The Commuting Derivations Conjecture

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Abstract

This paper proves the Commuting Derivations Conjecture in dimension three: if D_1 and D_2 are two locally nilpotent derivations which are linearly independent and satisfy $[D_1, D_2] = 0$ then the intersection of the kernels, $A^{D_1} \cap A^{D_2}$ equals $\mathbb{C}[f]$ where f is a coordinate. As a consequence, it is shown that $p(X)Y + Q(X, Z, T)$ is a coordinate if and only if $Q(a, Z, T)$ is a coordinate for every zero a of $p(X)$. Next to that, it is shown that if the Commuting Derivations Conjecture in dimension n, and the Cancellation Problem and Abhyankar-Sataye Conjecture in dimension n-1, all have an affirmative answer, then we can similarly describe all coordinates of the form $p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$. Also, conjectures about possible generalisations of the concept of "coordinate" for elements of general rings are made.

1 Introduction

In this article we will discuss the Commuting Derivations Conjecture (CD(n)) and its consequences. In short, the conjecture states that if one has $n-1$ independent commuting locally nilpotent derivations of $\mathbb{C}^{[n]}$, then the intersections of the kernels is generated by a coordinate. This conjecture is comparable to and connected with the Cancellation Problem (CP(n)) and the Abhyankar-Sataye Conjecture (AS(n)). This paper will show that if CP(n-1), AS(n-1) and CD(n) are all true, then we can describe all coordinates of the form $p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$. Ingredients in the proof of this last statement are a recent result of Edo-Vénéreau in [3] (see 2.5 below) and an idea of Derksen-Essen-Rossum in [2]. The main result of this paper is the proof of CD(3), which uses a recent result of Kaliman in [7]. Since CP(2) and AS(2) are true we can, as a consequence, describe all coordinates of the form $p(X)Y + q(X, Z_1, Z_2)$. A more general result by Kaliman-Vénéreau-Zaidenberg [9] on when $p(X, Z_1)Y + q(X, Z_1, Z_2)$ is a coordinate was achieved simultaneously to this article. The problem of recognising and characterising coordinates is of crucial importance for various questions in algebraic geometry, see for example [10], [16], [17], [18], [11], [8], [5].

Finally, at the end of this paper we discuss some possible definitions of the notion of coordinate in quotients of polynomial rings.

2 Preliminaries

Notations: In this article, $\mathbb{C}^{[n]}$ will denote a ring isomorphic over $\mathbb{C}$ to a polynomial ring in n variables. $LND(\mathbb{C}^{[n]})$ will be the set of all locally nilpotent $\mathbb{C}$-derivations on $\mathbb{C}^{[n]}$, i.e. the set of all $\mathbb{C}$-linear maps $D : \mathbb{C}^{[n]} \longrightarrow \mathbb{C}^{[n]}$ satisfying the Leibnitz rule $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathbb{C}^{[n]}$ and for all $a \in \mathbb{C}^{[n]}$ there exists an integer $n \in \mathbb{N}$ such that $D^n(a) = 0$. If A is some ring, A^* will be the set of invertible elements.

Definition 2.1. We say $F \in \mathbb{C}^{[n]}$ is a *coordinate in* $\mathbb{C}^{[n]}$ if there exist $F_2, \ldots, F_n \in \mathbb{C}^{[n]}$ such that $\mathbb{C}[F, F_2, \ldots, F_n] = \mathbb{C}^{[n]}$. Similarly, we say that $F \in \mathbb{C}^{[n]}$ is a *stable coordinate (in* $\mathbb{C}^{[n]}$*)* if there exist $m \in \mathbb{N}$ such that F is a coordinate in $\mathbb{C}^{[n+m]}$.

Not every polynomial is a coordinate, as can be seen by several examples. One can deduce the following:

Lemma 2.2. *If $F \in \mathbb{C}^{[n]}$ is a coordinate, then*

(i). F is irreducible, even $F + \alpha$ is irreducible for all $\alpha \in \mathbb{C}$,

(ii). $(\frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}) = (1)$,

(iii). $\mathbb{C}^{[n]}/(F) \cong \mathbb{C}^{[n-1]}$,

(iv). There exists a subring $A \subset \mathbb{C}^{[n]}$ such that F is algebraically independent over A, $A[F] = \mathbb{C}^{[n]}$, and $A \cong \mathbb{C}^{[n-1]}$.

