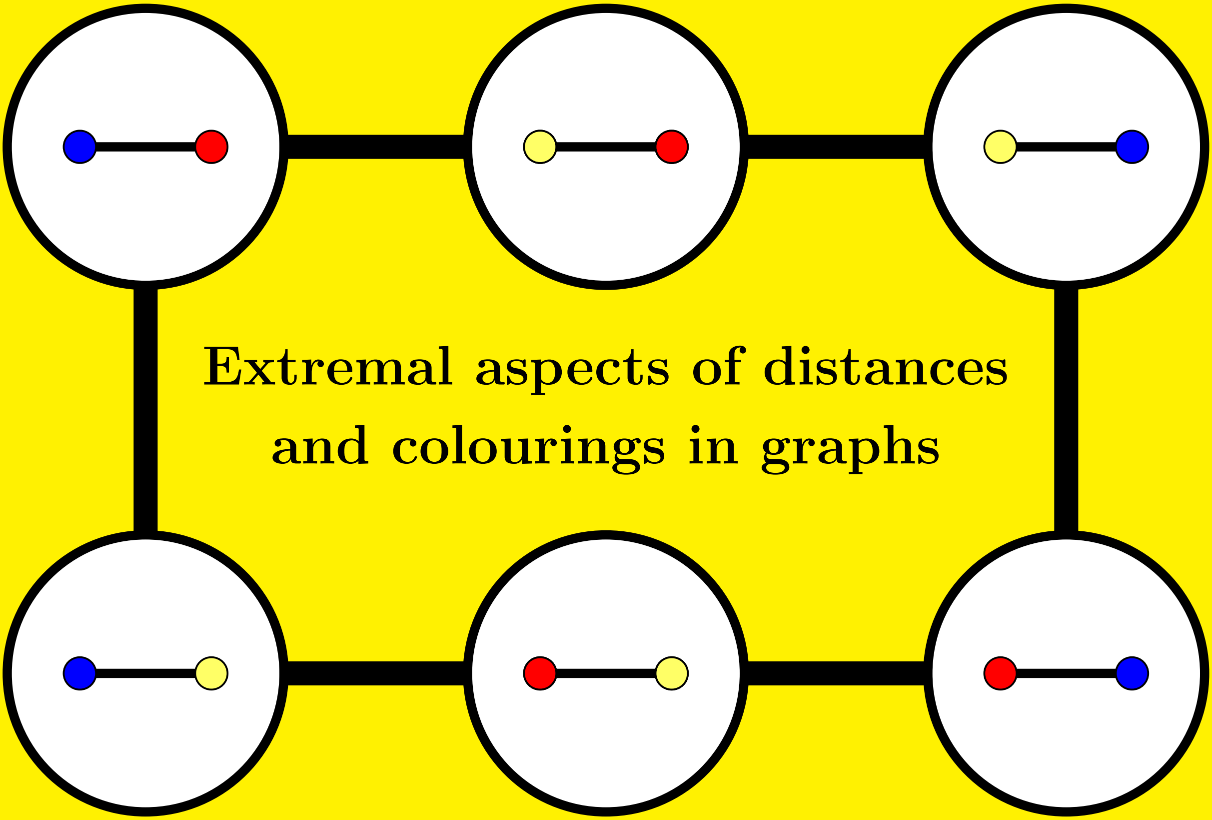
Extremal aspects of distances and colourings in graphs



Stijn Cambie

Extremal aspects of distances and colourings in graphs (short version)

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Stijn Cambie

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Promotor:
prof. dr. E. Cator

Copromotor:
dr. R.J. Kang

Manuscriptcommissie:
prof. dr. A.C.C. Coolen (voorzitter)
prof. dr. T. Szabó (Freie Universität Berlin, Duitsland)
prof. dr. P. Dankelmann (University of Johannesburg, Zuid-Afrika)

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Dedicated to all good hearted people,
and Tim Eppink in particular

Contents

0

Introduction

"Not feeling ready, does not make you less able to go through it!"
– Unknown

The main content is written in English, as most of research is an international occupation. As such, colleagues and friends who wanted this shorter version of the thesis can still enjoy a reasonable introduction. On the other hand, for family and friends who would like a full version in Dutch, the summary at the end should be sufficiently complete for you. This short version is still not that short, as including the main results and figures from each of 13 chapters (12 in this version), implied that the whole book is still reasonable thick. We tried to include the main pictures, such that at least the impression is as good as would be with the full version (which is 140 pages longer and already does not include all results of the PhD time).

For fellow PhDs working in more abstract fields (or with less pictures), it may be easier to write a short version which gives just the main ideas. Remember that if you do not have a look to all pages here, your readers won't do either. It may be a reminder that some paper can be saved.

The thesis deals with problems in what one can call extremal combinatorics, focused on graph theory. Defining combinatorics exactly is already a hard task, as noted in e.g. the series of interviews with famous combinatorialists in *Enumerative Combinatorics and Applications*. No definition will be entirely satisfactory according to Richard P. Stanley, see http://ecajournal.haifa.ac.il/Volume2021/ECA2021_S3I1.pdf. But let us try to do so. It is a subfield of mathematics in which one studies finite or at least discrete objects or patterns. So noting that almost everything is inherently finite, combinatorialists can study problems related to almost anything. When focusing on particular optimisation problems, one is working in extremal combinatorics. Here one tries to determine or estimate the minimum or maximum (or other optimum) under certain restrictions. Once knowing the optimum, we are also often interested in the structures that do attain that bound.

To summarise, in some sense we are just doing an exploration where we look for the best or worst cases, i.e. the most extreme configurations. These are the most beautiful ones to observe mathematically speaking. In many applications, one is also interested in the best strategy, the biggest profit or the cheapest way of transporting some goods.

We will explore some of these questions in a very fundamental setting, noticing connections with many other sub-areas of mathematics. Among other reasons, since a lot of the fundamental questions in other areas such as computer science and cryptography are in essence combinatorial, it is interesting to work in this field and add more understanding and intuition about the logic behind it. Hereby we are working on some problems which have some history and are open for a longer time. It would have been easy to present a thesis if we unravelled everything behind conjectures such as e.g. the Alon-Krivelevich or list-colouring conjecture, but here we only can add some insights to some of them.

As mentioned before, the main chunk of the thesis is about graphs. Graphs are important objects used to represent networks, e.g. contact networks (interesting when studying the spread of diseases) and transportation networks. Graph theory originates from the 18^{th} century when Leonhard Euler was thinking about the Seven Bridges of Königsberg. In the meantime, it has become a big field of research on its own related to many other topics.

Graphs and some related concepts will be formally defined in the next sections, Section 0.1 gives some ideas on the topic for those who are unfamiliar with graph theory and Section 0.2 summarises the formal definitions and notation. Once done that, we give a short overview of the content in Section 0.3. Since some content is also due to discussions and collaborations with others, in Section 0.4 all of them have been acknowledged.

0.1 An invitation to graph theory

Graphs are the main objects we will be working on in the thesis. So it is important to give some ideas about what they are and what they can represent.

A graph can be presented as a couple of sets V and E. Here V is a set of objects and E contains the pairs of objects that are related to each other. The objects in V will be called vertices (or nodes) and the pairs of vertices in E will be called edges. A graph can be depicted by having dots for the vertices and a line for an edge between the two vertices. In Figure 2 we have a depiction of graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{\{v_1, v_2\}, \{v_1, v_4\}\} \cup \{\{v_i, v_j\} \mid 2 \le i < j \le 5, (i, j) \neq (2, 4)\}$, which is the example we will use to explain some notions. For example, one can think of V being a set of people and E being the pairs of people that are in close contact with each other when one wants to study the spread of a virus. In such a case, the degree of a vertex v in V will be equal to the number of people person v sees regularly. So the degree of vertex v_1 is two for example.

As a rather concrete example, we can consider the Erdős collaboration graph. The vertices of this graph are mathematicians. Here two mathematicians are joined by an edge whenever they co-authored a paper together. So a new paper can introduce new edges. Hence we are not speaking about a fixed graph. In Figure 1 part of the Erdős collaboration graph is shown. The thick edges represent papers that are classified by MathSciNet at the moment of writing, while dashed edges represent collaborations that are not processed yet. Someone's Erdős number is nothing more than the minimum distance between that person and Erdős in the Erdős collaboration graph. As an example, Michiel de Bondt has currently an Erdős number of four by the sequence of collarobations Michiel de Bondt – Henk Don – F.M. Dekking – J.O. Shallit – Paul Erdős. Once some additional papers are fully classified, this number decreases to three as a shorter path is introduced: Michiel de Bondt – Stijn Cambie – Noga Alon – Paul Erdős.

Another option is that two people are connected (they form an edge in the graph) if an email has ever been sent between the two. When we are dealing with a computer virus that gets forwarded from one person to all its acquaintances, the distance between two people (vertices) v and u will be equal to the minimum number of forwarding emails it takes to reach u when initially only v has the computer virus on his or her computer.

The distance from v_1 to v_5 in our example (Figure 2) equals 2, as one can reach v_5 in two steps (and not in one step) from v_1 by taking the edges v_1v_2 and v_2v_5. Various types of graphs (and related versions) are used in e.g. modelling of networks, optimal planning in logistics and computing the shortest distance when travelling. There is an extension called digraph which can handle one-way traffic as well. In a digraph the edges are directed, i.e. ordered pairs (called arrows).

Degree and distance are very basic parameters for a graph and some fundamental questions related to these are studied in Part I. A clique is a set of vertices, all of which are connected to each other. One can think of a group of friends forming a clique, all of them knowing each other. For example $\{v_3, v_4, v_5\}$ form a clique on 3 vertices. In this case we have the maximum number of edges, the number of edges being the size among these vertices. Looking again to the part of the collaboration graph in Figure 1, we note that a paper with a group of people induces a clique between them by definition, e.g. with {S. Cambie, W. Cames van Batenburg , R.J. Kang} we have a clique of size 3, a triangle, due to e.g. the work [6]. That work actually inducing a clique on 6 vertices. But it is also possible to spot cliques in this graph with a certain number of vertices, for which this group of people have not collaborated in a single paper. E.g. {R.J. Kang, N. Alon, J. Pach} form a triangle when taking the recent papers into account as well, but the three edges are coming from different

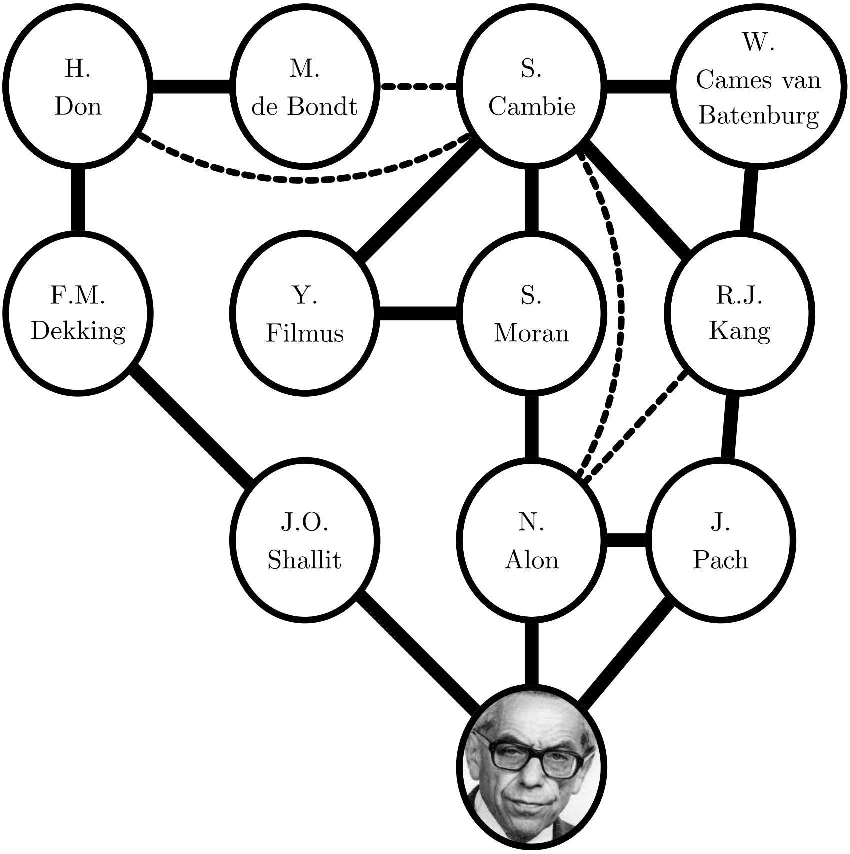


Figure 1: Part of the Erdős collaboration graph

collaborations.

In Part II we study some results on these concepts.

In Part III we also consider colouring of graphs. In its most basic form, one wants to assign a colour to every vertex $v \in V$ such that every two vertices connected by an edge receive a different colour. This would not be hard if one would not go for the least number of colours possible for this. The least number of colours needed for a certain kind of colouring will be called a chromatic number χ and multiple versions of this notion do exist.

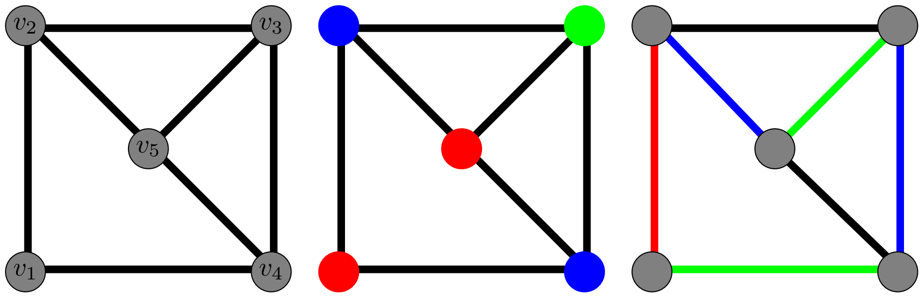


Figure 2: Example of a graph G and possible colourings

For our example, we have $\chi(G) = 3$ and the related chromatic index when we colour the edges properly, will be $\chi'(G) = 4$ This type of problem is related to for example scheduling problems (e.g. exam scheduling such that no student has two exams at the same time), frequency assignment problems (no two radio stations with an overlapping broadcast area may have the same frequency) or register allocation (in compiler optimisation, one cannot use the same variable twice). When colouring all vertices in a clique properly, every vertex needs a different colour. So there is a relation between the clique number ω of a graph (representing the largest clique) and the chromatic number χ of that graph as we see already that $\omega \le \chi$. Much more of these kinds of relationships will be mentioned in Part III.

0.2 Formal terminology and notation for graph theory

A graph will be denoted by $G = (V, E)$. Here V, the vertex set, and E, the edge set, are always considered to be finite in this thesis. A vertex is mostly assigned the letter v, but also other options such as e.g. u, w, x, y, z can be used, possibly with some index. An edge between two vertices u and v can be written as uv, but sometimes it is also presented by e (and f when an other edge comes into play, or possible with an index). When uv is an edge, u and v are said to be *adjacent*. Two edges e and f are incident if they share an end-vertex. It is somewhat common abuse of notation to use $v \in G$ and $e \in G$ instead of $v \in V$ and $e \in E$.

The order $|V|$ is generally denoted by n, while the size $|E|$ is denoted with m. Nevertheless, m is also used for the matching number of the graph, which is the maximum number of disjoint edges in G.

The complement of a graph G is the graph $\overline{G}$ with vertex set V and edge-set $\binom{V}{2} \backslash E$. A graph $H = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. Furthermore we say H is an induced subgraph of G, denoted by $H = G[V']$, if

$E' = E \cap \binom{V'}{2}$. Two subgraphs $H, H' \subset G$ are independent if they do not share a vertex and there is no edge between a vertex of H and a vertex of H'. If the vertex-set V can be partitioned as $V_1 \cup V_2$ where $G[V_1]$ and $G[V_2]$ are independent, the graph G is disconnected. If not, it is connected. In this thesis, we are almost only working with connected graphs.

With P_n, C_n, S_n and K_n we refer to a path, cycle, star and complete graph respectively on n vertices. A presentation of these graphs for $n = 5$ is shown in Figure 3. Note that edges do not need to have the same distance in a drawing, nor does a path have to be straight. Related to this, there is the notion of a planar graph, which is a graph that can be drawn without crossings on a plane. For example, K_5 is not planar, while K_5^-, the graph K_5 minus one edge, is planar. Planar graphs have a certain number of faces, f, which are regions bounded by some edges, except from one infinite region. For every connected planar graph it is true that $n - m + f = 2$ by a formula of Euler.

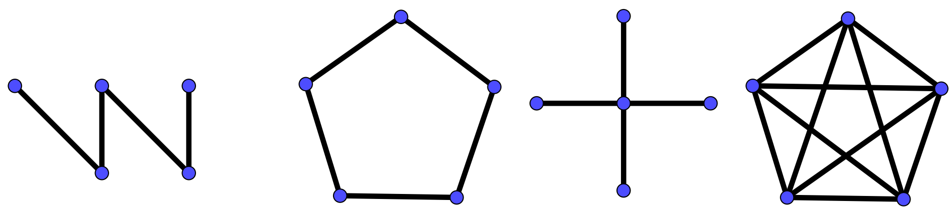


Figure 3: The graphs P_5, C_5, S_5 and K_5

Now we start with some more specific terminology for Part I.

Let $d(u, v)$ denote the distance between vertices u and v in a graph G, i.e. the number of edges in a shortest path from u to v. This notion can be infinite for disconnected graphs. The eccentricity of a vertex v, denoted $\mathrm{ecc}(v)$ or $\varepsilon(v)$, equals $\max_{u \in V} d(u, v)$. The diameter of a graph on vertex set V equals $\max_{u,v \in V} d(u, v)$, which is equal to $\max_{v \in V} \varepsilon(v)$, while the radius of G equals $\min_{v \in V} \varepsilon(v)$. The total distance, also called the Wiener index, of a graph G equals the sum of distances between all unordered pairs of vertices, i.e. $W(G) = \sum_{\{u,v\} \subset V} d(u, v)$. The average distance of the graph is $\mu(G) = \frac{W(G)}{\binom{n}{2}}$. The eccentricity of a graph G is the sum of the eccentricities over all vertices, i.e. $\varepsilon(G) = \sum_{v \in V} \varepsilon(v)$. For an edge $e = uv$, $\mathrm{n}_u(e)$ will be equal to the number of vertices x for which $d(x, u) < d(x, v)$. The degree (number of neighbours) of a vertex v will be denoted $\deg(v)$. This is equal to the order of the open neighbourhood $N(v) = \{u \in V \mid uv \in E\}$. We also will use the k^{th} neighbourhood $N_k(v) = \{u \in V \mid d(u, v) = k\}$. The maximum degree $\max_{v \in V} \deg(v)$ and minimum degree $\min_{v \in V} \deg(v)$ of a graph G, will be usually indicated by Δ and

δ respectively. If $\Delta = \delta = r$, the graph will be called r-regular.

