq 0$ and $ae + c - fa - f \geq 0$ brings up the problem of finding a polynomial G such that $G(r_1, r_2) = X_4(\ldots)$ which means $0 = G(r_1 (mod\ X_4), r_2 (mod\ X_4))$. But in this case r_1 depends on X_1 while r_2 doesn't. Hence in this case one has $R_0 = R_1$. In fact, in all remaining cases one has $R_0 = R_1$. Hence, triangular monomial derivations have finite kernel of at most 4 generators exactly as stated in the theorem. □

Lemma 3.3.
$$X_4^{fa+f-ae-c}exp(-sD)(X_1) =$$
$$X_4^{fa+f-ae-c}X_1 - \sum_{i=1}^{a}\mu_i X_2^{a-i}X_3^{b+1+i(d+1)}X_4^{(a-i)(f-e)}$$
where $\mu_i = \prod_{j=1}^{i}\left((\frac{a-j+1}{b+1+j(d+1)})(\frac{\lambda_2}{\lambda_3})^i\right)\frac{\lambda_1}{(b+1)\lambda_3}$.

Proof. One can ofcourse compute that the formula is correct, but that is not easy. We will use another method here. The ideas presented in this proof can be found in a more explained setting in [Maubach], especially the grading-theory used below. Define two degree functions on A by means of

$$deg_1(X_1^{t_1}X_2^{t_2}X_3^{t_3}X_4^{t_4}) = t_3 + (d+1)t_2 + (a(d+1)+b+1)t_1$$
$$deg_2(X_1^{t_1}X_2^{t_2}X_3^{t_3}X_4^{t_4}) = t_4 + ft_3 + (df+e)t_2 + (adf+ae+bf+c)t_1.$$

and define a multidegree $grad := (deg_1, deg_2)$ on A. So if we define $A_{n,m}$ as the linear k-span of the monomials M satisfying $grad(M) = (n,m)$ then $A := \bigoplus_{(n,m)\in\mathbb{N}^2} A_{n,m}$. Furthermore, a nice property of this grading is that $D(A_{n,m}) \subseteq A_{n-1,m}$, which can be easily checked. Using these properties it is an easy exercise to prove that for every monomial M occuring in $X_4^{fa+f-ae-c}exp(sD)(X_1)$ we have $grad(M) = grad(X_4^{fa+f-ae-c}X_1)$. Now if we restrict our map D to the linear space $A_{n,m}$ where $grad(X_4^{fa+f-ae-c}X_1) = (n,m)$ then D induces a linear map l from $A_{n,m}$ to $A_{n-1,m}$. Then since $X_4^{fa+f-ae-c}exp(-sD)(X_1) \in A_{n,m}$ we have $A_{n,m}^D = ker(l)$. The matrix of l with respect to the basis

$$\{X_1X_4^{fa+f-ae-c}, X_2^aX_3^{b+1}X_4^{a(f-e)}, X_2^{a-1}X_3^{b+1+(d+1)}X_4^{(a-1)(f-e)}, \ldots, X_3^{b+1+a(d+1)}\}$$

of $A_{n,m}$ and the basis
$$\{X_2^aX_3^bX_4^{a(f-e)}, X_2^{a-1}X_3^{b+d+1}X_4^{(a-1)(f-e)}, \ldots, X_3^{b+a(d+1)}\}$$
of $A_{n-1,m}$ we denote by $\mathcal{M}$. It has entries $m_{1,1} = \lambda_1$, $m_{i,i} = (a+1-i)\lambda_2$ for $i \geq 2$, $m_{i,i+1} = (b+1+(i-1)(d+1))\lambda_3$ for $i \geq 1$ and zeros elsewhere. It has dimension $(a+2)\times(a+1)$. The matrix has corank 1 and si of maximal rank. Hence the kernel is one dimensional. Some calculation proves that the kernel is spanned by $e_1 - \sum_{i=0}^{a}\mu_i e_{i+2}$ where $e_1,\ldots,e_{a+2}$ is the standard basis and μ_i is exactly as previously described. Hence $A_{n,m}^D$ is one dimensional and generated by $X_4^{fa+f-ae-c}X_1 - \sum_{i=1}^{a}\mu_i X_2^{a-i}X_3^{b+1+i(d+1)}X_4^{(a-i)(f-e)}$. We know that $X_4^{fa+f-ae-c}exp(-sD)(X_1)$ is in $A_{n,m}$ and also in $ker(D)$. Hence

$$X_4^{fa+f-ae-c}exp(-sD)(X_1) = X_4^{fa+f-ae-c}X_1 - \sum_{i=1}^{a}\mu_i X_2^{a-i}X_3^{b+1+i(d+1)}X_4^{(a-i)(f-e)}.$$

□

Now let us end, very poetically, with the title.

Corollary 3.4. Triangular monomial derivations on $k[X_1,X_2,X_3,X_4]$ have kernel generated by at most four elements.

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