It is an important question to be able to decide whether some polynomial is a coordinate. The question arises whether there exist sufficient properties which imply "coordinate". *(i)* and *(ii)* are by no means sufficient: take $F = XY + ZT + Z + T$, which satisfies both *(i)* and *(ii)* and is no coordinate (by corollary 4.2). Whether *(iii)* is sufficient, is still open for $n \geq 3$:

Abhyankar-Sathaye Conjecture (AS(n)): If $f \in \mathbb{C}^{[n]}$ and $\mathbb{C}^{[n]}/(f) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ then f is a coordinate.

AS(2) was proved by Abhyankar and Moh in [1].

Part *(iv)* of lemma 2.2 gives rise to the following problem:

Cancellation Problem (CP(n)): If $\mathbb{C}^{[n]} = A[T]$ then $A \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$.

This problem had been answered affirmatively for $n = 2$ ([14]) and $n = 3$ ([6]). The following conjecture is a new one. In the rest of the article its significance will become clear.

Commuting Derivations Conjecture (CD(n)) : If $D_1, \ldots, D_{n-1} \in LND(\mathbb{C}^{[n]})$ linearly independent over $\mathbb{C}^{[n]}$ such that $[D_i, D_j] = 0$ for all $1 \leq i, j \leq n-1$ (i.e. they all commute) then

$$\bigcap_{i=1}^{n-1} ker(D_i) = \mathbb{C}[f]$$

where f is a coordinate in $\mathbb{C}^{[n]}$.

The following lemma we will need in the next section.

Lemma 2.3. *Let R be a domain, and $r \in R$ such that rR is a prime ideal. Then r is irreducible in R.*

Proof. Let $I := rR$. Suppose r is reducible, i.e. $r = ab$ for some $a, b \in R$ not invertible. Since $ab \in I$, a prime ideal, we have a or b in I. We may assume $a \in I$, thus $a = rs$ for some $s \in R$, and thus $rsb = ab = r$ and since R is a domain we get $sb = 1$, which means b is invertible, a contradiction. Hence r must be irreducible. $\square$

The following theorem is a special case of the main theorem in [7].

Theorem 2.4. *Let $f \in \mathbb{C}[X, Y, Z]$ such that $\mathbb{C}[X, Y, Z]/(f - \lambda) \cong \mathbb{C}^{[2]}$ for all but finitely many $\lambda \in \mathbb{C}$. Then f is a coordinate.*

Proof. In the main theorem in [7] take $X' = \mathbb{C}^3$, $U := \{\lambda \mid \mathbb{C}[X, Y, Z]/(f - \lambda) \cong \mathbb{C}^{[2]}\}$, $Z := f^{-1}(U)$, $p = f$. Then this theorem states p is a coordinate. $\square$

The following is theorem 7 in [3]. $\eta(R)$ is the nilradical of some ring R.

Theorem 2.5. *Let A be a ring and let $p \in A^*$. Let $a \in A, G, F \in A[X]$ such that F is a coordinate in $A[X]$, $a \mod (pA)$ invertible, and $G(X) \mod (pA) \in \eta((A/pA)[X])$. Then $aF(X) + G(X) + pY$ is a coordinate in $A[X, Y]$.*

3 Proof of CD(3)

In the following lemma, the derivation δ_i (the restriction of D_i to A^{D_n}) is well-defined: for all $a \in A^{D_n}$ we have $0 = D_i(D_n(a)) = D_n(D_i(a))$, hence $D_i(A^{D_n}) \subseteq A^{D_n}$. We say that a $\mathbb{C}$-domain is a $\mathbb{C}$-algebra which is a domain.