A digraph will be denoted by $D = (V, A)$ and all concepts of a graph extend to digraphs, sometimes with an in- and out-version since the edges are directed in this case. The total distance of a digraph D, $W(D) = \sum_{u,v \in V} d(u, v)$ is now a sum of $n^2 - n$ distances and so the average distance is $\mu(G) = \frac{W(G)}{n^2-n}$. There are now two types of eccentricities. The outeccentricity is $\text{ecc}^+(v) = \max_{u \in V} d(v, u)$, while the ineccentricity equals $\text{ecc}^-(v) = max_{u \in V} d(u, v)$. The diameter is defined as before, but in the case of digraphs there is both an in- and outradius, $\text{rad}^-(v) = \min_{v \in V} \text{ecc}^-(v)$ and $\text{rad}^+(v) = \min_{v \in V} \text{ecc}^+(v)$. Also, there are indegrees and outdegrees.

While Part I is focused on questions related to the previous notions associated with distance, in Part II and III we will work with some other aspects of graphs. The clique number $\omega(G)$ is the order of the largest clique (as a subgraph) in a graph G. This is equal to the independence number of the complementary graph, where the independence number $\alpha(G)$ gives the order of the largest independent set in the graph G. The number of independent sets or cliques of order t are denoted by $i_t(G)$ or $k_t(G)$ respectively.

The line-graph of G, $L(G)$, is the graph with E as vertex set and the pairs of incident edges of G as edges of $L(G)$. For a graph G, the t^{th} power G^t is the graph on the same vertex set V and an edge set having exactly the pairs of vertices which are at distance at most t from each other. The chromatic number of G, $\chi(G)$, is the smallest number of partition classes in a partition of V into independent sets. These partition classes are also called colour classes as colouring every vertex with a representative of the class gives a proper colouring (no neighbours have the same colour). So alternatively, $\chi(G)$ is equal to the least number of colours needed in a proper vertex-colouring of G. Graphs for which $\chi(G) = 2$ are called bipartite graphs due to the bipartition into independent sets.

The degeneracy of a graph G, $\delta^\star(G)$ equals $\max_{H \subset G} \delta(H)$. If the degeneracy equals k, one can order the vertices as $v_1 v_2 \ldots v_n$ such that every vertex v_i has at most k neighbours v_j with $j > i$. For this, let v_i be iteratively chosen as a vertex of minimum degree in $G[V \backslash \{v_1, \ldots, v_{i-1}\}]$. The colouring number of G, $\text{col}(G)$ is equal to the degeneracy of G plus one. One can properly colour G with $\text{col}(G)$ colours by iteratively assigning colours to $v_n, v_{n-1}, \ldots, v_1$, by choosing a colour for v_i different from the ones assigned to its neighbours v_j with $j > i$. This implies that $\text{col}(G) \geq \chi(G)$. More notions and relations between them will be presented in Part III.

0.3 Overview of content

There are two main parts in the thesis, Part I and Part III. Here we deal with respectively distances and colourings of graphs. Part II contains some additional material that connects the main focus of distance and colouring of graphs. We note that there are connections between total distance and size, size and cliques and cliques and colourings. In Part IV we end our study with one neat result on graph reconfiguration. Here distances and colourings of graphs come nicely together, as we are interested in the diameter of a graph whose vertices are graph colourings. In that sense Part I and Part III come together in Chapter 11. As an addendum, we also have Chapter 12, which is about tiles that perfectly fit together in the sense that they cover and pack the three-dimensional space. This additional chapter is also disjoint from the rest and contains some material that is easier to understand. So mathematicians not into combinatorics still can have some fun with this and people not into mathematics may understand the beginning of the chapter.

To start the content of Part I, we first give some introductory results in Chapter 1, hereby starting with some folklore results and mentioning some basic examples of extremal problems related to it. More details can be found in [14] and [26]. Also [11], the first paper on this topic by the author, could fit here. In Chapter 2 we determine the gap between the diameter and the maximum average distance among graphs given large order and fixed diameter asymptotically. In that sense, it is an asymptotic resolution of an old and elementary question of Plesník [59] from 1984. This is based on [12]. For digraphs and n sufficiently large compared to the diameter d, we determine the precise maximum for the total distance and characterise the unique extremal digraph.

In Chapter 3 we investigate the extremal graphs that attain the minimum total distance given order and radius. Again, this question is answered asymptotically, i.e. for n sufficiently large in terms of the radius r. By doing so, we confirm a conjecture of Chen, Wu and An [33], even while we found a counterexample for $n = 8$ and $r = 3$. This chapter is based on the work in [13].

We start Part II with a continuation of Chapter 3, but in Chapter 4 we focus on the maximum size of a graph (and digraphs) given their radius and order. Here we note the relationships between the two extremal questions, on the maximum size and the minimum total distance. This chapter is based on [17].

In Chapter 5 we give two examples on relations between size and cliques. On the one hand, we work on Turán-like problems. Turán [64] proved upper bounds on the size when a clique is forbidden. A special case of this being the famous theorem of Mantel [54] that the balanced complete bipartite graph has the largest size among all triangle-free graphs of a given order. We prove a regular version of this. Here we conclude that for regular graphs with an odd number of vertices, the maximum size is much smaller. In the other direction, we also look for graphs maximising the clique count given size and order. More details can also be found in [23].

Chapter 6 deals mostly with strong cliques (edge sets such that every 2 edges are not too far from each other). Since the size of a strong clique gives a lower bound for the number of colours needed for a strong edge-colouring and there is a relation between ω, χ and Δ, we tackle problems related to both clique numbers and colouring.

In this chapter we study a problem posed by Erdős and Nešetřil [38]. In the chapter, we give some more history for the reader than has been done in the corresponding paper [20].

We start Part III with the definitions of list-colouring in Chapter 7 such that a short survey with some of the main and most elegant results on list-colouring can be presented. We also mention more related concepts and show how they relate.

In Chapter 8 we mention some examples of problems related to other important conjectures in the field of graph colouring. Here we show that a conjecture of Füredi et al. [44] would be a corollary of Hadwiger's conjecture [50]. Two sections have partially appeared as papers, see [6] and [15]. The third section in that chapter being inspired by an online workshop organised by "A Sparse (Graphs) Coalition"[1] in which a group was formed together with Vilhelm Agdur, Annika Heckel, Guillem Perarnau and Jan Volec.

Leaving a survey and some examples behind, in the last three chapters we do some deep-dive on list-colouring. In Chapter 9 we look to a conjecture of Alon and Krivelevich [4] and make some progress in an asymmetric setting, hereby showing that if the lists of vertices in one partition class have size $\log \Delta$ and in the other partition class have size $(1 + o(1))\frac{\Delta}{\log \Delta}$, one can properly colour the vertices with colours from their lists. We generalise this and also consider some interesting cases in the more general setting of independent transversals. These two (in the full thesis) chapters are based on the work of [2] and [27].

We finish Part III with the introduction and study of the concept of list packing in Chapter 10. Instead of finding one colouring, we want to have a certain collection of disjoint colourings, the number of disjoint colourings being equal to the length of the lists. More on this topic appears in the joint work [19].

Part IV brings together the two main notions in Chapter 11, as we are dealing with the diameter of a reconfiguration graph. A reconfiguration graph $\mathcal{C}_k(G)$ is a graph whose nodes are proper k-colourings of a fixed graph G. We prove a linear bound when $k \geq \Delta + 2$ and give a sharp result for $k \geq 2\Delta + 1$. By doing so, we prove that Cereceda's conjecture is not sharp when dealing with regular graphs. This study is initiated in the fourth online workshop on Graph Reconfiguration organised by "A Sparse (Graphs) Coalition". Here a group was formed together with Marcin Briański, Wouter Cames van Batenburg and Marc Heinrich. A more precise form will appear in further work together with Wouter Cames van Batenburg and Daniel Cranston.

The other chapter in the final part is on tilings. So here we bring everything together in a different sense, in the sense that we cover everything and have a perfect packing with copies of tiles. Chapter 12 is an independent chapter that starts with an introduction on tiling, intended to be (partially) accessible for everyone. The new results are published in [16].

Finally, the goal of the Dutch summary is to ensure that the ideas of the content

[1] See https://sparse-graphs.mimuw.edu.pl/doku.php?id=start

are understandable for every (Dutch-speaking) non-mathematician. Here we explain the topics investigated during the PhD comprehensively, without going into any technicalities.

0.4 Acknowledgement of contributions

To end this introduction, I would like to acknowledge all people who have contributed content-wise to the story of the thesis.

While the content of Part I is my work, of course, I want to thank Ross for the guidance here. Proofreading when writing my first articles and giving hints on what possible directions to explore are valuable contributions for sure which improved the final work as well.
Also, I thank John Haslegrave for some fruitful thinking together, which lead to completely addressing the full conjecture of [51]. As such, the results in Section 1.3 were extended in [26].

I have enjoyed the thinking process with Rémi and Ross as well, which lead to the content of Chapter 5 and the joint paper [23] which somewhat supersedes the chapter. Wouter has to be added to the list of collaborators when we continue with Chapter 6. With the exception of an overview of some previous research, the content of the chapter is mainly equal to our work in [20] (except for omitting the determination of $h_3(3)$). The communication with the three of them influenced the final presentation of Part II.

Part III was even more influenced by Ross than Part II. He came with more possible directions for research related to older conjectures and results. It was a pleasure to write the short survey in Chapter 7 based on some of his favourite literature.

The content of Chapter 8 has been influenced by many people. Even while only some of all our ideas have been presented to give a glimpse to the readers, I like to thank N. R. Aravind, Wouter Cames van Batenburg, Rémi de Joannis de Verclos, Ross J. Kang and Viresh Patel, as well as Vilhelm Agdur, Annika Heckel, Guillem Perarnau and Jan Volec with whom I formed a group in an online workshop organised by “A Sparse (Graphs) Coalition”[2].

Chapters 9 (and the extension) are without any doubt greatly influenced by ideas of Ross and Noga Alon. The full content can be found in the papers [2] and [27]. Also, Chapter 10 has only been written in the way it is, due to the collaboration with Ewan Davies, Ross and Wouter. Up to the addition of some examples we spoke about, the chapter is compiled from the selection of two of our main results in [19].

Referring to Chapter 11, I want to thank Marcin Briański, Wouter Cames van Batenburg, Daniel Cranston and Marc Heinrich for the discussions.

[2] See https://sparse-graphs.mimuw.edu.pl/doku.php?id=start

Part I

Extremal distance

1

Introductory results on extremal distance

You cannot solve a problem you do not understand.

– Zizi Abok

In general, mathematicians would prefer to solve certain hard, long-standing problems and conjectures. But before being able to do so, they at least need to understand the basics. That is why we will start with this chapter containing some folklore results and (sub)questions having shorter proofs. In this way, we may have a better feeling on the topic for the next chapters of this part where we deal with harder questions being open for a longer time.

In this part, we are mainly working with (distance-based) topological (graph) indices. These are numerical parameters characterising some properties or geometry of the abstract presentation of a graph (so they are independent of the presentation of the graph). They are e.g. used as predictor variables in quantitative structure-activity relationship models. These classification models are used for example in biology, chemistry, engineering and pharmacy. QSAR models are regression models that predict the performance based on known values for a number of test samples of some parameters such as graphical indices. Based on this, one can predict which substances or molecules will perform best. By only testing the most promising candidates at the end, one saves a lot of time and money as one does not have to test all candidate substances. Depending on the topological index, it can be related to different (sometimes unexpected) fields such as e.g. architecture and urbanism [61], network theory or random graph theory.

One indication of the growing interest in graphical or topological indices is the fact that this subfield got its own code ($05C09$) in the MathSciNet Classification of

2020.

The oldest and one of the most important topological indices is the total distance or Wiener index [68], which is linearly related for fixed order with the average distance of a graph. Being a natural and very useful concept, it has relations with other parameters of graphs as well. As an example, in [45] the authors showed that there is a relation between total distance and cover cost of a graph, a standard notion in the theory of Markov chains.

As mathematicians, we want to understand the underlying relationships. In extremal combinatorics, we want to know the best- and the worst-case under some assumptions. These are the extreme cases. Hereby, the more elementary, elegant and natural the question, the more important it is in general. We will be working towards some more fundamental questions on the average distance given radius and diameter, but as mentioned before, we start with a broader exploration with some folklore and some shorter proofs.

1.1 Starting with the folklore results

In this section, we start with some folklore results on some minimum and maximum distance measures for graphs, where we focus on connected graphs. We start with the extremal graphs (and trees) for the total distance given the order.

Theorem 1.1.1. *For any connected graph G, we have*

$$W(K_n) \le W(G) \le W(P_n)$$

and the extremal graphs are unique.

Proof. The lower bound is trivial as $d(u,v) \ge 1$ for every $u \ne v$ and K_n is the only graph for which all distances are equal to 1. For the upper bound, one can prove this by induction on the order noting that every connected graph G has a vertex v such that $G \backslash v$ is connected as well. Then

$$\begin{aligned} W(G) &\le W(G \backslash v) + \sum_{u \in V} d(u,v) \\ &\le W(P_{n-1}) + \sum_{i=1}^{n-1} i \\ &= W(P_n). \end{aligned}$$

□

Note that the first inequality is not necessarily an inequality, e.g. by removing a vertex in a cycle also other distances not involving v can increase. The inequality

$\sum_{u \in V} d(u,v) \le \sum_{i=1}^{n-1} i$ is a consequence from the observation that if $d(u,v) = i$, the distances from u to the vertices on a shortest path from u to v are exactly equal to $1, 2, \ldots, i$. The quantity $\sigma(v) = \sum_{u \in V} d(u,v)$ has been called the transmission of a vertex in the past. Here the transmission of a graph is just a synonym for the total distance of the graph.

Theorem 1.1.2. *For any tree T, we have*

$$W(S_n) \le W(T) \le W(P_n)$$

and the extremal trees are unique.

Proof. The upper bound was true for general graphs. For the lower bound, we note that in general $W(G) \sum_{u,v \in V} d(u,v) \ge |E| + 2\left(\binom{n}{2} - |E|\right)$, where equality only occurs if the graph has diameter 2. The size $|E|$ equals $n-1$ for any tree and the trees with diameter 2 are exactly the stars. □

One can observe that the extremal graphs/trees attaining the minimum or maximum Wiener index are exactly those having the minimum/maximum diameter. They are also extremal regarding the radius, but then they are not necessarily the unique extremal graphs.

The same graphs are also extremal when looking at the eccentricity of the graph.

Theorem 1.1.3. *For any connected graph G, we have*

$$\varepsilon(K_n) \le \varepsilon(G) \le \varepsilon(P_n)$$

and the extremal graphs are unique. For a tree T, we have $\varepsilon(S_n) \le \varepsilon(G)$.

At this point, we conclude that the graphs of a given order (without other constraints) attaining the maximum or minimum total distance, are exactly the graphs one would expect, being K_n and P_n and this is known for a long time.

1.2 Maximum difference of Wiener index and eccentricity

By the folklore results, we know that P_n and K_n maximise and minimise W and ε among all n-vertex connected graphs. In this section, we look to the connected graphs of order n minimising and maximising the difference of the two quantities, being $W - \varepsilon$.

We start determining the extremal graphs for the maximum and by doing so, we prove Conjecture 4.3 in [37]. For small values of n, the impact of the eccentricity is not neglectable and the extremal graphs behave differently. The intuition is that W is the sum of $\binom{n}{2}$ distances and ε the sum of only n distances. Hence the behaviour for large n can be guessed from the behaviour of W, but the extremal graphs are harder to guess for small n. For $n \le 6$, the extremal graphs are $K_2, K_3, \{K_4, S_4\}, S_5$ and S_6. For $n = 7$, the unique extremal graph is a star S_4 where every edge is subdivided, this graph being depicted in Figure 1.1. Here $(W - \varepsilon)(G) = 25$.

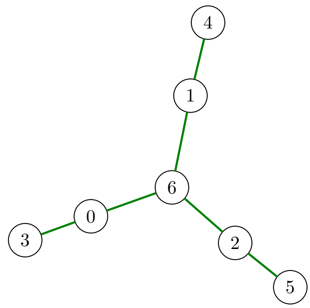


Figure 1.1: Extremal graph for $n = 7$ maximising $(W - \varepsilon)(G)$

There are four graphs attaining the maximum for order 8. These graphs are presented in Figure 1.2. All of them satisfy $(W - \varepsilon)(G) = 40$, being equal to respectively $74 - 34 = 79 - 39 = 67 - 27 = 84 - 44$.

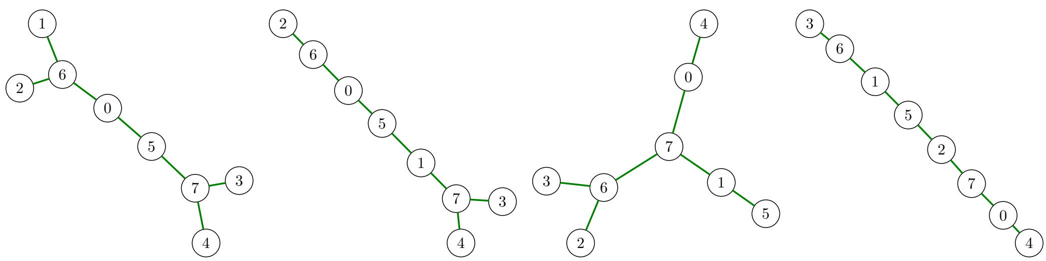


Figure 1.2: Extremal graphs for $n = 8$ maximising $(W - \varepsilon)(G)$

Nevertheless, for $n \ge 9$, the path P_n is the unique extremal graph and so our intuition based on the leading term W is correct.

Theorem 1.2.1. *For $n \ge 9$, among all graphs with order n, $W(G) - \varepsilon(G)$ is maximised by P_n. Here P_n is the unique extremal graph.*

To prove this theorem, we split the work in steps. As a first step, we prove that the extremal graphs are trees when the radius is at least 3.

Having proven this lemma, it is essentially enough to prove that the extremal graph cannot have radius at most 2 and then to check it for trees. For this, a bit of work is needed (as one can expect due to the behaviour of the extremal graphs for $n \le 8$).

We now also determine this minimum among all graphs.

Proposition 1.2.1. *For every graph G of order n, we have $(W-\varepsilon)(G) \ge \left\lceil \frac{n(n-4)}{2} \right\rceil$ and this bound is sharp.*

Proof. Fix a vertex v. There is a vertex u with $d(u,v) = \varepsilon(v)$ and an other one with $d(w,v) \ge \varepsilon(v) - 1$. Hence we know that $\sum_{u \in V \setminus v} d(u,v) - 2\varepsilon(v) \ge \varepsilon(v) + (\varepsilon(v)-1) + (n-3) - 2\varepsilon(v) = n-4$. Summing over all vertices v, we conclude that $2(W-\varepsilon)(G) \ge n(n-4)$. Dividing by 2 and observing that $(W-\varepsilon)(G)$ is always an integer, we obtain the desired result.

Furthermore, this $(W-\varepsilon)(G)$ is attained by the complement of

$$\begin{cases} \frac{n}{2}K_2 & \text{if } n \text{ is even,} \\ \frac{n-1}{2}K_2 \cup K_1 \text{ or } \frac{n-3}{2}K_2 \cup P_3 & \text{if } n \text{ is odd.} \end{cases}$$

□

For $n \ge 6$, this is the exact characterisation of the extremal graphs. The graph P_4 for $n = 4$ and the one sketched in Figure 1.3a for $n = 5$ are also extremal.