Lemma 3.1. *Let A be a $\mathbb{C}$-domain and $D_1, \ldots, D_n$ be commuting locally nilpotent derivations which are linearly independent over A. Let $\delta_i := D_i|_{A^{D_n}}$. Then $\delta_1, \ldots, \delta_{n-1}$ are linearly independent over A^{D_n}.*

Proof. Suppose that $\sum a_i \delta_i = 0$ for some $a_i \in A^{D_n}$. Since D_n is nonzero there exists a preslice $p \in A$ for D_n, i.e. an element p which satisfies $d := D_n(p) \neq 0$ and $D_n^2(p) = 0$ (i.e. $d \in A^{D_n}$). Let $s := pd^{-1} \in A[d^{-1}]$. Then $D_n(s) = 1$. Furthermore, by [4] pages 27-28, $A[d^{-1}] = A^{D_n}[d^{-1}][s]$. Let $a := \sum a_i D_i(s) \in A[d^{-1}]$, say $\tilde{a} := d^m a \in A$. So

$$(\sum_{i=1}^{n-1} a_i d^m D_i)(s) = d^m a = \tilde{a} = \tilde{a} D_n(s).$$

Also by our hypothesis

$$\sum_{i=1}^{n-1} a_i d^m D_i - \tilde{a} D_n = 0$$

on A^{D_n}. Since $A \subset A^{D_n}[d^{-1}][s]$ it follows that $\sum a_i d^m D_i = \tilde{a} D_n$. From the linear independence of the D_i over A we deduce that $d^m a_i = 0$ for all i, whence $a_i = 0$ for all i. $\square$

Proposition 3.2. *Let A be a $\mathbb{C}$-domain with $trdeg_{\mathbb{C}}Q(A) = n(\geq 1)$. Let $D_1, \dots, D_n$ be commuting locally nilpotent $\mathbb{C}$-derivations on A which are linearly independent over A. Then*

(i). There exist s_i in A such that $D_i s_j = \delta_{ij}$ for all i, j and

(ii). $A = \mathbb{C}[s_1, \dots, s_n]$ a polynomial ring in $s_1, \dots, s_n$ over $\mathbb{C}$.

Proof. We use induction on n. The case $n = 1$ is well-known (cor. 1.3.33 [4]). So let $n \geq 2$. $trdeg_{\mathbb{C}}(A^{D_n}) = n - 1$ and according to lemma 3.1 the derivations $\delta_i := D_i|_{A^{D_n}}$ $1 \leq i \leq n-1$ satisfy the hypothesis of the proposition. So by induction there exist $s_i \in A^{D_n}$ such that $\delta_i s_j = \delta_{ij}$ and $A^{D_n} = \mathbb{C}[s_1, \dots, s_{n-1}]$. So the first $n-1$ derivations have a slice in A. Similarly D_n has a slice s_n in $A^{D_1} \subset A$. Then from $A = A^{D_n}[s_n]$ the result follows. □

Lemma 3.3. *Let A be a $\mathbb{C}$-domain with $trdeg_{\mathbb{C}}(Q(A)) = n$. If $D_1, \dots, D_p$ are commuting locally nilpotent $\mathbb{C}$-derivations which are linearly independent over A , then $trdeg_{\mathbb{C}}Q(A^{D_1} \cap \dots \cap A^{D_p}) = n - p$.*

Proof. The case $p = 1$ is well-known. Let $B := A^{D_p}$. By lemma 3.1 the derivations $\delta_i := D_i|_B$ for all $1 \leq i \leq p-1$ are linearly independent over B. Hence by induction $trdeg_{\mathbb{C}}Q(B^{\delta_1} \cap \dots \cap B^{\delta_{p-1}}) = trdeg_{\mathbb{C}}Q(B) - (p-1) = n - 1 - (p-1) = n - p$. Since $B^{\delta_1} \cap \dots \cap B^{\delta_{p-1}} = A^{D_1} \cap \dots \cap A^{D_{p-1}} \cap A^{D_p}$ the result follows. □

Proposition 3.4. *Let A be an affine $\mathbb{C}$-domain such that $trdeg_{\mathbb{C}}Q(A) = n$ and $A^* = \mathbb{C}^*$. If A is a* UFD *and $D_1, \dots, D_{n-1}$ are commuting locally nilpotent $\mathbb{C}$-derivations on A which are linearly independent over A, then $\cap A^{D_i} = \mathbb{C}[g]$ for some $g \in A$ which satisfies $g - c$ is irreducible in A for all $c \in \mathbb{C}$.*