(a) Additional graph minimising $W - \varepsilon$ for $n = 5$

(b) S_6 with subdivided edge

Figure 1.3: Extremal graph and tree minimising $W - \varepsilon$

We also prove that the extremal tree is the path P_n when $n \le 6$ and if $n \ge 7$, the extremal tree is attained for the star S_{n-1} with one edge subdivided, this graph being depicted in 1.3b for $n = 7$. This is mainly a corollary of the following lemma, as it implies that we only have to compare a few possible trees. The main idea of the proof is to consider a fixed shortest path between two vertices at maximum distance and using the triangle-inequality multiple times.

Lemma 1.2.2. *Among all trees with fixed diameter d and order n, the minimum of $(W-\varepsilon)$ occurs if and only if we have a path P_{d+1} of diameter d with the $n-d-1$ remaining vertices connected to the same vertex on the path, which is a central vertex if d is odd, or a central vertex or neighbour of it, if d is even.*

1.3 Comparing the (variable) Wiener index and Szeged index

In the full thesis, we prove also more technical results. One section uses concepts such as the variable versions of the Wiener index and Szeged index $W^\alpha(G)$ and $\mathrm{Sz}^\alpha(G)$. We won't explain these. The authors from [51] had 2 conjectures on the difference $h(\alpha) = \mathrm{Sz}^\alpha(G) - W^\alpha(G)$. They claimed that $h(\alpha) \geq 0$ for every graph when $\alpha \geq 1$. This turned out to be true. Essentially, we proved that the necessary condition of the inequality of Karamata [52] does hold for the input, which are certain graph parameters. We also proved that their stronger conjecture, that $h(\alpha)$ can cross the x-axis only once, was true for 100 % of the graphs. As such, it may sound surprising that we also know there are infinitely many counterexamples to their conjecture. The construction we used is presented in 1.4 and is basically a composition of two graphs having a different behaviour.

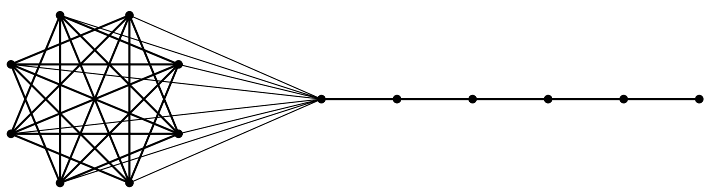


Figure 1.4: The graph $G_{k,\ell}$ for $k = 8$ and $\ell = 5$.

1.4 Maximum total distance given matching number

We also could prove certain results for the extremal graphs in a general way. Instead of solving a very specific problem, we show that a broader class of problems has the same solutions. One elegant idea we use there is the one of pruning and regrafting. We show that we can construct a better graph by essentially resplacing some subgraph at a new place. By doing so, the possibilities for the extremal graphs are reduced.

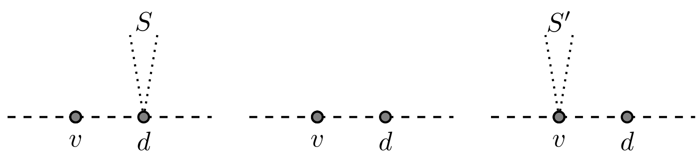


Figure 1.5: the graph G, S being pruned from G and S being regrafted at v

2

An asymptotic resolution of a problem of Plesník

You must tell yourself 'No matter how hard it is, or how hard it gets, I'm going to make it.'

– Les Brown

In this chapter, we look to the natural question finding the extremal graphs that attain the maximum and minimum Wiener index (total distance), equivalently average distance, among all graphs of order n and diameter d. As the diameter and average distance, together with eccentricity and radius, are the most important notions related to distances in graphs, this really is one of the most fundamental questions in the subfield. In 1984, Jàn Plesník [59] determined the minimum average distance among all graphs of order n and diameter d. He did this both for graphs and digraphs and characterised the extremal graphs.

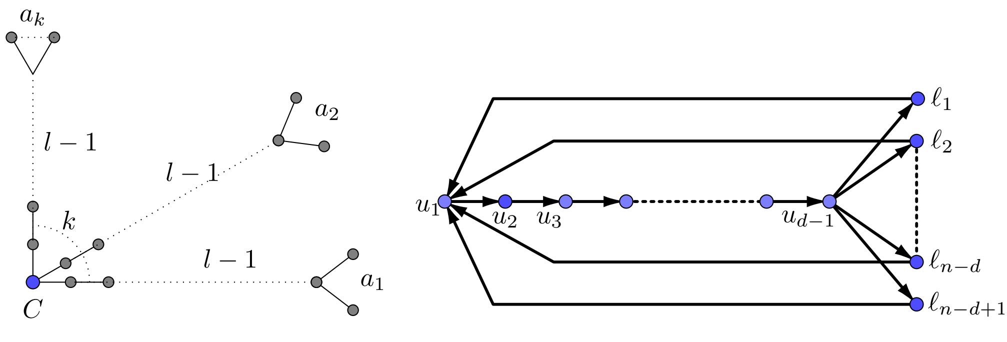
Determining sharp upper bounds depending on n and d has proven to be much more difficult. An open problem that Plesník had already asked was, 'What is the maximum average distance among graphs of order n and diameter d?', both in the case of graphs and digraphs.

We present an asymptotic solution to this longstanding open problem of Plesník. The diameter d, being the maximum distance is an upper bound for the average distance μ. When n goes to infinity, the difference tends to zero. Our contribution[1] was to determine the order of the gap. That difference behaves like $\frac{d^{3/2}}{\sqrt{n}}$.

The main first step in the proof of each of these results is to devise a graph or digraph which is almost extremal. For this, we want many pairs of vertices which

[1]For the full proof, see [12]

are of distance d from one another. In the graph case, we take many subtrees with many leaves. When the diameter is even, we just combine them into one tree. When the diameter is odd, we use a central clique so that the distance between leaves of different subtrees are of distance d. The construction is sketched in Figure 2.1a.



(a) Graph case (b) digraph case

Figure 2.1: Graph and digraph obtaining upper bound

For some intuition about this construction, take two vertices at random. Since the number of leaves is large, the probability that both vertices are leaves is large. Similarly, since we have many subtrees, the probability that both leaves are in different subtrees is large. Hence the probability that two vertices are at a maximal distance is large, implying that the average distance is close to d for this construction. In the digraph case, the construction is even simpler. See Figure 2.1b. Every two vertices ℓ_i and ℓ_j are at distance d. When n is large and we choose two random vertices, the probability that they are both labelled with ℓ is large. Hence the average distance will be close to d again.

In the other direction, we take a graph of diameter d and order n. The idea is that many pairs of vertices cannot be at distance d from each other. If almost all vertices are at distance d from a certain vertex v, their paths towards v have many points in common and so the distance between these vertices is small. To make this rigorous, we apply the pigeonhole principle.

For the digraph case, we need another strategy, since we cannot use the edges in both directions to get short paths between vertices. In this case, we see that if there are many ordered pairs of vertices at distance d, then the distance between some ordered pairs of vertices on the shortest paths are smaller than d. We use this fact in a rigorous, structured way to find a vertex u such that for almost all other vertices v we have $d(u,v) = d(v,u) = d$. From that, we can recover the structure of the extremal digraph.

3

Extremal total distance given radius

Learn to be indifferent to what makes no difference.

– Marcus Aurelius

In this section, we wonder about the minimum and maximum of a graph or digraph with given order and radius.

3.1 A counterexample and an asymptotic result for the conjecture of Chen, Wu and An

We start defining the following graphs and digraphs, which we will prove to be the solutions for the above problem (for n sufficiently large in terms of r).

Let $G_{n,r,s}$, where $n \geq 2r$ and $1 \leq s \leq \frac{n-2r+2}{2}$, be the graph obtained by taking two blow-ups of two consecutive vertices in a cycle C_{2r} by cliques K_s and $K_{n-2r+2-s}$ respectively. Note that $\omega(G_{n,r,s}) = n - 2r + 2$. An example is presented in Figure 3.1.

Let $D_{2r,r,1}$ be a digraph with $2r$ vertices $v_1, v_2, \dots v_r$ and $w_1, w_2, \dots, w_r$, such that there are directed edges from v_i to v_j and from w_i to w_j if and only if $j \leq i+1$ and a directed edge from any v_i to w_1 and from any w_i to v_1.

Let $D_{n,r,s}$, $n \geq 2r$ and $1 \leq s \leq \frac{n-2r+2}{2}$, be the digraph obtained by taking the blow-up of v_1 by a bidirected clique K_s and a blow-up of w_1 by a bidirected clique $K_{n-2r+2-s}$.

In the graph case, the solution was already conjectured by Chen, Wu and An [33].

Conjecture 3.1.1 ([33]). *For any graph G of order $n \geq 2r$ with radius r, $W(G) \geq W(G_{n,r,1})$. Equality holds if and only if $G \cong G_{n,r,s}$ for some s.*

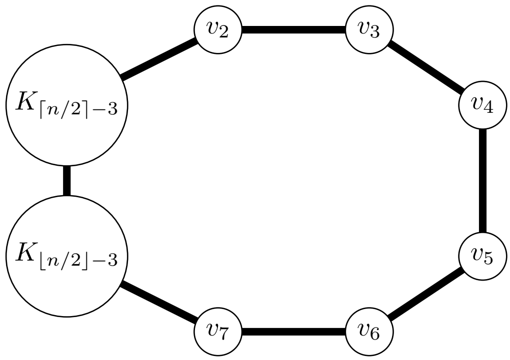


Figure 3.1: The graph $G_{n,4,\lfloor n/2\rfloor-3}$, a graph with radius 4 maximising size

For small values of n for a fixed r, there might be a few exceptions to Conjecture 3.1.1. The graph Q_3 is a counterexample for the equality statement when $r = 3$ and $n = 8$, as it also has a total distance equal to 48. A computer check[1] has shown that this is the only counterexample for $n < 10$.

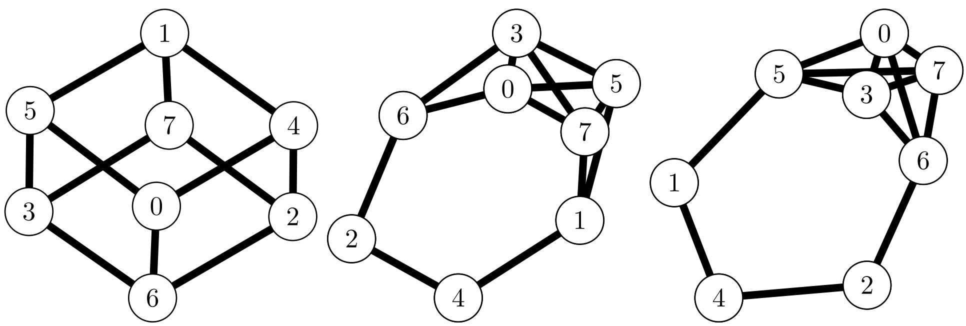


Figure 3.2: The three extremal graphs for $r = 3$ and $n = 8$: $Q_3, G_{8,3,2}$ and $G_{8,3,1}$

Although there are counterexamples to Conjecture 3.1.1 when n is small, we can show that Conjecture 3.1.1 is asymptotically true.

Theorem 3.1.2. *For any $r \geq 3$, there exists a value $n_1(r)$ such that for all $n \geq n_1(r)$, any graph G of order n with radius r satisfies $W(G) \geq W(G_{n,r,1})$. Equality holds if and only if $G \cong G_{n,r,s}$ where $1 \leq s \leq \frac{n-2r+2}{2}$.*

For $r \geq 3$, we propose the digraph analogue to Conjecture 3.1.1. Figure 3.3 shows $D_{n,r,s}$ for $r = 3$.

[1] See https://github.com/StijnCambie/ChenWuAn, document SmallN_CWA.

Conjecture 3.1.3. *Let n and r be two positive integers with $n \geq 2r$ and $r \geq 3$. For any digraph D of order n with outradius r, $W(D) \geq W(D_{n,r,1})$. Equality holds if and only if $D \cong D_{n,r,s}$ for $1 \leq s \leq \frac{n-2r+2}{2}$.*

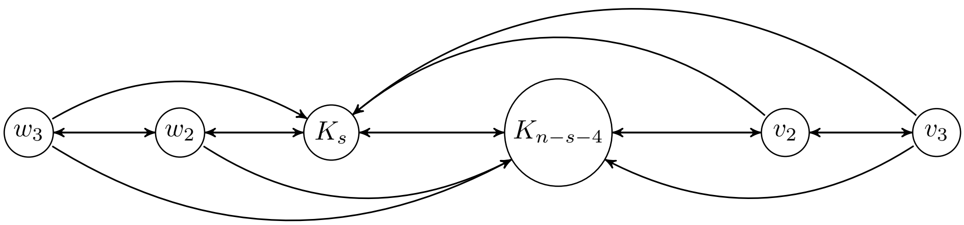


Figure 3.3: The digraph $D_{n,r,s}$ for $r = 3$

Just as in the graph case, like with Conjecture 3.1.1, for fixed r there may be a few counterexamples for small n. Also, Conjecture 3.1.3 is asymptotically true, i.e. for $n \geq n_1(r)$ for some function $n_1(r)$. The interested reader can find the full proof in [13].

3.2 Conjecture of Chen-Wu-An for large order

In this section, we sketch the ideas for the proof of Theorem 3.1.2. Essentially, step by step we reveal more structure of the extremal graphs.

- The total distance of $G_{n,r,s}$ equals $W(G_{n,r,s}) = \binom{n}{2} + (r-1)^2 n - r(r-1)^2$
- If $W(G) < \binom{n}{2} + an$, for some positive constant a, then $\omega(G) \geq \frac{n}{8a}$.
- For n large enough and G a graph of order n and radius r with $W(G) < \binom{n}{2} + an$, we can assume that $G \backslash v$ satisfies the same properties.
- In that case, we have $W(G) \geq W(G \backslash v) + n - 1 + (r-1)^2$ and equality is only possible if some specific substructure is present.
- We take $a = a(r) = (r-1)^2$ and $n_1 := n_1(r) = n_0(r) + a(r)n_0(r)$. Using induction, the previous lemmas and some case analysis, we conclude.

3.3 Maximum total distance of digraphs with outradius 1

We include one elegant proof characterizing the extremal digraphs with $rad^+(D) = 1$. For larger outradius, one can get estimates, but the precise results are unknown.

Theorem 3.3.1. *Given a digraph D of order n with outradius equal to 1, the Wiener index of D is at most $\frac{n^3-n}{3}$ with equality if and only if the graph is isomorphic to one of the two configurations (taking the blue or red arc respectively) given in Figure 3.4.*

Proof. Let v be the center, the vertex for which $d(v,V)=1$. We denote the number of vertices u with $d(u,v)=i$ with X_i. Call this path $uv_{i-1}v_{i-2}\dots v_1v$. Then $d(u,v_j)=i-j$, while for every vertex x different from any such v_j and v,u, we have $d(u,x)\le d(u,v)+d(v,x)=i+1$. Hence $\sum_{x\in V(G)} d(u,x)\le \sum_{j=1}^{i} j+(n-i-1)(i+1)=\frac{1}{2}(2n-i-2)(i+1)$. Note that the parabolic function $f(i)=\frac{1}{2}(2n-i-2)(i+1)$ is an increasing function in i up to $\frac{2n-3}{2}$ and f obtains the same values in $n-1$ and $n-2$. Since i is at most $n-1$, we find that

$$W(G)\le \sum_{i=1}^{n-1} X_i f(i)+(n-1)\le \sum_{i=1}^{n-1} f(i)+(n-1)=\frac{n^3-n}{3}$$

with equality if and only if $X_i=1$ for every $i\le n-3$, from which one can conclude that there are only the two given cases of equality. □

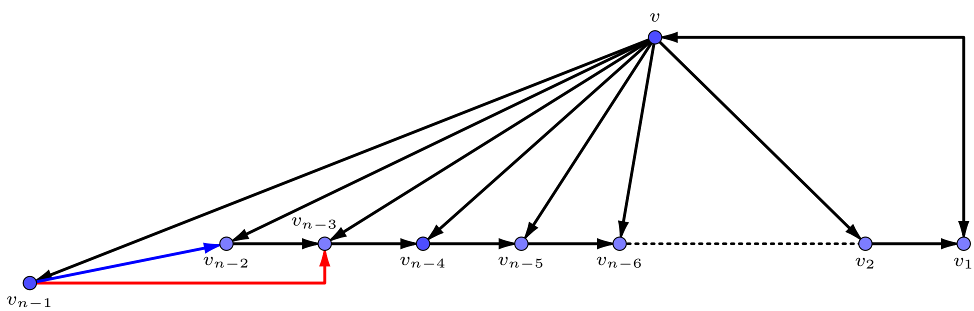


Figure 3.4: The two extremal digraphs with outradius 1 and maximal Wiener index

We end Part I with an overview on the minimum and maximum total distance.

	min $\mu(G)$	max $\mu(G)$	min $\mu(D)$	max $\mu(D)$
rad$^+$/rad$^-$			$1+r^2\Theta\left(\frac{1}{n}\right)$	$\frac{n}{3}+r\Theta(1)$
rad	$1+r^2\Theta\left(\frac{1}{n}\right)$	$2r-r^{1.5}\Theta\left(\frac{1}{\sqrt{n}}\right)$	$1+r^2\Theta\left(\frac{1}{n}\right)$	$2r-r^2\Theta(\frac{1}{n})$
d	$1+d^2\Theta\left(\frac{1}{n}\right)$	$d-d^{1.5}\Theta\left(\frac{1}{\sqrt{n}}\right)$	$1+d^2\Theta\left(\frac{1}{n}\right)$	$d-d^2\Theta(\frac{1}{n})$

Table 3.1: minimum and maximum average distances for digraphs

Part II

From Distance to Colouring over Size and Cliques

4

On total distance and size

It's only by saying "No" that you can concentrate on the things that are really important.

– Steve Jobs

The order and size are the first quantities mentioned in almost any introduction on graphs. As such, it is natural that some of the most fundamental questions are related with the size of a graph. So now we will study the similar-looking question to the one mentioned in Chapter 2, where one is interested in the maximum size instead. In particular, here we give an impression about the extremal graphs and digraphs in a few cases and some observations about possible analogies.

The graphs and digraphs maximising the size given order and diameter were determined by Ore [58, Thr. 3]. The extremal digraphs are depicted in Figure 4.1.

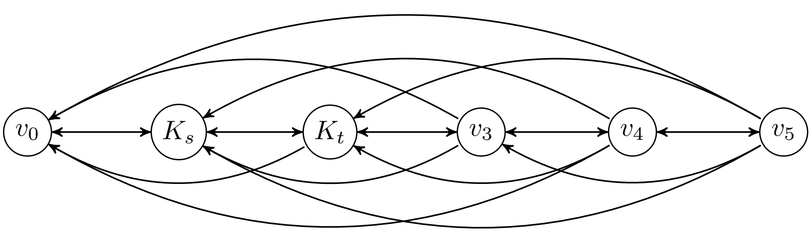


Figure 4.1: The digraph $\overline{\Gamma}_{n,d,1,s}$ for $d = 5$

The graph $\overline{\Gamma}_{n,d,i,s}$ is formed by having $d+1$ vertices $v_i, 0 \le i \le d$, with v_iv_j being an arc if and only if $i \ge j-1$, where 2 non-end vertices are replaced by bidirected cliques K_s and K_t (such that $s + t + d - 1 = n$). When $d = 1$, the unique extremal digraph is obviously (the bidirected) K_n. We include a short alternative prove for the upper bound.