Proof. Put $B := \cap A^{D_i}$. By lemma 3.3 we have $trdeg_{\mathbb{C}}B = n - (n-1) = 1$. Also B is a UFD (see [4] cor. 1.3.36) and $B = A \cap Q(B)$. Since $trdeg_{\mathbb{C}}Q(B) = 1$ it follows from special case of Hilbert 14 (using B is normal since it is a UFD) that B is a finitely generated $\mathbb{C}$-algebra. So B is an affine domain of krull dimension one. It is a well-known result that if $B^* = \mathbb{C}^*$, B is a UFD and B is an affine domain of krull dimension one, that $B = \mathbb{C}[g] \cong_{\mathbb{C}} \mathbb{C}^{[1]}$. (See for example [13].) Since $g - c$ is irreducible in $\mathbb{C}[g]$ for all $c \in \mathbb{C}$ and B is factorially closed in A it follows that $g - c$ is also irreducible in A (see [4] exercise 6, 1.3). □

Proposition 3.5. *Let D_1, D_2 be two linearly independent (over $\mathbb{C}[X,Y,Z]$) commuting locally nilpotent $\mathbb{C}$-derivations. Then there exists $g \in \mathbb{C}[X,Y,Z] \backslash \mathbb{C}$ such that*

(i). $\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[g]$

(ii). $\mathbb{C}[X,Y,Z]_{b(g)} = \mathbb{C}[f,g,p]_{b(g)}$ for some $f, p \in \mathbb{C}[X,Y,Z]$ and $b(g) \in \mathbb{C}[g] \backslash \{0\}$

(iii). $\mathbb{C}[X,Y,Z]/(g - \lambda) \cong_{\mathbb{C}} \mathbb{C}^{[2]}$ for all $\lambda \in \mathbb{C}$ with $b(\lambda) \neq 0$.

Proof. (i) $\mathbb{C}[X,Y,Z]^{D_1} = \mathbb{C}[f,g]$ and $\mathbb{C}[X,Y,Z]^{D_2} = \mathbb{C}[p,q]$ by [12]. Since D_1, D_2 commute we have $D_2(\mathbb{C}[f,g]) \subseteq \mathbb{C}[f,g]$. Write $d_2 := D_2|_{\mathbb{C}[f,g]}$. By lemma 3.1 it follows that $d_2 \neq 0$ on $\mathbb{C}[f,g]$. So by Rentschler's theorem we may assume that $d_2 = a(g)\frac{\partial}{\partial f}$ i.e. $D_2(g) = 0$ and $D_2(f) = a(g) \neq 0$. So $\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[f,g]^{d_2} = \mathbb{C}[g]$ i.e.

$$\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[g]. \tag{1}$$

Similarly we get $D_1(\mathbb{C}[p,q]) \subset \mathbb{C}[p,q]$ and putting $d_1 := D_1|_{\mathbb{C}[p,q]}$ this gives by Rentschler that we may assume $d_1 = b(q)\frac{\partial}{\partial p}$ for some $b(q) \neq 0$. So

$$\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[p,q]^{d_1} = \mathbb{C}[q]. \tag{2}$$

From (1) and (2) we deduce that $\mathbb{C}[g] = \mathbb{C}[p]$, whence $g = \lambda q + \mu$ form some $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. Replacing q by g (and hence $b(q) = b(\lambda^{-1}(g-\mu) = \tilde{b}(g)$ by $\tilde{b}(g)$ we get that we may assume the following

$$\begin{aligned} &\mathbb{C}[X,Y,Z]^{D_1} = \mathbb{C}[f,g], D_1 f = D_1 g = 0, D_1 p = b(g) \neq 0 \\ &\mathbb{C}[X,Y,Z]^{D_2} = \mathbb{C}[p,g], D_2 f = a(g) \neq 0, D_2 g = D_2 p = 0. \end{aligned}$$

(ii) Also $\mathbb{C}[f,g,p] \cong_{\mathbb{C}} \mathbb{C}^{[3]}$ (for if p depends on $\mathbb{C}[f,g]$ then $D_1 p = 0$, contradiction). Observe that $D_1 p = b(g) \neq 0$ and $D_1^2 p = D_1 b(g) = 0$, so $s := p/b(g) \in \mathbb{C}[X,Y,Z]_{b(g)}$ satisfies $D_1 s = 1$, whence $\mathbb{C}[X,Y,Z]_{b(g)} = \mathbb{C}[f,g]_{b(g)}[s] = \mathbb{C}[f,g,p]_{b(g)}$.
(iii) It remains to show the last statement. Since $g - \lambda$ is irreducible in $\mathbb{C}[f,g]$, for all $\lambda \in \mathbb{C}$ and since $\mathbb{C}[f,g] = \mathbb{C}[X,Y,Z]^{D_1}$ is factorially closed in $\mathbb{C}[X,Y,Z]$, it follows that $g - \lambda$ is irreducible in $\mathbb{C}[X,Y,Z]$ for all $\lambda \in \mathbb{C}$. Now assume $b(\lambda) \neq 0$ i.e. $(g-\lambda)$ does not divide $b(g)$. We will show that $A := \mathbb{C}[X,Y,Z]/(g-\lambda) \cong_{\mathbb{C}} \mathbb{C}^{[2]}$. According to 3.2 it suffices to show that $\bar{D}_1$ and $\bar{D}_2$ are linearly independent derivations over A. Suppose that $a_1, a_2 \in \mathbb{C}[X,Y,Z]$ are such that $\bar{a}_1\bar{D}_1 + \bar{a}_2\bar{D}_2 = 0$. (" $\bar{\ }$ " means mod $(g-\lambda)$.) Then $(a_1D_1 + a_2D_2)(\mathbb{C}[X,Y,Z]) \subset (g-\lambda)\mathbb{C}[X,Y,Z]$. In particular, $a_1(X,Y,Z)b(g) + 0 = a_1D_1(p) + a_2D_2(p) \in (g-\lambda)$. Since $g-\lambda$ is irreducible in $\mathbb{C}[X,Y,Z]$ and $g-\lambda \nmid b(g)$ it follows that $(g-\lambda)|a_1$ i.e. $\bar{a}_1 = 0$. So $\bar{a}_2\bar{D}_2 = 0$ i.e. $a_2D_2(\mathbb{C}[X,T,Z]) \subset (g-\lambda)$. If $(g-\lambda) \nmid a_2$, then $g-\lambda | D_2(X), D_2(Y), D_2(Z)$. In this case let $(g-\lambda)^e | D_2(X), D_2(Y), D_2(Z)$, $e \geq 1$ maximal. Then replace D_2 by $\tilde{D}_2 := (g-\lambda)^{-e}D_2$. It then follows that $\bar{D}_1$ and $\bar{\tilde{D}}_2$ are independent over A. Obviously $D_1, \tilde{D}_2$ have the same properties as the pair D_1, D_2 and $\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[X,Y,Z]^{D_1,\tilde{D}_2}$ which concludes the proof. ☐

Theorem 3.6. *CD(3) is true, i.e. let D_1, D_2 be two linearly independent (over $\mathbb{C}[X,Y,Z]$) commuting locally nilpotent $\mathbb{C}$-derivations, then $A^{D_1,D_2} = \mathbb{C}[g]$ and g is a coordinate in $\mathbb{C}[X,Y,Z]$.*

Proof. Combining 3.5 and 2.4 gives exactly this result. ☐

4 Coordinates

Theorem 4.1. *Assume AS(n-1), CD(n) and CP(n-1). Let $F := p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ where $p(X) \neq 0$. Then equivalent are:*

(i). F is a coordinate in $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]$

(ii). $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$

(iii). $q(a,Z_1,\ldots,Z_{n-1})$ *is a coordinate in* $\mathbb{C}[Z_1,\ldots,Z_{n-1}]$ *for every zero* a *of* $P(X)$.

(iv). F *is a coordinate over* $\mathbb{C}[X]$ *in* $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]$

Proof. (*of theorem 4.1*)
From 4.5 we have *(iii)*$\Longrightarrow$*(iv)*. *(iv)*$\Longrightarrow$*(i)* and *(i)*$\Longrightarrow$*(ii)* follow since they are weaker statements in general. *(ii)*$\Longrightarrow$*(iii)* follows from 4.7. □

From the fact that AS(2), CP(2) and CD(3) (see 3.6) are true, we can deduce the following corollaries:

Corollary 4.2. *The above equivalences hold for* $F = p(X)Y + q(X,Z_1,Z_2)$.

Corollary 4.3. *AS(4) is true if restricted to polynomials of the form* $p(X)Y+q(X,Z,T)$.