Theorem 4.0.1. *Let D be a digraph of order n and diameter $d \geq 2$. Then its size*

$$|A(D)| \leq (n - d - 1)(n + 2) + \binom{d+2}{2} - 1.$$

Proof. Let D be an extremal digraph. As D has diameter d, it has two vertices u_0 and u_d with $d(u_0, u_d) = d$. Take a shortest directed path $\mathcal{P} = u_0 u_1 \dots u_{d-1} u_d$ between them.

For every vertex v not belonging to this path $\mathcal{P}$, let

$$i_{max} = \max\{i \mid \overrightarrow{vu_i} \in A\} \text{ and } i_{min} = \min\{i \mid \overrightarrow{u_i v} \in A\}.$$

By definition of the distance function, we have $d(u_{i_{min}}, u_{i_{max}}) \leq 2$ and as such $i_{max} \leq i_{min} + 2$. This implies that there are at most $d + 4$ different arrows using v and some $u_i \in \mathcal{P}$. For every two vertices $v, w \in V(G \backslash \mathcal{P})$, there can be at most 2 arrows between them. $\square$

By comparing the extremal (di)graphs attaining the minimum average distance (Plesník [59]) and maximum size (Ore [58]), it turns out that the set of (di)graphs attaining the minimum average distance is a subset of the set of (di)graphs attaining the maximum size.

It is natural to think again that there is a relation between maximising the size and minimising the total distance. More edges may imply smaller distances, in particular, more distances are equal to the minimum of 1. Nevertheless, the correspondence is not exact since the cubical graph Q_3 also minimises the total distance among the graphs of order 8 and radius 3, but it does not maximise the size. The reader can check this while making use of Figure 3.2. On the other hand, we proved that this intuitive idea is true is when n is sufficiently large in terms of r. In 1967, Vadim Vizing [66] determined the maximum size among all graphs of given order and radius. From his proof, one can conclude that the extremal graphs, for $r \geq 3$, are exactly the graphs of the form $G_{n,r,s}$ (defined in Section 3.1). So the correspondence between the extremal graphs is exact for n sufficiently large by Theorem 3.1.2. Also in the digraph case, for n sufficiently large in terms of r, the extremal biconnected digraphs are precisely $D_{n,r,s}$ (Figure 3.3). This has also been proven for $r = 3$ and every $n \geq 6$. Without the restriction of biconnectedness, the digraph can have infinite total distance, so we would not be able to compare at all. Fridman [43, Thr. 5&6] showed that the maximum size of a digraph with given order and outradius is actually attained by digraphs that are not biconnected. They are of the form $\overline{\Gamma}^{\star}_{n,r,i,s}$. An example of that construction is presented in Figure 4.2.

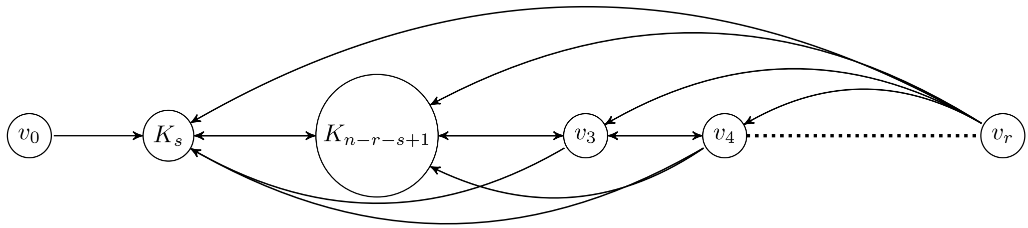


Figure 4.2: An extremal digraph $\overline{\Gamma}^{\star}_{n,r,1,s}$ maximising the size given outradius r and order n

The maximisation problem has been considered under some additional constraints as well, for example when restricting to bipartite graphs. This has been done in [36, Thr. 2]. Note that $G_{n,r,s}$ would become bipartite if one would remove the edges in the cliques K_s and $K_{n-2r+2-s}$ in the extremal graph $G_{n,r,s}$. It turns out that this gives an extremal graph when performing this on $G_{n,4,\lfloor n/2\rfloor-3}$, but the characterisation of the extremal graphs is different. The extremal graphs are obtained by taking blow-ups in 3 consecutive vertices of a cycle C_{2r} with independent sets of order a, b and c respectively where $a+b+c=n-2r+3$ and $0 \le |a+c-(b+1)| \le 1$. This has been done in Figure 4.3 for $n=20$ and $r=4$. Here $1 \le a \le 4$.

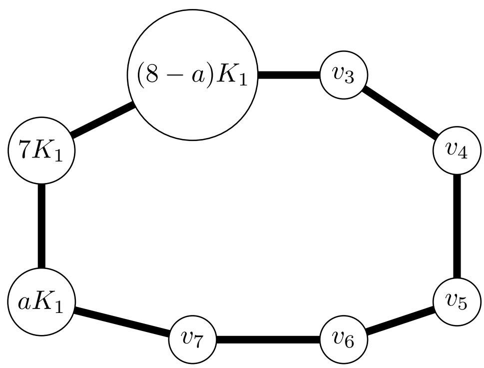


Figure 4.3: The bipartite graph(s) with radius 4 and order 20 maximising size

Knowing the result for graphs, it is natural to wonder about digraphs as well.

Dankelmann [35] considered the same question in the case of bipartite digraphs and determined the sharp upper bound for the size. First note that the bipartite digraph with maximum size is the bidirected $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, the digraph containing every arrow between vertices of two balanced independent sets. The extremal digraph with maximum size given outradius, was of the form $\overline{\Gamma}^{\star}_{n,r,i,s}$

For any bipartite digraph with order n and outradius r, the size can be upper bounded by the size of the intersection of the digraph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ (maximising size given bipartiteness) and some $\overline{\Gamma}^{\star}_{n,r,i,s}$ (maximising size given outradius), where $s \in \{\lfloor \frac{n-r+1}{2} \rfloor, \lceil \frac{n-r+1}{2} \rceil\}$. Nonetheless, as was the case in the graph case, there are way more extremal digraphs. Again there are blow-ups with three stable sets aK_1, bK_1 and cK_1 with $|a + c - (b + 1)| \le 1$. An example is presented in Figure 4.4.

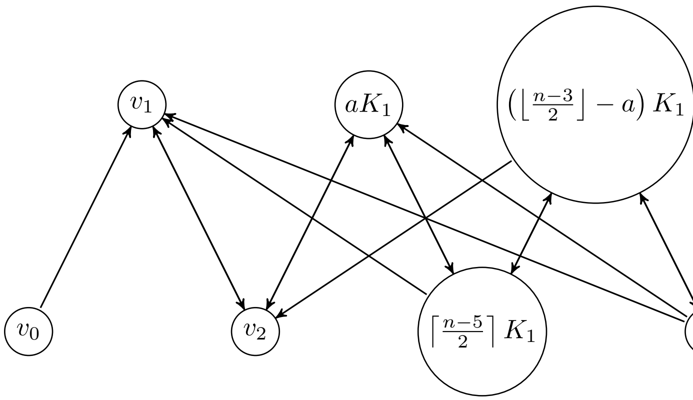


Figure 4.4: An extremal bipartite digraph of outradius 6 and order n maximising the size

For biconnected digraphs, we conjecture that the extremal (bipartite) digraphs are a bipartite subdigraph of a balanced $D_{n,r,s}$. An example of such an extremal digraph is presented in Figure 4.5. This has been proven for n sufficiently large in terms of r, when r is even. Even while not all cases are proven, we observe a clear difference in behaviour. In the biconnected case, there are at most 2 extremal graphs, while there can be $\Theta(n)$ in the general case.

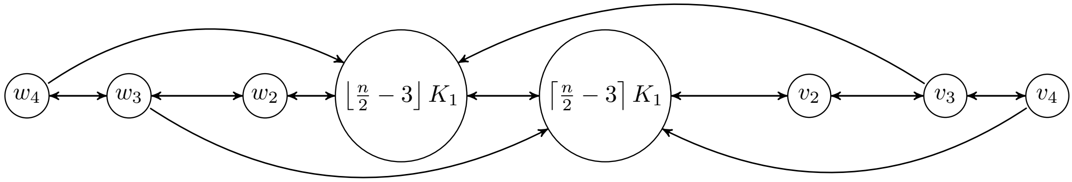


Figure 4.5: The bipartite digraph $D_{n,4,\lfloor \frac{n}{2}-3 \rfloor} \cap K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

5

On size and cliques

The possession of knowledge does not kill the sense of wonder and mystery. There is always more mystery.

– Anaïs Nin

Mantel's theorem [54] and Turán's theorem [64] are foundational theorems in extremal graph theory proving that some information (a lower bound) on the size (for fixed order) is sufficient to know that certain cliques are present in a graph. The theorem of Mantel states that the balanced complete bipartite graph is the graph maximising the number of edges under the condition that it is triangle-free.

Theorem 5.0.1 (Mantel's theorem [54]). *If a graph G on n vertices contains no triangle then it contains at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. Equality is possible if and only if $G \sim K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

In this chapter, we only want to give some indication of the relation between size of a graph and cliques.

We do so by giving two examples. We prove an analogue of Mantel's theorem [54] for regular graphs and give a saturation result as well. This is done in Section 5.1. Since the size is determined by order and regularity in the case of regular graphs, this is an example indicating the main message of the chapter for K_3s.

In Section 5.2 we investigate a question posed in [53] on the clique count given order and size in the critical regime (where the result of [32] does not apply) and present some differences with the regime where the result of [32] does apply.

5.1 Regular Mantel's theorem and its supersaturation version

Here we mention a regular version of Mantel's theorem and a saturation version. For full proofs, see [23]. The main property that has been explored in this way, is a large dependency on the parity of the order. We state our result in terms of the so-called regular Turán number (which we have defined slightly differently to what has been done in [46]).

Definition 5.1.1. The *regular Turán number* of a graph H is

$$\mathrm{ex}_r(n, H) = \max\{k : |V(G)| = n, G \text{ is } k\text{-regular and does not contain } H \text{ as a subgraph}\}.$$

For a family of graphs $\mathcal{H}$, $\mathrm{ex}_r(n, \mathcal{H})$ is defined similarly, so G must not contain any $H \in \mathcal{H}$.

Theorem 5.1.2 (Regular Mantel's theorem)**.** *Let G be a k-regular, triangle-free graph on n vertices. When n is even, we have $k \le \frac{n}{2}$. When n is odd, we have $k \le 2\left\lfloor \frac{n}{5} \right\rfloor$. Moreover, these bounds are sharp. Put in another way,*

$$\mathrm{ex}_r(n, K_3) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 2\left\lfloor \frac{n}{5} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

We can also prove a supersaturation result of the regular version of Mantel's theorem, showing that once the regularity is larger than the threshold of the theorem, we are sure that there are at least $\Omega(n^2)$ triangles present.

Theorem 5.1.3 (Supersaturated regular Mantel's theorem)**.** *Let G be a k-regular graph on n vertices. If n is odd and $k > 2\left\lfloor \frac{n}{5} \right\rfloor$, then G contains at least $\frac{1}{300}n^2$ triangles.*

We can show slightly more.

Theorem 5.1.4. *When n is odd and k is an even number with $2\left\lfloor \frac{n}{5} \right\rfloor < k \le 2\left\lfloor \frac{n}{4} \right\rfloor$, every k-regular graph on n vertices has $\Omega(n^2)$ triangles. Moreover, this is sharp up to the multiplicative constant.*

Proof. The lower bound is proven in Theorem 5.1.3. So now we prove sharpness of the result. Let $n = 2x + 1$ and $2\left\lfloor \frac{n}{5} \right\rfloor < k = x - y \le 2\left\lfloor \frac{n}{4} \right\rfloor$. We construct a k-regular graphs with $O(n^2)$ triangles. For this take a $K_{x,x}$, delete y disjoint perfect matchings and delete another disjoint matching of size $\frac{k}{2}$ (i.e. on k vertices). Now connect the end-vertices of that last matching with an additional vertex v. Every triangle in the resulting graph contains the additional vertex v, from which the conclusion follows. For $k = 2\left\lfloor \frac{n}{5} \right\rfloor + 2$, this gives a construction with approximately $\frac{n^2}{50}$ triangles. □

5.2 Maximising clique number given order and size

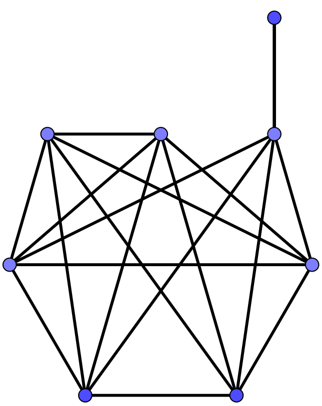
In this section, we focus on the question posed in [53] of maximising the number $k_t(G)$ of cliques K_t in G given both the order and size as well as the maximum degree.

Due to the result in [32] proving the main conjecture in [53] being true, the extremal graph with $n = a(r+1)+b$ (here $b \leq r$) are known when $m \leq a\binom{r+1}{2} + \binom{b}{2}$. It is natural to pose the following analogous conjecture for the remaining cases, i.e. in the critical regime (where one cannot have a copies of K_{r+1}).

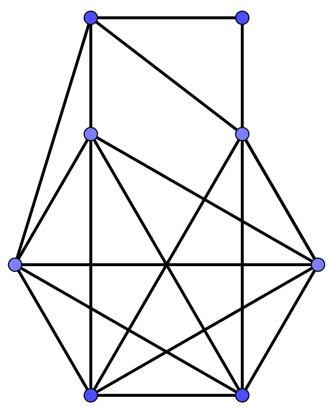
Conjecture 5.2.1. *Let $n = a(r+1)+b$ and $\frac{nr}{2} \geq m > a\binom{r+1}{2} + \binom{b}{2}$. Any graph maximising k_t for a fixed t or $k = \sum_{t\geq 2} k_t$ among all graphs of order n, size m and maximum degree at most r can be represented as $(a-1)K_{r+1} + H$.*

There are some obstructions to a tidier conjecture. Examples 1 and 2 show that there might be several different kinds of an extremal graph H, and for distinct t the extremal graphs might not correspond. This is in stark contrast to the cases of prescribed size and order alone.

Example 1. The graph G in Figure 5.1a satisfies $k_3(G) = 16, k_4(G) = 4, k_5(G) = 0$ and $k(G) = 20$. It is the unique graph maximising $k_3(G)$ among all graphs with $(n, m, r) = (8, 18, 5)$. On the other hand, the graph G in Figure 5.1b satisfies $k_3(G) = 15, k_4(G) = 6, k_5(G) = 1$ and $k(G) = 22$. It is the unique graph maximising $k(G)$ among all graphs with $(n, m, r) = (8, 18, 5)$ and maximises k_4 and k_5 as well. For k_4 and k_5 there are respectively 2 and 3 extremal graphs.



(a) Extremal graph for k_3



(b) Extremal graph for k

Figure 5.1: Graphs with $(n, m, r) = (8, 18, 5)$

As the $t = 3$ case was the main interest in [53], we can further focus on this case.

Describing the extremal graphs in general seems to be hard as they are not unique and also $k_3(\overline{G})$ and the degree sequences can be different for different extremal graphs, as the next example shows.

Example 2. There are three graphs with the maximum number of triangles, 16, among all graphs of order 8, size 17 and maximum degree at most 5. The number of triangles in their complement $\overline{G}$ is equal to $4, 1$ and 0 respectively, implying also that their degree sequences are different.

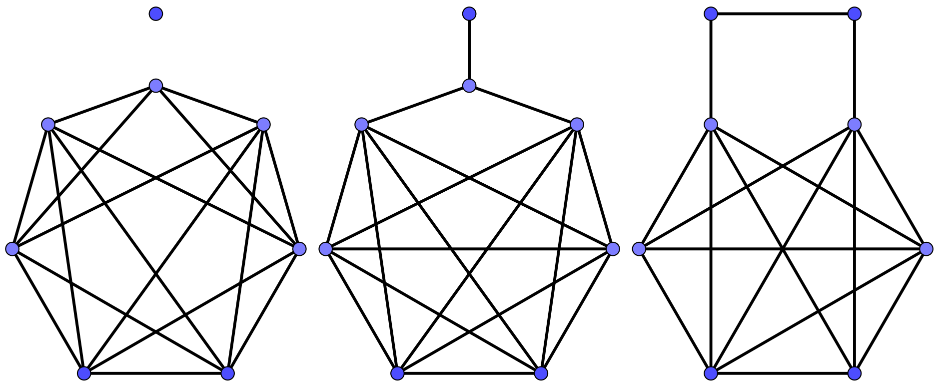


Figure 5.2: Graphs with $(n, m, r) = (8, 17, 5)$ maximising k_3

We also remark that in the critical regime, increasing m can imply both a decrease or increase in the number of triangles. This is also the case if you increase both m and n by 1.

Example 3. When $r = 4$, the maximum number of triangles among all graphs of order n and size m in the critical regime are given below in Table 5.1.

$n\backslash m$	11	12	13	14	15	16
6	7	8				
7		8	7	7		
8				8	8	8

Table 5.1: maximum $k_3(G)$ given n and m when $r = 4$

6

On cliques and colouring

Act as if what you do makes a difference. It does.

– William James

In this chapter, we give an overview on results related to a general problem of Erdős and Nešetřil, as written in [38]. They wonder about the smallest integer $h_t(\Delta)$ so that every G with $h_t(\Delta)$ edges and maximum degree $\leq \Delta$ contains two edges so that the shortest path joining these edges has length $\geq t$. Equivalently, $h_t(\Delta) - 1$ is the largest number of edges inducing a graph of maximum degree Δ whose line graph has diameter at most t. It is easy to see that $h_t(\Delta)$ is always at most $2\Delta^t$. In a graph of maximum degree Δ whose line graph has diameter at most t, every vertex is at distance at most t from at least one of the end-vertices of a fixed edge $e = uv$. This implies that (for intuition, see Figure 6.1) $h_t(\Delta) - 1 \leq 1 + 2 \cdot \sum_{i=1}^{d} (\Delta - 1)^i < 2\Delta^d$.

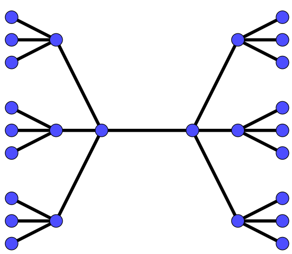


Figure 6.1: A greedy upper bound of 25 for $h_2(4) - 1$ obtained by looking locally

But one might imagine that this upper bound one obtains by looking locally is not sharp in general. For instance, the $t = 1$ case is easy and $h_1(\Delta) = \Delta + 1$, with the exception that $h_1(2) = 4$. The extremal graphs being the star $K_{1,\Delta}$ (when $\Delta \neq 2$) and triangle K_3 (for $\Delta = 2$). For $t = 2$, it is known due to the work by Chung, Gyárfás, Tuza and Trotter [34] that that $h_2(\Delta) \sim 5\Delta^2/4$. Furthermore they found that the extremal graphs, being blown-ups of a C_5, are unique. These blown-up C_5s are presented in Figure 6.2 for $\Delta \in \{4, 5\}$.

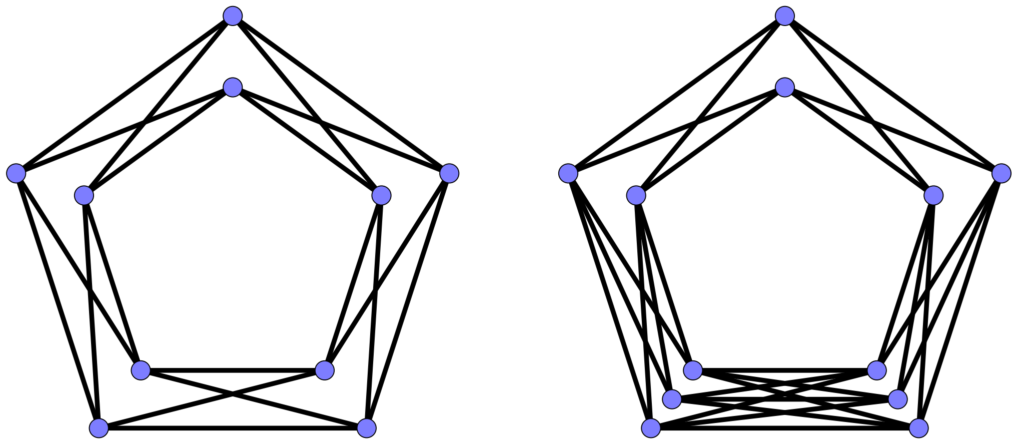


Figure 6.2: Blow-up of a C_5 with $\Delta = 4$ and the variant with $\Delta = 5$

For bipartite graphs of maximum degree Δ, the extremal value of $\omega(L(G)^2)$ was determined already in [41]. It is attained by the complete balanced bipartite graph $K_{\Delta,\Delta}$. Nevertheless, as noted in [30] the different bound is due to forbidding the graph C_5.

As a next step in the sequence of stronger versions of the question of Erdős and Nešetřil, in [40] there have been stated a few conjectures that also the chromatic number of the square of the line-graph (being the strong chromatic index) is bounded by the same expressions. This is stated fully as the Strong edge-colouring conjecture.

We will consider the same variants of the problem of Erdős and Nešetřil for larger t, mainly by investigating the $\omega(L(G)^t)$ version. Here we will make a distinction between general graphs and $C_{2t+1}-$free graphs. One could argue that historically and intuitively, considering bipartite graphs separately would be natural as well. But we note that the bounds for bipartite graphs and $C_{2t+1}-$free graphs are the same (at least for $t \in \{1, 2, 3, 4, 6\}$), so it seems to be the most natural and correct distinction. Furthermore for $C_{2t+1}-$free graphs we can prove the sharp bounds (for certain t) and these bounds do not hold any more for the general case. An example for $\Delta = t = 3$ is shown in Figure 6.3, where the red vertex is the additional vertex used in the subdivision. This graph is known to be the only extremal graph up to isomorphism, as shown by a careful case analysis as presented in [20]. When $\Delta \geq 4$, there are plausibly multiple non-isomorphic graphs attaining our conjectured bound, due to a local modification where one deletes an arbitrary vertex v and replaces it with an edge whose end vertices are connected to i and $\Delta - i$ of the original neighbours of v, where $1 \leq i \leq \frac{\Delta}{2}$. The case $i = 1$ corresponds to a subdivision of an edge. An example of such a replacement is indicated in Figure 6.4.

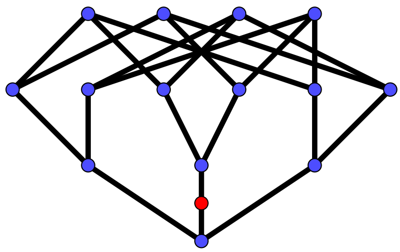


Figure 6.3: The extremal graph of size 22

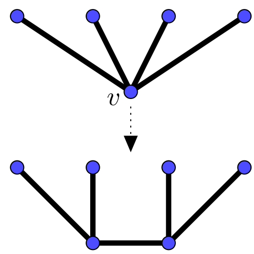


Figure 6.4: A local modification

We conjecture that in general, the trivial upper bound is in essence a factor 2 from the correct bound.

Conjecture 6.0.1. *For $t \neq 2$ and any $\varepsilon > 0$, $h_t(\Delta) \leq (1+\varepsilon)\Delta^t$ for all large enough Δ.*

In the results, we will compare the maximum size with the size of $T_{k,\Delta}$, a rooted tree of height k. In Figure 6.5 an examples has been drawn. In essence, an extremal graph will be equal to $T_{k,\Delta}$, with some end vertices glued together in a very structured way.

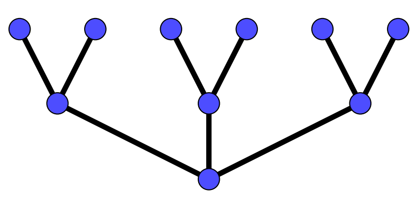


Figure 6.5: Constructions of $T_{2,3}$

6.1 Bounds on $\omega(L(G)^t)$

In this section, we mention our main results.

Instead of verifying that the shortest path between any two edges of the graph is short, we can do so for any pair of edges where one of them has a fixed vertex v of maximum degree. This implies that one can look more locally. This is a big relaxation on the number of conditions that one is checking and as such, the proof is somewhat easier. Nevertheless, for C_{2t+1}-free graphs this settled Conjecture 6.0.1 in the special case of graphs containing no cycle C_{2t+1} of length $2t+1$ as a subgraph.

Proposition 6.1.1. *For fixed Δ and t, let G be a graph with maximum degree Δ. Let v be a vertex with maximum degree j and let $u_1, u_2, \ldots, u_j$ be its neighbours. Suppose that in $L(G)^t$, every edge of G is adjacent to vu_i for every $1 \leq i \leq j$, i.e. for every edge e and any edge of the form vu_i, there is a path of length at most $t-1$ connecting e and vu_i.*
If G is C_{2t+1}-free, then $|E(G)| \leq |E(T_{t,\Delta})|$.
In the general case, $|E(G)| \leq \frac{3}{2}\,|E(T_{t,\Delta})| - \frac{1}{2}\,|E(T_{t-1,\Delta})|$.

For example when $t = 2$, the following examples in Figure 6.6 shows that the blow-up of a C_5 is not extremal any more and our result under the relaxed condition is the best one can aim for, i.e. the proposition itself is sharp.

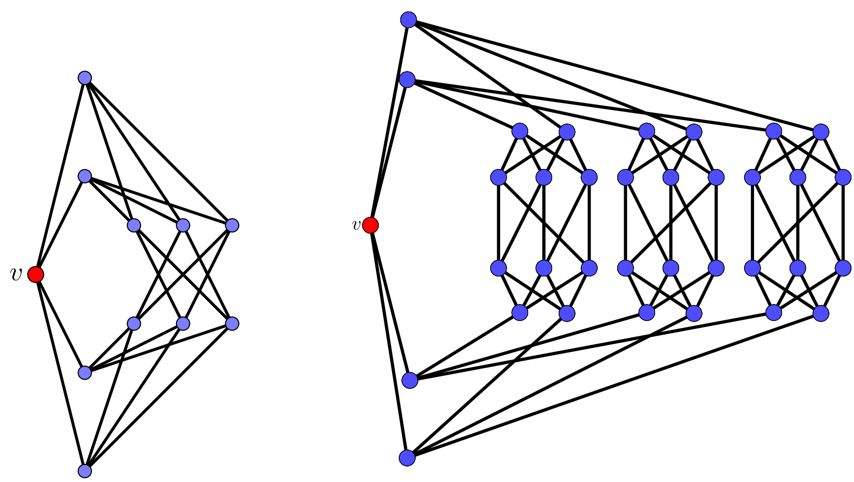


Figure 6.6: Extremal graphs under the relaxed condition for $\Delta = 4$ and $t = 2, 3$.

Part III

Graph colouring

7

A survey on list-colouring

Mathematics, rightly viewed, possesses not only truth, but supreme beauty
– Bertrand Russell

In this chapter, we give one fundamental result on list-colouring.

First, we start with the definitions of all the necessary concepts. For a graph $G = (V, E)$ and a positive integer k, a mapping $c : V \to [k] = \{1, 2, \ldots, k\}$ is called a proper *k-colouring* of G if $c(u) \neq c(v)$ whenever $uv \in E$. The smallest value k for which a proper k-colouring does exist is the chromatic number $\chi(G)$ of G. This is the minimum number of colours needed to obtain a proper colouring.

For a positive integer k, a mapping $L : V \to \binom{\mathbb{Z}^+}{k}$ is called a k-list-assignment of G; a colouring $c : V \to \mathbb{Z}^+$ is called an L-colouring if $c(v) \in L(v)$ for any $v \in V$. We say G is k-choosable if for any k-list-assignment L of G there is a proper L-colouring of G. The choosability $\mathrm{ch}(G) = \chi_\ell(G)$ (or choice number or list chromatic number) of G is the least k such that G is k-choosable.

In sections 7.1 we present some foundational work on list-colouring for planar graphs. Here we will present Thomassen's proof that every planar graph G has list chromatic number $\chi_\ell(G)$ at most 5, in contrast with the chromatic number which is at most 4 by the four colour theorem. In general, we know that $\chi_\ell(G)$ can be arbitrarily larger than $\chi(G)$, but at most with a factor that behaves like $\log n$.

7.1 List-colouring of planar graphs

You don't have to believe in God, but you should believe in The Book.
– Paul Erdős

One of the most famous results in graph theory is the four colour theorem, conjectured in the 19^{th} century and proven by Appel and Haken [5], with technical assistance from Koch. It states that every planar graph G can be coloured with four colours, i.e. satisfies $\chi(G) \leq 4$. It was the first theorem proven with the help of some computer verification and as it was impossible to check by hand, it was questioned by others on philosophical grounds.

This was some inspiration for Erdős, Rubin and Taylor [39] to state some conjectures on the choosability of planar graphs. They conjectured that $\chi_\ell(G) \leq 5$ for every planar graph and that there do exist planar graphs which do attain the upper bound. We will highlight the two results confirming these conjectures. The proof that $\chi_\ell(G) \leq 5$ holds for every planar graph is proven in a two-page paper by Thomassen and is so elegant, it is one of the "Proofs from the Book" [1].

Theorem 7.1.1 ([63]). *Every planar graph G is 5-choosable.*

Proof. First, one observes that it is sufficient to assume that all inner faces are triangles (adding edges makes it harder to colour) For this, we prove the following statement by induction on the number of vertices of a *near-triangulations.*

Let G be a near-triangulated graph, and let $C = v_1v_2 \dots v_pv_1$ be the cycle bounding the outer region, i.e. all other edges and vertices are within C. Let $|L(v)| = 3$ for every $v \in C$ and $|L(v)| = 5$ for every $v \in V \backslash C$. Let v_1 and v_2 be coloured (with different colours) in their lists. Then this can be extended to a proper L-colouring of G. The base case where the order $n = 3$ is trivial. So assume the statement is true for every order smaller than n and we have a graph G on n vertices. First we consider the case where the cycle C has a chord v_iv_j, i.e. C is not an induced cycle and v_iv_j is an edge which is not part of the cycle (so $\mid i-j \mid \not\equiv 1 \pmod p$). This is presented in the left in Figure 7.1.

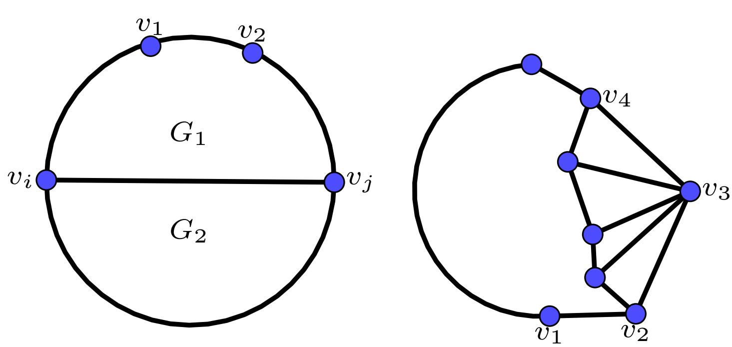


Figure 7.1: The two scenarios in the proof of Thomassen

Under this assumption, the edge $v_i v_j$ divides the graph G in two subgraphs G_1, G_2 (having circumferences $v_i v_{i+1} \dots v_j v_i$ and $v_j v_{j+1} \dots v_p v_1 \dots v_i v_j$), which do have only the edge $v_i v_j$ in common. Assume without loss of generality that G_1 contains v_1 and v_2. Then by induction the subgraph G_1 can be properly L-coloured and since v_i and v_j are pre-coloured, also G_2 can be extended to a proper L-colouring.

If the cycle C has no chord, then $G \backslash v_3$ has a boundary $C \backslash \{v_3\} \cup N(v_3)$. This is depicted in the right part of Figure 7.1. Pick two colours α, β in $L(v_3)$, different from the colour of v_2. For every $u \in N(v_3) \backslash \{v_2, v_4\}$, take $L'(u) = L(u) \backslash \{\alpha, \beta\}$. For the remaining vertices u, we let $L'(u) = L(u)$. As such, the lists L' for boundary vertices of $G \backslash v_3$ have length 3 again and vertices not on the boundary still have length 5. This implies that by induction $G \backslash v_3$ can be properly L'-coloured. Since v_4 (possibly $v_4 = v_1$) can be assigned at most one colour from α, β, we can assign one of these colours to v_3 which is different than the colour given to v_4 to obtain a proper L-colouring of G. □

Mirzakhani [57][1] constructed a planar graph for which $\chi_\ell(G) = 5$

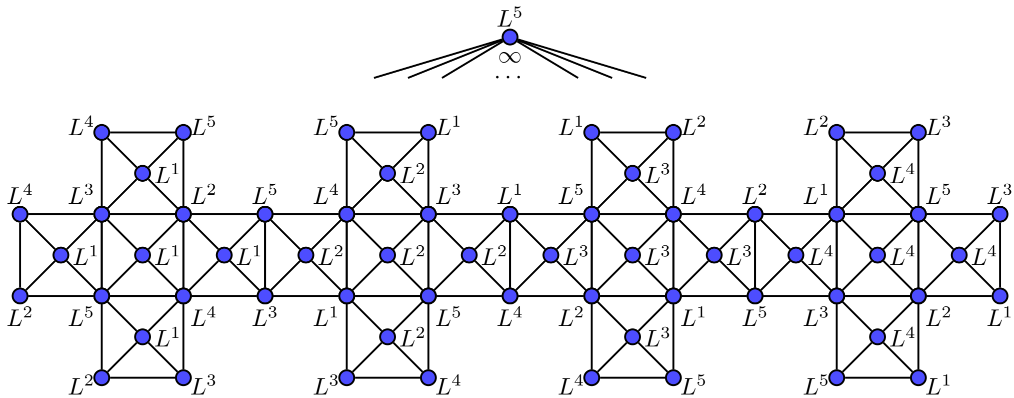


Figure 7.2: The Mirzakhani graph M, a planar 3-colourable graphs which is not 4-choosable.

Theorem 7.1.2 ([57]). *There do exist planar graphs G which are not 4-choosable.*

Proof. Consider the graph with 63 vertices depicted in Figure 7.2, with associated lists $L^i = [5] \backslash \{i\}$ of length 4 for every vertex. Here we have adapted the figure from [55] in such a way that even more symmetry is clear. One can recognise four blocks in the form of a +sign which are translated versions from each other, in such a way that the

[1] She was the first female mathematician to be awarded the Fields Medal

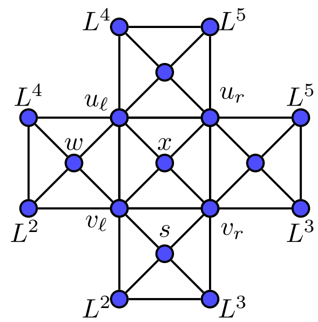


Figure 7.3: One block of the Mirzakhani graph M which is not list-colourable

elements in the lists are also translated with the same constant. Assume the graph is list-colourable, i.e. there exists a list-colouring c. The vertex ∞ which is connected to all vertices at the outside has list $L^5 = [4]$ and due to symmetry we may assume it is coloured with 1.

Therefore, we can focus on the first block. Now only the colours in $[5]\backslash\{1\}$ are allowed for the vertices in this block and thus $L^i = [5]\backslash\{1, i\}$ in Figure 7.3. After some case analysis, we conclude.

□

One other result we want to mention, is that $\chi_\ell(K_{n,n}) \sim \log_2 n$ as $n \to \infty$.

The vertices of $K_{n,n}$ have degree n and $\chi_\ell(K_{n,n}) \sim \log_2 n$. Alon and Krivelevich noticed similar behaviour for random bipartite graphs with all degrees around Δ and conjectured the following upper bound. Here C can plausibly be replaced with an expression of the form $1 + o(1)$.

Conjecture 7.1.1 ([4]). *There is some absolute constant $C > 0$ such that any bipartite graph of maximum degree at most $\Delta \geq 2$ is k-choosable if $k \geq C \log \Delta$.*

There do exist many related notions, which can be ordered and for many of these inequalities. Furthermore, every parameter can be unbounded in terms of the previous one, except for $\chi_\ell(G), \chi_{DP}(G), \delta^\star(G) + 1$.

Proposition 7.1.2. *For every graph G,*

$$\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G) \leq \delta^\star(G) + 1 \leq \Delta(G) + 1 \leq n.$$

8

Introductory results on graph colouring

You must go on adventures to find out where you truly belong.
– Sue Fitzmaurice

In this chapter, we want to give some additional diverse examples in the field of graph colouring related to famous conjectures in the field. Already by investigating the different examples here, one can get a feeling that graph colouring is a wide area. On the other hand, we are considering problems that have connections to long-standing conjectures which are similar in flavour, all of them being related to structural graph theory where some substructure is forbidden. These conjectures claim that by forbidding a certain substructure, one can conclude something about the chromatic number of the graph or a relation between the chromatic number and the list chromatic number of it. The forbidden substructures are an induced tree (and clique), a clique minor and a claw (induced $K_{1,3}$) respectively.

8.1 On rainbow induced paths in triangle-free graphs

Sumner [62] proved that for any fixed tree T on n vertices, any colouring of a graph G with $\chi(G) \geq n$ colours contains a rainbow copy of T, i.e. a copy of T which is coloured with n different colours. Here it is sufficient to consider the case where $\chi(G) = n$.

An analogous algorithm works to prove the general statement. A more detailed survey can be found in the thesis of Wass [67]. In the same paper of Sumner, the Gyárfás-Sumner conjecture [49, 62] has been stated.

Conjecture 8.1.1 (Gyárfás-Sumner [49, 62])**.** *For a fixed tree T, if a graph G does* *not contain an induced copy of T, then $\chi(G)$ is bounded by $f(\omega(G))$ for some function f.*

In this section, we consider a beautiful conjecture of Aravind that claims that an analogue of the result of Sumner for induced paths is true if we impose that G is triangle-free.