Lemma 4.4. *Let* $q(Z_1,\ldots,Z_{n-1}) \in \mathbb{C}[Z_1,\ldots,Z_{n-1}]$. *Suppose AS(n-1) and CP(n-1) are true. If* $\mathbb{C}[Z_1,\ldots,Z_{n-1},Y]/(q) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ *then* q *is a coordinate in* $\mathbb{C}^{[n-1]}$.

Proof. $\mathbb{C}[Z_1,\ldots,Z_{n-1}]/(q)[Y] \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ so by CP(n-1) we have $\mathbb{C}[Z_1,\ldots,Z_{n-1}]/(q) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ and by AS(n-1) we have q is a coordinate in $\mathbb{C}^{[n-1]}$. □

Write

$$p(X) = \Pi_{i=1}^r (X-\alpha_i)^{e_i}$$

for some $e_i \in \mathbb{N}$, and $F := p(X)Y+q(X,Z_1,\ldots,Z_{n-1})$ for some $q \in \mathbb{C}[X,Z_1,\ldots,Z_{n-1}]$.

Theorem 4.5. *Let* $q(X,Z_1,\ldots,Z_{n-1})$ *be such that* $q(\alpha_i,Z_1,\ldots,Z_{n-1})$ *is a coordinate in* $\mathbb{C}[Z_1,\ldots,Z_{n-1}]$ *for every* $1 \le i \le r$. *Then* $F := p(X)Y + q(X,Z_1,\ldots,Z_n)$ *is a coordinate in* $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]$ *over* $\mathbb{C}[X]$.

Proof. Using theorems 2.1.1 part 4 and 3.7.11 from [15], we see that it suffices to prove that F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y,Z_1,\ldots,Z_{n-1}]$ over $\mathbb{C}[X]_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset \mathbb{C}[X]$. Let $\mathfrak{m} = (X-\alpha)$ for some $\alpha \in \mathbb{C}$. Notice that if $a(X) \in \mathbb{C}[X]$ we have $a \in \mathbb{C}[X]^*_{\mathfrak{m}}$ if and only if $a(\alpha) \neq 0$. In case $\alpha \neq \alpha_i$ we have $p(\alpha) \neq 0$ and hence F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y,Z_1,\ldots,Z_{n-1}]$. Left to prove the case $\alpha = \alpha_1$ ($\alpha = \alpha_i$ has the same proof). Let $q_1(Z_1,\ldots,Z_{n-1}) := q(\alpha,Z_1,\ldots,Z_{n-1})$ (hence a coordinate in $\mathbb{C}^{[n-1]}$), and define

$$\tilde{p} := \Pi_{i=2}^r (X-\alpha_i)^{e_i} = p(X)(X-\alpha)^{-e_1}.$$

Now

$$F = (X-\alpha)^{e_1}\tilde{p}(X)Y + q_1 + (X-\alpha)h(X,Z_1,\ldots,Z_{n-1})$$

for some h. Notice $\tilde{p} \in \mathbb{C}[X]^*_{\mathfrak{m}}$. But now, using 2.5 we have F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y,Z_1,\ldots,Z_{n-1}]$. □

Lemma 4.6. *Let* $F = p(X)Y+q(X,Z_1,\ldots,Z_{n-1})$ *irreducible. Then there exists* $\lambda \in \mathbb{C}$ *such that* $X-\lambda \mod (F)$ *is irreducible in* $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]/(F)$.

Proof. Take λ such that $p(\lambda) \neq 0$. Then

$$\mathbb{C}[X, Y, Z_1, \ldots, Z_{n-1}]/(F, X-\lambda) = \mathbb{C}[Y, Z_1, \ldots, Z_{n-1}]/(p(\lambda)Y + q(\lambda, Z_1, \ldots, Z_{n-1})) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$$

which is a domain: hence $(X - \lambda, F)$ is prime, and thus $X - \lambda \mod F$ is irreducible by lemma 2.3. $\square$

Lemma 4.7. *Assume CD(n), CP(n-1) and AS(n-1). Let* $F := p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ *and assume* $\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$*. Then* $q(a, Z_1, \ldots, Z_{n-1})$ *is a coordinate in* $\mathbb{C}[Z_1, \ldots, Z_{n-1}]$ *for all zeros* a *of* $p(X)$*.*