Conjecture 8.1.1 (Aravind [7]). *Every properly coloured triangle-free graph of chromatic number χ contains a rainbow induced path of length χ.*

We give two graphs showing that some stronger forms of Conjecture 8.1.1 are not true. First, we present a proper colouring of the Chvátal graph (this graph has order 12 and chromatic number 4) in Figure 8.1 for which it is impossible to find a rainbow induced P_4 starting in B or B', nor such that A or A' are centres of the path (have degree 2). As the graph is rather small, the reader who likes to puzzle can check this by hand. This example already implies that greedy or heuristic algorithms starting to find a rainbow induced path P_4 from any random vertex cannot assume anything about the neighbours of that first vertex.

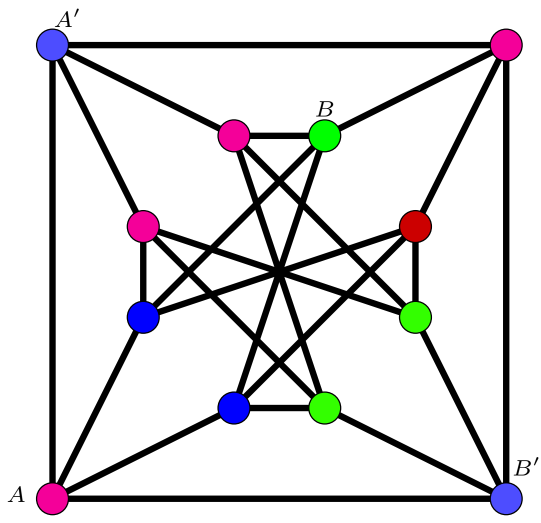


Figure 8.1: A proper colouring of the Chvátal graph

In Figure 8.2, we have an example of a colouring of a graph of order 22 and chromatic number 5. This is a minimal graph since no triangle-free graph with chromatic number 5 has order less than 22. Nevertheless, a computer check verified that no rainbow induced P_5 contains one of the vertices from $\{2, 7, 16\}$.

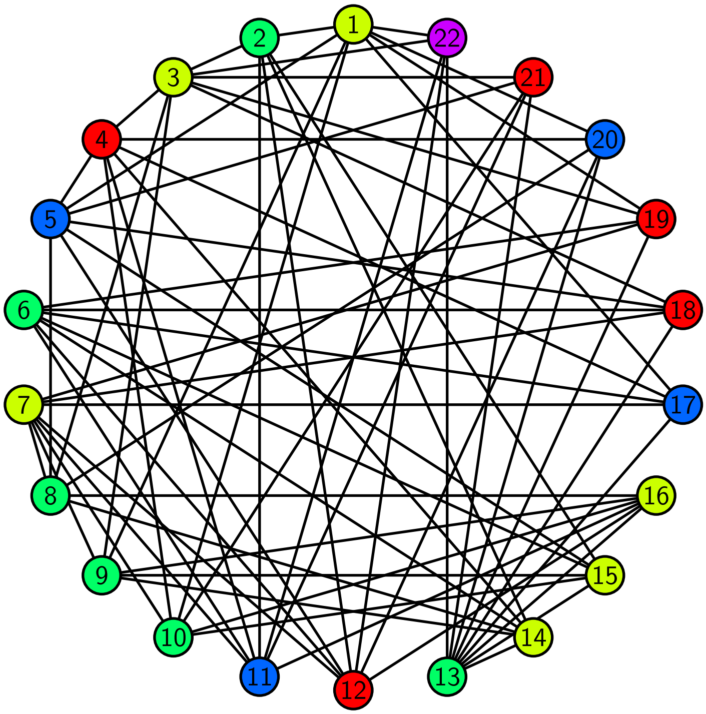


Figure 8.2: Coloured graph with no induced rainbow P_5 through vertices $2, 7$ or 16

We also prove a weaker statement, i.e. it is a corollary of the conjecture if true).

Theorem 8.1.2. *Every properly coloured K_3-free graph of chromatic number χ contains a rainbow independent set of size $\lceil \frac{1}{2}\chi \rceil$.*

Proof of Theorem 8.1.2. Let G be a K_3-free graph of chromatic number χ and let $\varphi : V(G) \to \mathbb{Z}^+$ be a proper colouring. We seek a rainbow independent set of size $\lceil \frac{1}{2}\chi \rceil$. Initialise $G' = G$ and $X = \varnothing$, and iterate the following until G' is empty (if needed).

1. Take an arbitrary vertex $v \in V(G')$ and add it to X.

2. Let $S(v) = \varphi^{-1}(\varphi(v))$ and delete the vertices of $S(v)$ from G'.

3. Delete the vertices of the neighbourhood $N = N_{G'}(v)$ from G'.

When this procedure ends, by definition X is a rainbow independent set. We also know that the chromatic number $\chi(G)$ is at most $2|X|$, since the vertex set has been partitioned as $\cup_{v\in X}(S(v)\cup N(v))$ and both $S(v)$ and $N(v)$ are independent sets. $\square$

8.2 A relation between a conjecture of Füredi et al and Hadwiger's conjecture

One of the most important conjectures in graph colouring is Hadwiger's conjecture. It generalises for example the four colour theorem. Leaving the general definition of a graph minor aside, the conjecture says that if one cannot find t disjoint connected subgraphs $H_1, H_2, \ldots, H_t$ of a graph G such that for every $1 \le i < j \le t$ there is an edge between H_i and H_j, then one can colour the graph G with t colours. We denote with $\eta(G)$ the largest number η such that there are η disjoint connected subgraphs H_i of G which are pairwise connected. Hadwiger's conjecture [50] from 1943 states that a graph G having no K_{t+1} minor, can be coloured with t colours.

Conjecture 8.2.1 (Hadwiger, [50]). *For every graph G, we have $\eta(G) \ge \chi(G)$.*

It is one of the deepest and most famous unsolved problems in graph theory. By investigation of the random graph $G_{n,\frac{1}{2}}$, it is known that the conjecture is true for almost all graphs by [8]. A survey on the conjecture can be found in [60].

One neat problem which is a corollary of Hadwiger's conjecture is the so-called Seagull problem. This is due to the relation with induced $K_{1,2}$s, which are called seagulls.

Conjecture 8.2.2 (Seagull problem). *For every graph G with $\alpha(G) = 2$, we have $\eta(G) \ge \frac{n}{2}$.*

In general, it is still unsolved and it might be a possible direction to disprove Hadwiger's conjecture.

We proved a relation between Conjecture 8.2.2 and another conjecture dealing with graphs with independence number 2, which was considered to be risky according to Füredi et al. [44]. But if the conjecture is false, Hadwiger's conjecture would be false as well by Theorem 8.2.4. To state their conjecture, we need the notion $\mathrm{cm}(G)$ being the size of a largest connected matching of G. This is a matching $e_1, e_2, \ldots, e_{\mathrm{cm}}$ such that every two edges in the matching are connected.

Conjecture 8.2.3 ([44]). *Let G be a graph with independence number $\alpha = 2$ and order $n = 4t-1$. Then $\mathrm{cm} \ge t$.*

Theorem 8.2.4. *If Conjecture 8.2.1 is true, then so is Conjecture 8.2.3. Equivalently, if Conjecture 8.2.3 is false, then so is Hadwiger's conjecture.*

More precisely we will prove that if Conjecture 8.2.3 is not true for a certain value of t, the Seagull's problem is false for $n = 4t-1$.

8.3 Generalising Vizing's theorem

In a paper of 1964, Vizing [65] proved that any simple graph G with maximum degree Δ admits a proper $(\Delta+1)$-edge colouring, i.e. the edge chromatic number $\chi'(G) \leq \Delta + 1$. Since the Δ edges ending in the maximum degree vertex all need a different colour, the result of Vizing implies $\Delta \leq \chi'(G) \leq \Delta + 1$. So there are only two options. Class 1 graphs are the graphs with $\chi'(G) = \Delta$ and class 2 graphs satisfy $\chi'(G) = \Delta + 1$.

Now we want to introduce the notion of *simultaneous colouring* of multiple graphs, such that we can consider some generalisations of Vizing's theorem.

Let ℓ graphs $G_1, G_2, \ldots, G_\ell$ and $G = \cup_{i=1}^{\ell} G_i$ be their (edge) union. The *multiplicity* of an edge e is the number of graphs G_i with $i \leq \ell$ on which e appears. An edge-colouring of G is *simultaneous with respect to* $G_1, \ldots, G_\ell$ if its restriction to each graph G_i is a proper edge-colouring. Cabello asked how many colours are needed to ensure the existence of a simultaneous colouring of G with respect to each G_i. We denote by $\chi'(G_1, \ldots, G_\ell)$ the minimum number of colours needed to obtain a simultaneous colouring. We define $\chi'(\ell, \Delta)$ to be the largest integer k such that $k = \chi'(G_1, \ldots, G_\ell)$ for some graphs $G_1, \ldots, G_\ell$ of maximum degree (at most) Δ.

For $\ell = 2$, an open problem from Cabello asks whether $\chi'(\ell, \Delta)$ can be significantly larger than Δ. There are examples of class 1 graphs G_1 and G_2, for example two P_4s in a C_5, for which $\chi'(G_1, G_2) > \max\{\chi'(G_1), \chi'(G_2)\}$. Nevertheless it is already unclear if $\chi'(\ell, \Delta) > \Delta + 1$ might be true.

We improve the bounds from [9] for $\chi'(\ell, \Delta)$ by adapting their proofs.

Theorem 8.3.1.

$$\sqrt{\ell}\Delta \times (1 + o_{\ell,\Delta}(1)) \leq \chi'(\ell, \Delta) \leq \sqrt{2\ell}\Delta + 1,$$

where $o_{\ell,\Delta}(1) \to 0$ *if* $\Delta >> \ell >> 1$.

Inspired by [9], we also wonder about the following generalisation of a theorem of Vizing [65].

Conjecture 8.3.2. *Let $G_1, \ldots, G_\ell$ be ℓ graphs of maximum degree Δ such that every edge of their union G has multiplicity at most k. Then $\chi'(G_1, \ldots, G_\ell) \leq k(\Delta - 1) + 2$.*

So note that for $k = 1$, this is exactly Vizing's theorem. The intuition behind is that every edge $e = uv$ is in at most k graphs G_i and hence v belongs to at most $k(\Delta - 1) + 1$ edges from these graphs.

If true, Conjecture 8.3.2 is sharp, at least when $k = \Delta = q + 1$ for a prime power q. In Figure 8.3 we have presented the example for $k = \Delta = 2$.

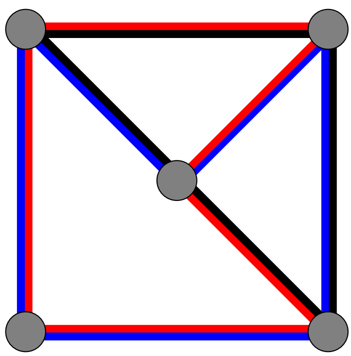


Figure 8.3: Example of three graphs on the same vertex set showing $\chi'(3, 2) = 4$

9

Asymmetric list sizes in bipartite graphs

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere - Archimedes did it in a bathtub.

– Richard J. Trudeau

In this chapter, we investigate asymmetric versions for list-colouring and conjecture 7.1.1.

Given a bipartite graph with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, consider a list-assignment such that every vertex in A or B is given a list of colours of size k_A or k_B, respectively.

We prove some general sufficient conditions in terms of Δ_A, Δ_B, k_A, k_B to be guaranteed a proper colouring such that each vertex is coloured using a colour from its list. These are asymptotically nearly sharp in the very asymmetric cases. We derive some of these necessary conditions through an intriguing connection between the complete case and hypergraph Turán numbers. From another angle, we also show that the complete case cannot give the precise sharp bounds, as one possibly would expect.

We establish one sufficient condition in particular, where $\Delta_A = \Delta_B = \Delta$, $k_A = \log \Delta$ and $k_B = (1 + o(1))\Delta/\log \Delta$ as $\Delta \to \infty$. This amounts to partial progress towards conjecture 7.1.1.

9.1 Introduction

Given a bipartite graph $G = (V{=}A \cup B, E)$ with parts A, B and positive integers k_A, k_B, a mapping $L : A \to \binom{\mathbb{Z}^+}{k_A}, B \to \binom{\mathbb{Z}^+}{k_B}$ is called a (k_A, k_B)-*list-assignment* of G. We say G is (k_A, k_B)-*choosable* if there is guaranteed a proper L-colouring of G for any such L. We investigate the following problem.

Problem 9.1.1. *Given Δ_A and Δ_B, what are optimal choices of $k_A \le \Delta_A$ and $k_B \le \Delta_B$ for which any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is (k_A, k_B)-choosable?*

We have the upper bounds on k_A, k_B, since the problem is trivial if $k_A > \Delta_A$ or $k_B > \Delta_B$.

One hope here is that further study of these problems may yield insights into Conjecture 7.1.1.

Our first main result provides general progress towards Problem 9.1.1.

Theorem 9.1.2. *Let the positive integers Δ_A, Δ_B, k_A, k_B, with $k_A \le \Delta_A$ and $k_B \le \Delta_B$, satisfy one of the following conditions.*

1. $k_B \ge (ek_A\Delta_B)^{1/k_A}\Delta_A$.
2. $e\Delta_A(\Delta_B - 1)\left(1 - (1 - 1/k_B)^{\Delta_A \min\{1, k_B/k_A\}}\right)^{k_A} \le 1$.

Then any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is (k_A, k_B)-choosable.

We complement our sufficient conditions for (k_A, k_B)-choosability with necessary ones. An easy boundary case is the next result for complete bipartite graphs.

Proposition 9.1.1. *For any $\delta, k \ge 2$, the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = \delta^k$ and $|B| = k$ is not (k, δ)-choosable.*

Proof. Let the vertices of B be assigned k disjoint lists of length δ, and let the vertices of A be assigned all possible k-tuples drawn from these k disjoint lists. □

This is best possible in the sense that the conclusion does not hold if $|A| < \delta^k$ or $|B| < k$. However, we know that there do exist non-complete non-(k, δ)-choosable graphs that are slightly more efficient (in the sense that it has $\Delta_A = k$ and $\Delta_B < \delta^k$).

I may not be there yet, but I am closer than I was yesterday.

– Jose N. Harris

In the full version of the thesis, there is also a chapter on a more technical version of list-colouring, called correspondence colouring. We omit the definitions here. That chapter was mainly based on [27].

10

Packing list-colourings

A man ceases to be a beginner in any given science and becomes a master in that science when he has learned that he is going to be a beginner all his life.

– Robin G. Collingwood

Graph colouring is inspired by e.g. resource allocation problems. Here it is natural to wonder about several such allocations in parallel, such that they collectively cover all possible resource usage. In the graph colouring terminology, we will wonder not only about finding one possible list-colouring, but also about finding many disjoint ones, such that they form a packing. It has been already 25 years since Alon, Fellows and Hare [3] suggested the study of this type of problem. We first formally define the additional notions needed in this study.

Given a list-assignment L of G, an *L-packing of G of size k* is a collection of k mutually disjoint L-colourings $c_1, \dots, c_k$ of G, that is, $c_i(v) \neq c_j(v)$ for any $i \neq j$ and any $v \in V(G)$. We say that an L-packing is proper if each of the disjoint L-colourings is proper. We define the *list (chromatic) packing number* $\chi_\ell^\star(G)$ of G as the least k such that G admits a proper L-packing of size k for any k-list-assignment L of G. Note that $\chi_\ell^\star(G)$ is necessarily at least $\chi_\ell(G)$, but an inequality in the other direction might be harder.

As a first example, we consider the fan F_7, which can be built starting with a path of length 6, $u_1u_2 \dots u_6$, and connecting all its vertices to an additional vertex c. This is a graph with degeneracy equal to 2.

Now assume there is a 3-list-assignment, as depicted in Figure 10.1.

With some case analysis, one can conclude that no list packing is possible for this assignment and hence $\chi_\ell^\star(F_7) > 3$. We can order the colours in $L(c)$ as $(1, 2, 3)$ and consequently look how one can extend the three disjoint list-colourings of F_7. This has been summarised in Table 10.1

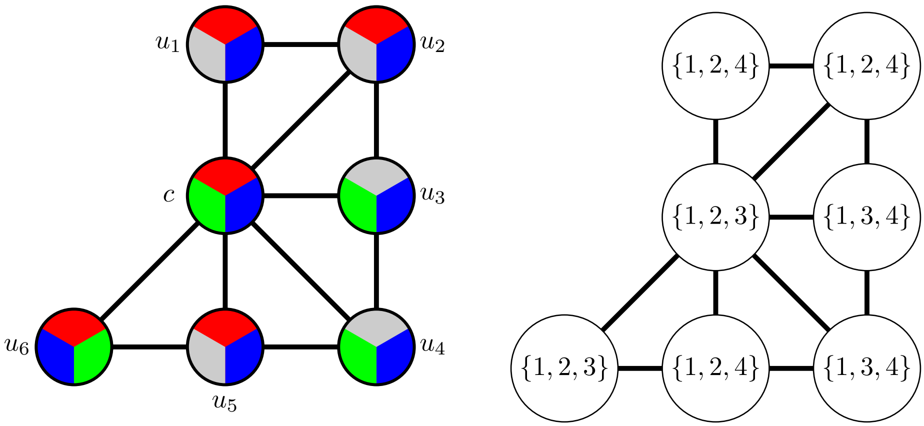


Figure 10.1: Two presentations of a colour-assignment of F_7 showing $\chi_\ell^\star(F_7) = 4$

vertex v	possible permutation colours $(c_1(v), c_2(v), c_3(v))$
c	$(1,2,3)$
u_1	$(4,1,2), (2,4,1), (2,1,4),$
u_2	$(2,4,1), (4,1,2)$
u_3	$(3,1,4), (3,4,1)$
u_4	$(4,3,1)$
u_5	$(2,1,4)$
u_6	-

Table 10.1: Partial list packings for F_7

We can conclude for our example F_7 that $\chi_\ell^\star(F_7) = 4$.

Theorem 10.0.1. *$\chi_\ell^\star(G) \le 2\delta^\star(G)$ for any graph G.*

This theorem can be proven greedily by applying Hall's marriage theorem.

This theorem is sharp for $\delta^\star(G) \in \{1,2\}$ and so it would be interesting to know if it is for larger degeneracy as well. Since $\chi_\ell(G) \le \delta^\star(G)+1$, the answer could possible show that $\chi_\ell^\star(G) - \chi_\ell(G)$ can be arbitrary large. We do not know the latter but conjecture the following, actually for $C = 2$.

Conjecture 10.0.2. *There exists $C > 0$ such that $\chi_\ell^\star(G) \le C \cdot \chi_\ell(G)$ for any graph G.*

Some more results can be found in [19]. One big result is that $\chi_\ell^\star(G) \le (1+o(1)) \cdot \Delta/\log(\Delta)$ if G is bipartite. The proof is the most technical one in the thesis and the result generalises the best result known for Conjecture 7.1.1.

Part IV

Extra: The pieces fit together

11

Diameter of some reconfiguration graphs

The enemy of art is the absence of limitations

– Orson Welles

In this short chapter, we will present some results on the diameter of some reconfiguration graphs. The inspiration comes from the conjecture of Cereceda [31, Conj. 5.21]. This topic deals with both distance and colouring of graphs and so it is a nice result to finish the main content of the thesis.

For this, we need the notion of a reconfiguration graph $\mathcal{C}_k(G)$ for a number k and graph G. The vertices of $\mathcal{C}_k(G)$ are exactly the (proper) k-colourings of the graph G (so $k \geq \chi(G)$ is necessary) and two k-colourings α and β are connected if they are equal on all but exactly one vertices of G. This also implies that the distance between k-colourings α and β is at most j if one can obtain β by at most j recolourings of one vertex at a time, starting with colouring α, in such a way that every intermediate colouring is a proper k-colouring as well. Such a reconfiguration graph is not necessarily connected, but it was proven that it is for $k \geq \deg(G) + 2$, where $\deg(G)$ is the degeneracy of the graph. Furthermore, it was conjectured that for $k \geq \deg(G) + 2$ the diameter of $\mathcal{C}_k(G)$ is polynomial in the order, more precisely that it is quadratic.

Conjecture 11.0.1 (Cereceda)**.** *For a graph G with n vertices and $k \geq \deg(G) + 2$, the diameter of $\mathcal{C}_k(G)$ is $O(n^2)$.*

The best known bound at this point for this conjecture is $O(n^{\deg(G)+1})$ by Bousquet and Heinrich [10]. When $k \geq \Delta(G) + 2$, an $O(n^2)$ bound is known by Cereceda [31, Prop. 5.23] and so the conjecture is known to be true for regular graphs. As we show in this chapter, in the regular case, the bound is even linear.

11.1 A linear bound in Cereceda's conjecture for regular graphs

Theorem 11.1.1. *For a graph G with n vertices and for any $k \geq \Delta(G) + 2$, the diameter of $\mathcal{C}_k(G)$ is at most $2n$.*

In particular, this gives a linear bound for Cereceda's conjecture for regular graphs, where the constant does not depend on the degree.

Corollary 11.1.2. *For a regular graph G with n vertices and for any $k \geq \deg(G)+2$, the diameter of $\mathcal{C}_k(G)$ is at most $2n$.*

Furthermore, for k sufficiently large, we can prove a sharp bound.

Theorem 11.1.3. *For a graph G with n vertices and for any $k \geq 2\Delta(G) + 1$, the diameter of $\mathcal{C}_k(G)$ is at most $\left\lfloor \frac{3n}{2} \right\rfloor$. This bound is sharp for many graphs G.*

The bound is sharp for $G = K_n$ and $k = n + 1$ or $G = K_{m,m}$ and $k \geq 3$ (try to switch the colours of the two bipartition classes).

Example 11.1.4. Let $G = C_4$ with $V = [4]$, $\alpha(i) = i$ and $\beta(i) \equiv i + 1 \pmod 4$. Then one needs at least 6 recolourings, i.e. $d(\alpha, \beta) = 6$. Part of this is presented in Figure 11.1

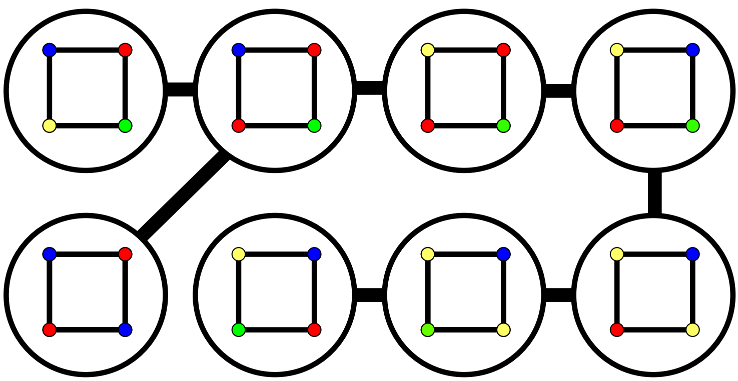


Figure 11.1: Part of the reconfiguration graph $\mathcal{C}_4(C_4)$

More on this topic, can be found in [18].

12

Tiling in the least dimension

"The key to success is to be enthusiastic about your subject"

– Noga Alon

12.1 General introduction on tiling

We start the exploration of a different topic, tiling, also called tessellation. This is something everyone knows from a very young age, as jigsaw puzzles are examples of tiling. It is the process of partitioning some object or space into predefined copies, which are the tiles. So the union of the tiles is exactly equal to the object and the intersection of any two tiles is empty. From a different perspective, a tiling can be considered as a covering with no overlap or as a packing with no gaps.

One can observe tilings everywhere. In nature, one may be fascinated by the hexagonal tiling of a honeycomb or some checkered patterns from flowers in botanic gardens. Note that many reptiles contain tiles, e.g. look to the scales of a snake (please pay attention if you are looking to a venomous snake). The ball in football is made up of hexagons and pentagons and so one sees a tiling at the surface. An optimal packing of small boxes in a big box is a three-dimensional tiling and one can consider more optimal packings in logistics as examples for tiling. As a tourist, when spotting some nice architectural patterns, one may be mainly enjoying some tiling pattern that is adapted creatively. The most well-known artist for being creative with tilings is without any doubt M.C. Escher. So the Escher Museum is a good place to visit if you cannot imagine a sufficient amount of creative tiling patterns.

Thinking about tiling happened already long ago. The word tessellation is derived from the Greek word *Tesseres*, which is translated as four. Indeed the most common tessellation is the one with square tiles. For sure, there is such a pattern in the house you live in, look to the bathroom floor or the brick pattern outside (note that a rectangle is a basic transformation of a square). When one wants to tile the plane with regular polygons, one can soon conclude there are only three working configurations. These being equilateral triangles, squares and regular hexagons. These are depicted in Figure 12.1. This is a folklore result. For example, the Arabic, Indian and Chinese

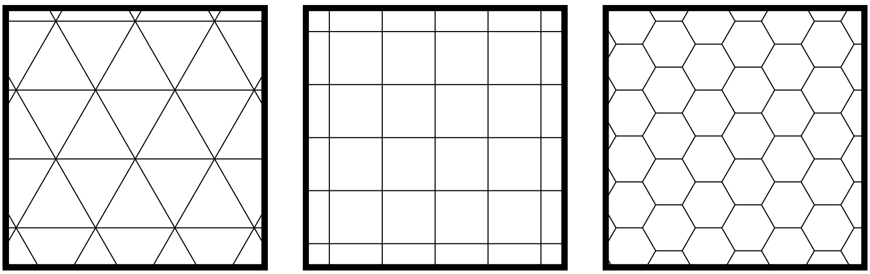


Figure 12.1: The three regular tesselations of the plane

cultures already practised the art of tiling a thousand years ago.

These three regular tessellations show some symmetry, both translational and rotational. Looking to a vertex (corner of the polygon), when rotating the whole tiling around this vertex one can end up with the same tiling again. For the triangles, one can rotate the tiling around such a vertex over $60°$. One can repeat this 6 times before having the exact initial configuration. For this reason, we say there is 6-fold rotational symmetry here. There is also translational symmetry as one can move the pattern along e.g. a side of a triangle. So the three patterns have both translational symmetry and n-fold rotational symmetry, where $n = 6, 4$ and 3 respectively.

For the one interested, we will briefly sketch that n-fold rotational symmetry cannot occur together with translational symmetry when $n > 6$. Assume you have a wonderful tessellation that has both n-fold rotational symmetry and translational symmetry, where $n = 5$ or $n > 6$. Pick the minimum distance h such that there is a translation $\overrightarrow{AB}$ over distance h such that the translated tiling is the same as the original one. Rotating over an angle $\frac{360}{n}$ would give the same tiling due to the n-fold rotational symmetry. The idea is presented in Figure 12.2. Translating over the vector $\overrightarrow{BC}$ would also yield the same configuration and so does the composite of two translations, which is a translation over $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Since $n \geq 7$, we have that $\frac{360}{n} < 60$ and the inclined angle $A\hat{B}C$ is the smallest angle and so AC is the shorter side of the triangle ABC. But this is a contradiction as we obtained translational symmetry (along $\overrightarrow{AC}$) over a distance less than h.

The remaining and most interesting case, was the case of pentagons. It became especially interesting to know which pentagons could tile the plane. People have

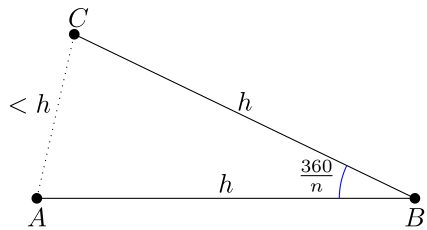


Figure 12.2: Translational - and n-fold rotational symmetry is impossible for $n > 6$

searched for this for almost a century and the final classification has been proven only recently by Michaël Rao in 2017. The main idea being some system of equalities for a finite number of scenarios. The equalities being based on the fact that the sum of angles in a pentagon equals 540° and the sum of angles in the plane being 360°. The 15 families of convex pentagons that can be used to tile the plane, is shown in Figure 12.3.

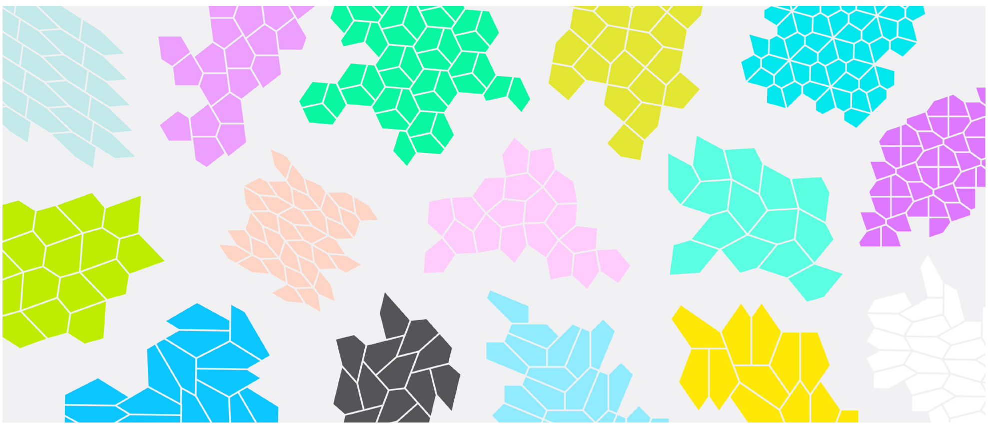


Figure 12.3: The 15 families of pentagons that tile the plane

Source: https://www.quantamagazine.org/pentagon-tiling-proof-solves-century-old-math-problem-20170711/

Another interesting path was taken by Kepler. In his book *Harmonices Mundi* he had thought about relationships between nature and properties of tiling. Nevertheless, for a short time, we will focus on the work of Penrose. Penrose found pairs of two tiles that can tile the plane, but only in a non-periodic way. Part of such a Penrose tiling is shown on the left in Figure 12.4. As there is fivefold rotational symmetry, there cannot be translational symmetry as derived before. A last interesting thing to note is that no fivefold symmetry or aperiodic structures were systematically observed in nature before. Dan Shechtmann started to study quasicrystals shortly after. Quasicrystals are structures in which the atoms are arranged in a non-periodic pattern. This was pretty new for the community and Shechtmann got the Noble prize in chemistry for these findings in 2011.

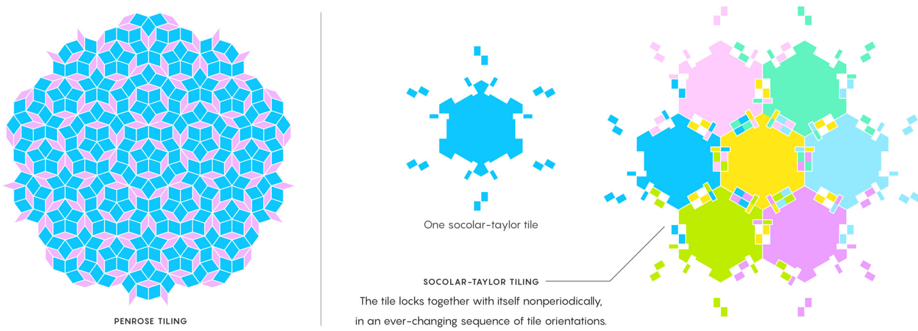


Figure 12.4: A Penrose tiling and a disconnected einstein

Source: https://www.livescience.com/50027-tessellation-tiling.html

There are also relationships between tilings and other problems more closely related to computer science. A single tile that can be used to tile the plane, but only in a non-periodic manner, is called an *einstein*. The existence of a (connected) *einstein* is equivalent to a single-tile decision problem. A disconnected einstein is presented on the right in Figure 12.4. The Socolar-Taylor tile contains multiple pieces that are in the same configuration when rotating or reflecting it in a tiling. In the remainder of this chapter, we will also be mainly focusing on a problem on disconnected tiles, but in a more discrete setting.

For some intuition for what is coming, one can think first to the famous game *Tetris*. In Tetris, one has building blocks that consist of multiple squares. One tries to put them down in such a way to tile a big rectangle, or at least fill some lines. These building blocks consisting of multiple squares are polyominoes and in the game Tetris more specifically tetrominoes as they consist of exactly 4 squares. If one has better graphics on the phone, one can imagine them to be three-dimensional blocks, so in that way, the two-dimensional tiles are embedded in a three-dimensional space. See also Figure 12.6 for a visualisation of the previous sentences. Mathematicians wondered already long ago about the possibility to tile a certain region with specific polyominoes, an exposition is e.g. written in [47]. As one folklore result, we mention the following result.

Theorem 12.1.1. *A $1 \times d$-tile does tile a $k \times n$-board if and only if $d|k$ or $d|n$.*

The *if*-statement is obvious, but the other direction is not. As an example, one can check this for $d = 4, k = n = 6$ in Figure 12.5. As $4|6^2$, one can expect to be able to tile the square with 9 rectangles. Now as every rectangle would cover every number in $\{0, 1, 2, 3\}$ exactly once, every number should appear 9 times to make this plausible. Nevertheless, there are only 8 number 3s in the square and so we get a contradiction. We hope this is a satisfying result for the reader less familiar with mathematics. We warn those readers that the content will be more mathematical from now on in this chapter.

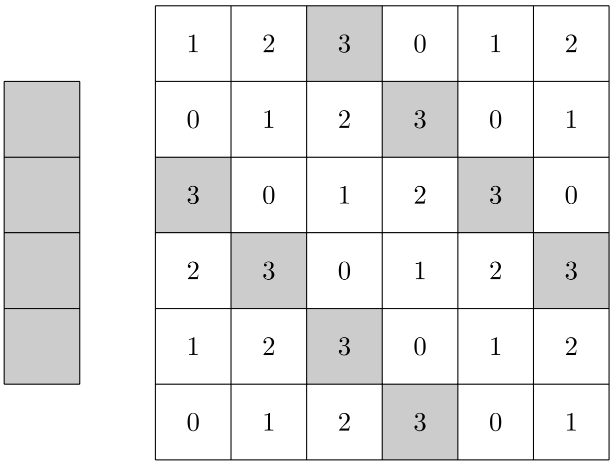


Figure 12.5: One example of the folklore result

12.2 Specific introduction

Now we make the translation from the accessible ideas to the more precise statements for which we obtained new results. The tiles we focus on will be the union of unit squares or (hyper)cubes and by putting these in a Cartesian coordinate system $\mathbb{R}^n$, we can associate the tiles with subsets of $\mathbb{Z}^n$. Formally, we define a tile T as any (finite) subset of $\mathbb{Z}^n$. Such a tile can be embedded in a higher-dimensional space $\mathbb{Z}^m$ by considering it as $T \times \{0\}^{m-n}$. In Figure 12.6 an example is shown for a typical tetromino in Tetris.

Given n, let T be a tile in $\mathbb{Z}^n$. The cardinality of T, $|T|$, is the size of T, i.e. the number of elements of the subset. Confirming a conjecture of Chalcraft that was

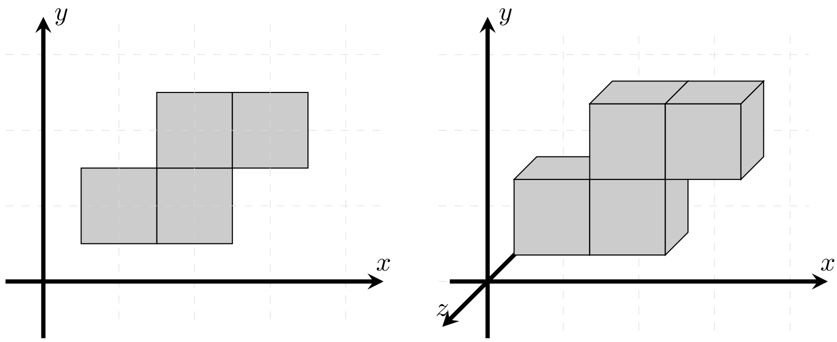


Figure 12.6: The tile $T = \{(1,1),(2,1),(2,2),(3,2)\}$ in $\mathbb{Z}^2$ and T embedded in $\mathbb{Z}^3$

posed on MathOverflow, Gruslys, Leader and Tan [48] showed that T tiles $\mathbb{Z}^d$ for some d. This is an existence result and they wondered about better bounds in terms of the dimension n and the size $|T|$. They conjectured the following for the case $n = 1$.

Conjecture 12.2.1 (Gruslys, Leader, Tan [48])**.** *For any positive integer t there is a number d such that any tile T in $\mathbb{Z}$ with $|T| = t$ tiles $\mathbb{Z}^d$.*

The main example we will be working with, will be the punctured interval $[k(1)\ell]$ which denotes an interval of $k + \ell + 1$ points with one point removed, i.e. separating two intervals of length k and l.

Wondering about Conjecture 12.2.1, one may wonder if the dimension d only depends on the genus (number of holes) of the tile instead of the size. Leading to the following question.

Question 12.2.2. *Does there exist a function $d\colon \mathbb{N} \to \mathbb{N}$ such that any tile $T \subset \mathbb{Z}$ with genus g tiles $\mathbb{Z}^{d(g)}$?*

Answering this affirmatively, would confirm Conjecture 12.2.1 since $g \leq t - 1$. As observed in e.g. [48], for any fixed d, there are one-dimensional tiles with large genus which cannot tile $\mathbb{Z}^d$, see Section 12.6. In particular we can prove that $d(g) \geq \frac{g}{2} + 1$. We note that Question 12.2.2 is false for n-dimensional tiles with $n \geq 2$, even for genus 0. In full detail, we prove in Section 12.4 that punctured intervals do tile $\mathbb{Z}^3$ as our main result.

Theorem 12.2.3. *Every punctured interval $T = [k(1)\ell]$ does tile $\mathbb{Z}^3$.*

This theorem answers two concrete questions posed by Metrebian [56, Qu. 10,11]. As a corollary the least d for which $T = [k(1)k]$ tiles $\mathbb{Z}^d$ equals $\min\{k, 3\}$, answering [48, Qu. 21].

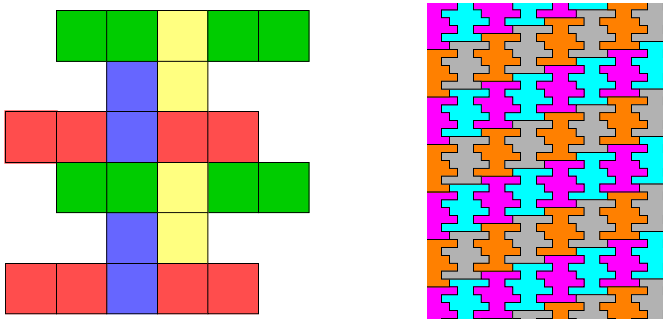


Figure 12.7: Tiling of the plane with [2(1)2]

Source: “Bicycle or unicycle: a collection of intriguing mathematical puzzles”

For $k = 1$, this is trivial as two tiles of the form [1(1)1] can be combined to a 4×1-rectangle. For $k = 2$, one cannot fill the gap in the one-dimensional line, while it is possible to tile the plane, as illustrated in Figure 12.7. For $k \geq 3$, as mentioned in [56], it is a tedious job to show that $k(1)k$ cannot tile the plane. An idea for that case distinction is given in Figure 12.8.