Proof. Let

$$D_i := \frac{\partial q}{\partial Z_i}\frac{\partial}{\partial Y} - p\frac{\partial}{\partial Z_i}$$

for all $1 \leq i \leq n-1$. These derivations are triangular derivations since

$$\begin{gathered} D(Y) \in \mathbb{C}[Z_1, \ldots, Z_n, X], \\ D(Z_i) \in \mathbb{C}[Z_{i+1}, \ldots, Z_n, X] \\ \text{and } D(X) \in \mathbb{C} \end{gathered}$$

and it is not difficult to see that a triangular derivation is locally nilpotent (see for example [4], corollary 1.3.17). It is clear that $[D_i, D_j] = 0$, and that the D_i are linearly independent over $\mathbb{C}[X, Y, Z_1, \ldots, Z_{n-1}]$. Now we know

$$\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}.$$

Furthermore $D_i(F) \subset (F)$, so the derivations $\bar{D}_i := D_i \mod (F)$ are well-defined on $\mathbb{C}^{[n+1]}/(F) \cong \mathbb{C}^{[n]}$. Also they are independent over $\mathbb{C}^{[n+1]}/(F)$. Since we assumed CD(n) we have

$$\bigcap_{i=1}^{n-1} ker(\bar{D}_i) = \mathbb{C}[g]$$

for some coordinate g. Since $ker(\bar{D}_i) \supset \mathbb{C}[X]$ we see $\mathbb{C}[g] \supset \mathbb{C}[X]$. By lemma 4.6 we see that $X - a$ is irreducible in $\mathbb{C}^{[n+1]}/(F)$ for some $a \in \mathbb{C}$. Now $X - a = Q(g)$ for some polynomial $Q(T) \in \mathbb{C}[T]$. Decomposing $Q(T)$ into linear factors $T - \lambda_i$ and observing that $g - \lambda_i$ is irreducible in $\mathbb{C}^{[n+1]}/(F)$ (since g is a coordinate in it), it follows that $g - \lambda_i$ divides the irreducible element $X - a$. So $X - a = bg + c$ for some $b \in \mathbb{C}^*, c \in \mathbb{C}$. Thus $\mathbb{C}[g] = \mathbb{C}[X]$, and $X - \alpha$ is a coordinate in $\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$ for every $\alpha \in \mathbb{C}$. So

$$\mathbb{C}^{[n-1]} \cong_{\mathbb{C}} \mathbb{C}^{[n+1]}/(F, X - \alpha) \text{ for all } \alpha \in \mathbb{C}.$$

In case $p(\alpha) = 0$ we have

$$\mathbb{C}^{[n-1]} \cong_{\mathbb{C}} \mathbb{C}[Y, Z_1, \ldots, Z_{n-1}]/(q(\alpha, Z_1, \ldots, Z_{n-1})$$

and thus by CP(n-1) and AS(n-1) and lemma 4.4 we have $q(\alpha, Z_1, \ldots, Z_{n-1})$ is a coordinate in $\mathbb{C}[Z_1, \ldots, Z_{n-1}]$. $\square$

5 An extension of the concept of coordinate

This section deals with a lot of conjectures, and an attempt to generalise the concept of stable coordinate for elements in a quotient ring of a polynomial ring.

Definition 5.1. Let $I = (f_1, \ldots, f_m)$ be an ideal in $\mathbb{C}[X_1, \ldots, X_n] = \mathbb{C}^{[n]}$. Let $r \in \mathbb{C}^{[n]}$. Define $r + (I) \in \mathbb{C}^{[n]}/I$ is a *generalised coordinate in* $\mathbb{C}^{[n]}/I$ if $f_1 Y_1 + \ldots + f_m Y_m + r \in \mathbb{C}^{[n+m]}$ is a stable coordinate.

The definition does not depend on the generators of I as can be seen from

Lemma 5.2. *Let $I = (f_1, \ldots, f_m) = (g_1, \ldots, g_l)$ be an ideal in $\mathbb{C}[X_1, \ldots, X_n] = \mathbb{C}^{[n]}$. Let $r \in \mathbb{C}^{[n]}$. Then $f_1 Y_1 + \ldots + f_m Y_m + r \in \mathbb{C}^{[n+m]}$ can be mapped to $g_1 Z_1 + \ldots + g_l Z_l + r$ by an automorphism of $\mathbb{C}[X, Y, Z] = \mathbb{C}^{[n+m+l]}$.*

Proof. Let $F := f_1 Y_1 + \ldots + f_m Y_m + r$ and $G := g_1 Z_1 + \ldots + g_l Z_l + r$. We will show that there is an automorphism of $\mathbb{C}[X, Y, Z]$ sending F to G. Since $(g_1, \ldots, g_l) = (f_1, \ldots, f_m)$ in $\mathbb{C}[X]$ we have $g_i = a_{i1} f_1 + \ldots + a_{im} f_m$ for some $a_{ij} \in \mathbb{C}[X]$. Let $L_j := a_{1j} Z_1 + \ldots + a_{lj} Z_l$ for $1 \leq j \leq m$.Notice that

$$G = f_1 L_1 + \ldots + f_m L_m + r.$$

Now let φ be the elementary automorphism sending Y_j to $Y_j + L_j$ for each j and leaving other variables invariant. Then

$$\begin{aligned} \varphi(F) = \; & f_1 \varphi(Y_1) + \ldots + f_m \varphi(Y_m) + r \\ = \; & f_1 (Y_1 + L_1) + \ldots + f_m (Y_m + L_m) + r \\ = \; & F + f_1 L_1 + \ldots + f_m L_m \\ = \; & F + G - r \end{aligned}$$

In the same way we can make an automorphism τ sending G to $G + F - r$, so F can be mapped to G by $\tau^{-1} \varphi$. □

Conjecture 5.3. "Generalised coordinate" is an extension of the concept of "stable coordinate". In other words, if I is an ideal in $\mathbb{C}^{[n+m]}$ and if $r \in \mathbb{C}^{[n+m]}/I$ is a generalised coordinate, and $\mathbb{C}^{[n+m]}/I \cong_{\mathbb{C}} \mathbb{C}^{[n]}$ then r is a stable coordinate.

Examining polynomials of the form $P(X_1, \ldots, X_n) Y + Q(X_1, \ldots, X_n)$ might be a good idea in combination with the next question:

Question: Is there an algorithm which decides of (lots of) $F \in \mathbb{C}[X_1, \ldots, X_n]$ if there exists a ringautomorphism φ such that $\varphi(F)$ is linear in X_n ?

Another possible different approach of extending the concept of (stable) coordinate to a more general ring is looking for (stable) slices in such a ring:

Definition 5.4.

(i). Let R be a finitely generated $\mathbb{C}$-algebra. Say $s \in R$ is a *slice in* R if there exists a locally nilpotent $\mathbb{C}$-derivation on R such that $D(s) = 1$.

(ii). Let R be a finitely generated $\mathbb{C}$-algebra. Say $s \in R$ is a *stable slice in* R if there exists some $n \in \mathbb{N}$ and a locally nilpotent $\mathbb{C}$-derivation on $R[T_1, \ldots, T_n]$ such that $D(s) = 1$.

"Slice" and "stable slice" are extensions of the concept of coordinate, since every coordinate over a polynomial ring induces a locally nilpotent derivation having the coordinate as slice. Compare also lemma 2.2 part 4. So we can ask the same question for "stable slice" as we did for "generalised coordinate" (conjecture 5.3):

Conjecture 5.5. "Stable slice" is an extension of the concept of "stable coordinate". In other words: let $(f_1, \ldots, f_m) = I \subset \mathbb{C}^{[n]}$ be an ideal. Let $s \in \mathbb{C}^{[n]}$. Then s is a stable slice in $\mathbb{C}^{[n]}/I$ if and only if $s + f_1T_1 + \ldots + f_mT_m$ is a stable coordinate in $\mathbb{C}^{[n+m]}$.

Independently of the conjectures 5.3 and 5.5 one can make the following (two) conjecture(s):

Conjecture 5.6. Let $s \in \mathbb{C}[X_1, \ldots, X_n]$. Then

(i). s is a stable slice $\Longrightarrow$ s is a generalised coordinate.

(ii). s is a generalised coordinate $\Longrightarrow$ s is a stable slice.

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