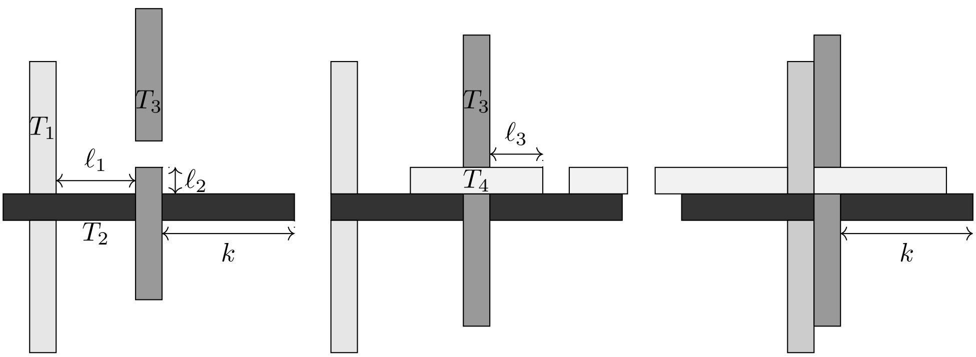


Figure 12.8: Subconfigurations that cannot occur in a tiling of $\mathbb{Z}^2$ with $[k(1)k]$

12.3 From partial to complete tilings

In this section, we will prove that finding certain partial tilings is enough to conclude that a whole tiling does exist. This is done in Lemma 12.3.1 which is a generalisation of Lemma 4 in [56].

Lemma 12.3.1. *Let T be the one-dimensional tile $[k(m)\ell]$. Suppose there are three disjoint subsets A, B, C of $\mathbb{Z}^d$ with the same cardinality such that one can tile $\mathbb{Z}^d \setminus (A \cup B)$, $\mathbb{Z}^d \setminus (A \cup C)$ and $\mathbb{Z}^d \setminus (B \cup C)$ with T. Then T tiles $\mathbb{Z}^{d+1}$.*

The construction has been sketched in Figure 12.9 for $\{1, 2, \ldots, 3(k+\ell)\} \times \{0, 1, 2\}$. By gluing infinitely many copies of that picture together, one gets the full construction of Y.

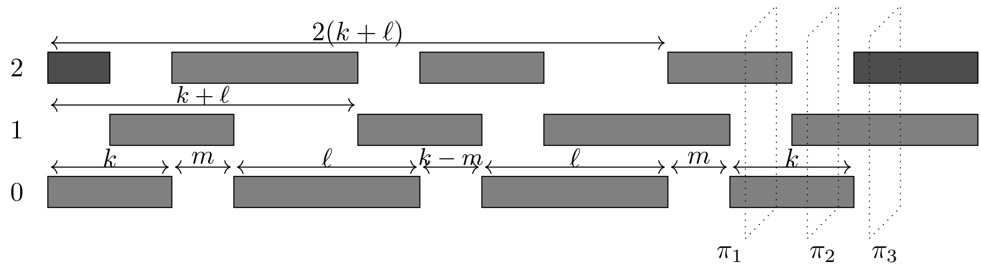


Figure 12.9: Construction of Y.

When $m \geq \min\{k, \ell\}$, where we assume without loss of generality $k = \min\{k, \ell\}$, one can glue two copies T_1, T_2 of T together to a tile T' with $k' = \ell' = k + \ell$ and $m' = m - k$. See Figure 12.10 for a depiction.

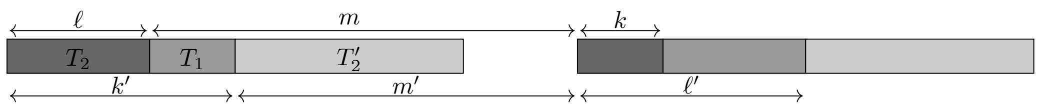


Figure 12.10: Gluing T_1 and T_2 and copies T'.

12.4 Punctured intervals tile $\mathbb{Z}^3$

Throughout this section, we let T be a punctured interval tile, which is the union of an interval of length k and an interval of length ℓ with a gap of size 1. So $T = [k(1)\ell]$ equals a translate of $\{-k, -k+1, \ldots, -1, 1, 2, \ldots, \ell\}$ as a subset of $\mathbb{Z}$. By applying Lemma 12.3.1, we will prove that T tiles $\mathbb{Z}^3$ for any k, ℓ.

First, we prove that there do exist partial tilings of the plane satisfying the conditions of Lemma 12.3.1 when T is the symmetric punctured interval $[k(1)k]$. Here infinitely many different constructions are used, depending on the number of factors

2 in the prime factorisation of n. Using some number theory, we can verify that everything works out. An example when k is odd is given in Figure 12.11.

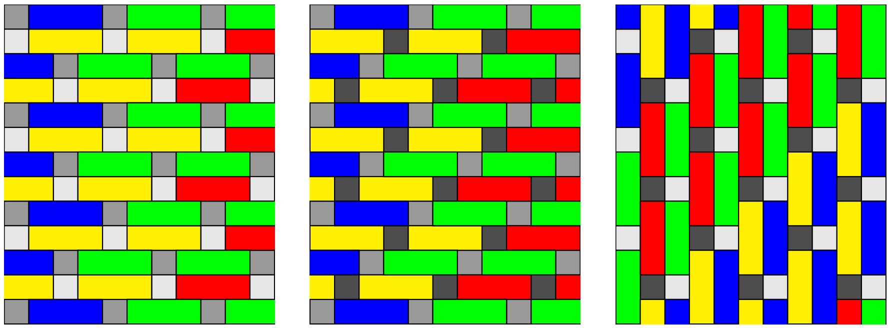


Figure 12.11: Partial tilings for $k \equiv 1 \pmod 2$.

When $k > \ell$, there are other choices for A, B and C to conclude. A sketch of an example is given in Figure 12.12.

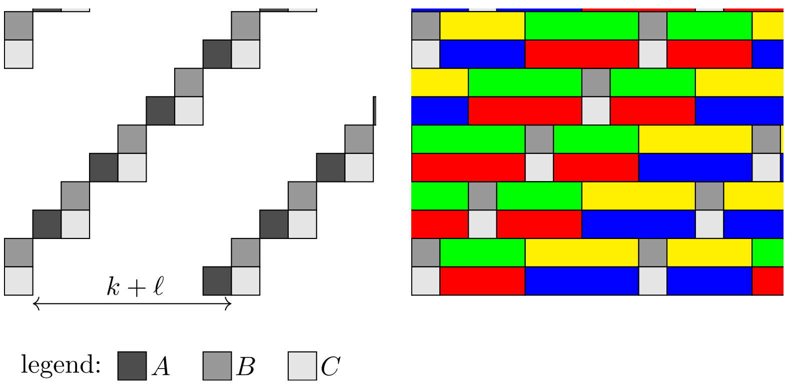


Figure 12.12: Construction of A, B, C for $T = [k(1)\ell]$ where $k = \ell + 1$ with a partial tiling for $\mathbb{Z}^2 \backslash (B \cup C)$.

12.5 The tiles $[k(2)\ell]$ do tile $\mathbb{Z}^3$ as well

Again, a general tiling strategy has been constructed. We only present a construction for a particular case.

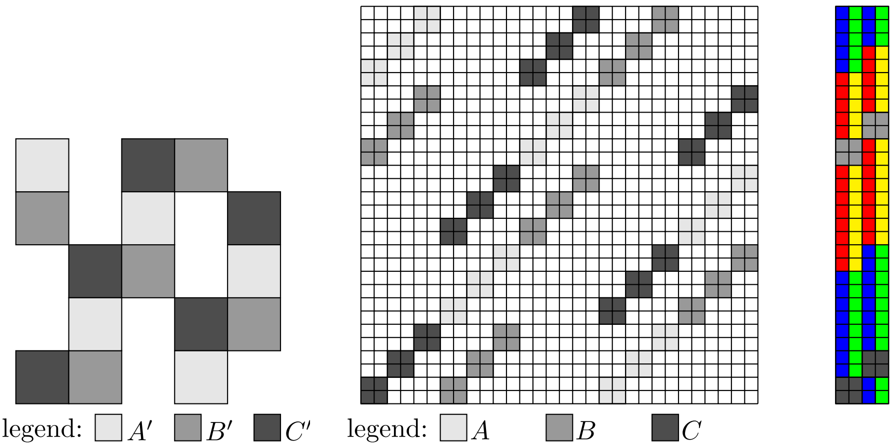


Figure 12.13: Construction of A, B, C for $T = [8(2)5]$

12.6 Impossible tilings

We end with the example of the hook-tile H_k. This tile does not tile the any fixed d dimensional plane, once k is sufficiently large. The main idea is that the area inside is much larger than the boundary and the inside cannot be partially covered by tiles lying in the same plane. As there are only $\binom{d}{2}$ directions for a plane, we can conclude. More precisely, the tile H_k does not tile $\mathbb{Z}^d$ if $k \geq 8\binom{d}{2} - 6$.

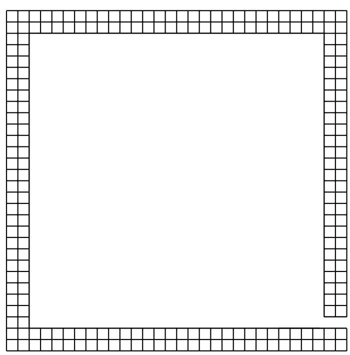


Figure 12.14: An example of a hook-tile H_{26}

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Index

Summary

"If we cannot be masters of our mouth, then we can end up as slaves to our mouth"

– Dhar Mann

This English summary will be much shorter than the Dutch version. This is because there are more sections (written in English) that are intended to be sufficiently elementary. Besides, the introduction, Chapter 0 contains already an overview of the content and some preliminaries. In this summary, we do not explain any proof and so we only focus on the main story.

In Part I we are mostly looking for the (extremal) graphs attaining the maximum or minimum total distance, under certain conditions. In Section 1.1 this has been done without any other constraint than the order (and being connected). The path P_n is the graph that is spread out the most and hence it is not surprising it has the largest total distance among all connected graphs with a fixed order. The clique K_n is the densest one and as all distances are equal to one, trivially it is the extremal graph attaining the least total distance.

In Chapter 2 we solve a problem of Plesník asymptotically, in the sense that we know the behaviour of the maximum for the total distance when n is much larger than d. For this, we construct graphs that have large total distance and we prove that these give nearly the right answer. In Chapter 3 we determine the minimum total distance given the radius r and characterise the extremal graphs, for n large compared with r. These being cycles C_{2r} where some vertices are replaced by cliques. As the cycle is part of the graph, the radius is r and due to the cliques, there are many distances equal to 1. Due to the example Q_3, see Figure 3.2 we know that more is needed to characterise the extremal graphs for all values of r.

Continuing in Chapter 4 we note that the graphs with the maximum size given radius are somewhat identical, at least for n large compared with r. Again Q_3 indicates that there are counterexamples to this.

In all of these chapters (2 to 4), there are also some results for digraphs. In certain cases, the outcome is not as elegant, but for example, the old problem of Plesník [59]

has been solved precisely for $n \gg d$ in the digraph case.

Chapter 5 deals with the old problem of Mantel [54]. His famous result says that among triangle-free graphs, the balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has the largest size. If we consider regular graphs, this implies that for n even there are at most $\frac{n^2}{4}$ edges, which is sharp of course. Nonetheless, for n odd, a regular triangle-free graph can have no more than $\frac{n^2}{5}$ edges. So this is an interesting difference. Extremal graphs can be found among the blow-up of a C_5, see Figure 6.2, and slightly adapted versions of this graph when n is not a multiple of 5.

These blow-ups of a C_5 also appear in Chapter 6, being the extremal graphs in the strong edge-colouring conjecture. Wondering about t-strong colouring for $t > 2$, we study $\omega(L(G)^t)$ instead of $\chi(L(G)^t)$ and in particular prove this for C_{2t+1}-free graphs for $t \in \{3, 4, 6\}$. This allows us to make progress on a problem of Erdős and Nešetřil.

In Part III the big chunk on graph colouring is presented, with an overview in Chapter 7 of some important results on list colouring and related extensions. Next, in Chapter 8 we collect some results related to famous conjectures. The most interesting one here is a reduction from Hadwiger's conjecture to a conjecture of Füredi et al. Hadwiger's conjecture is one of the most important conjectures in graph colouring or even graph theory as it would give a connection between structural graph theory and graph colouring that is far from immediate. Being conjectured almost 80 years ago, it is still open. The conjecture of Füredi et al. was considered to be risky according to the authors when they posed it but is still open as well. While they are claiming that one graph specifically is extremal, $2K_{2t-1}$, it remains unsolved.

In the remaining three chapters, among other interesting things, we are looking to generalisations of the minimum list size needed to colour a bipartite graph. While the conjecture of Alon-Krivelevich claims that this is possible with list size $O(\log \Delta)$, the best known bound is still of the form $O\left(\frac{\Delta}{\log \Delta}\right)$. The latter bound is true for the more general correspondence setting and for triangle-free graphs. So while bipartite graphs should be much simpler, it seems non-trivial to beat the bound $(1+o(1))\frac{\Delta}{\log \Delta}$. This is the bound one gets when applying the local lemma. Looking to asymmetric settings, as has been done in Chapter 9 one can lower the list size for one of the two partition classes to $\log \Delta$. In the asymmetric case, we also construct examples of bipartite graphs $G = (V = A \cup B, E)$ with maximum degree Δ_A, Δ_B which are not k-choosable while K_{Δ_B, Δ_A} is k-choosable. This implies that some intuition behind the Alon-Krivelevich conjecture is not completely true. Intuitively, in the complete bipartite graph, the edges are connected in the densest (most optimal) way and hence are the hardest for list-colouring. Nevertheless, it is still unclear if the derived bounds for complete bipartite graphs are sufficiently close to the right answer.

Once we can find one proper colouring using colours from the list, one can wonder about finding multiple disjoint ones. In Chapter 10 we wonder about the most extreme case for this, the possibility of a precise packing of colourings of a graph. Having defined $\chi_\ell^\star(G)$ as the minimum list size k such that for any k-list-assignment there are k disjoint proper colourings using the colours from the lists, we prove multiple results for this, but in particular that $\chi_\ell^\star(G) \le (1+o(1))\frac{\Delta}{\log \Delta}$ for bipartite graphs. The main question we wonder about here is the possibility that $\chi_\ell^\star(G) \le 2\chi_\ell(G)$ or even $(1+o(1))\chi_\ell(G)$. This chapter is based on [19].

Finally, in Chapter 11 we prove that the diameter of a reconfiguration graph $\mathcal{C}_k(G)$ is linear when $k \ge \Delta + 2$. This is related to Cereceda's conjecture. In particular this proves that the quadratic bound in Cereceda's conjecture is not sharp for regular graphs.

Chapter 12 deals on the problem of tiling with certain disconnected tiles.

Samenvatting

If you can't explain it simply you don't understand it well enough
– Albert Einstein

Aangezien wetenschappelijk onderzoek hoofdzakelijk in het Engels gepresenteerd wordt voor de experts, is deze korte Nederlandstalige samenvatting net bedoeld voor het algemene publiek. In het bijzonder richt ik me hier tot familieleden en vrienden. Het technische deel mag dan wel eens mooi zijn om door te bladeren, maar het is toch ook leuk als een deel helemaal[1] begrijpbaar is. Gedurende vier jaar kreeg ik de kans om op uitdagende vragen binnen de combinatoriek te werken. Met deze samenvatting wil ik jullie een idee geven waarover dat ging. Combinatoriek op zich is een breed begrip dat over alles kan gaan dat eindig is, maar hierin kan men de grootte van hetgeen bestudeerd worden nog steeds willekeurig groot maken. Ikzelf focuste vooral op extremale vragen. Dit zijn vragen waarbij gezocht wordt naar het minimum of maximum van iets en de voorbeelden die dat minimum of maximum ook bereiken. Wanneer men zo efficiënt mogelijk de post wil rondbrengen of het beste vaccin wil creëeren, is men ook bezig met het zoeken van een extremum. In mijn geval werd hierbij vooral gefocust op een deelgebied van combinatoriek, genaamd grafentheorie. Een **graaf** $G = (V, E)$ bestaat uit twee verzamelingen van elementen. Er is een verzameling van knopen V en een verzameling van zijden E, waarbij een **zijde** een paar is bestaande uit 2 knopen. Hierbij komt in eerste instantie ieder paar maximaal 1 keer voor. Indien dit niet het geval is, heeft een zijde een multipliciteit en spreken we over een multigraaf. Een zijde noteren we vaak als $e = uv$, waarbij u en v de bijhorende knopen zijn. In zo'n geval zijn u en v **buren** en worden ze adjacent genoemd in het Engels. Het aantal buren van een knoop v wordt zijn graad genoemd. Twee zijden worden buren genoemd als ze een eindknooppunt gemeen hebben. Het aantal knopen, $|V|$, wordt de **orde** genoemd en in het algemeen genoteerd met het symbool n. Een graaf kan steeds grafisch voorgesteld worden door iedere knoop te tekenen als een kleine cirkel en iedere zijde als een lijnstuk (of kromme) die de twee corresponderende knopen verbindt. In Figuur 12.15 is een voorbeeld getekend van een

[1] Of toch bijna helemaal.

graaf met $n = 5$ knopen. De knopenset V is gelijk aan $\{v_1, v_2, v_3, v_4, v_5\}$ en de set met de zijden E bevat o.a. v_1v_2, maar niet v_1v_5. Grafen kun je bijna overal construeren. Zo werd een collaboratiegraaf getekend in Figuur 1, waar wiskundigen verbonden worden als ze een paper samen gepubliceerd hebben[2]. In deze thesis beschrijven we vooral resultaten die gaan over twee belangrijke noties van een graaf, afstand en kleuringen. De **afstand** tussen 2 knopen u en v is het minimale aantal zijden dat je nodig hebt om van u naar v te gaan. Dat is de lengte van een kortste pad startende in u en eindigend in v. Zo is de afstand tussen v_1 en v_5 in Figuur 12.15 gelijk aan 2. Een voorbeeld van zo'n kortste pad van v_1 naar v_5 gebruikt de zijden v_1v_2 en v_2v_5 (en merk op dat er geen korter pad, van lengte 1, bestaat). Als de afstand tussen elke twee knopen eindig is, wordt de graaf **samenhangend** genoemd. In deze thesis focusen we bijna enkel op samenhangende grafen.

Het tweede begrip is **kleuring** van een graaf. Hiervan bestaan verschillende versies, zo zie je in het midden en rechts in Figuur 12.15 voorbeelden waarbij de knopen, respectievelijk de zijden gekleurd zijn. Dit zodanig dat buren geen gelijke kleur hebben. Zo'n kleuring heet dan een **propere kleuring**.

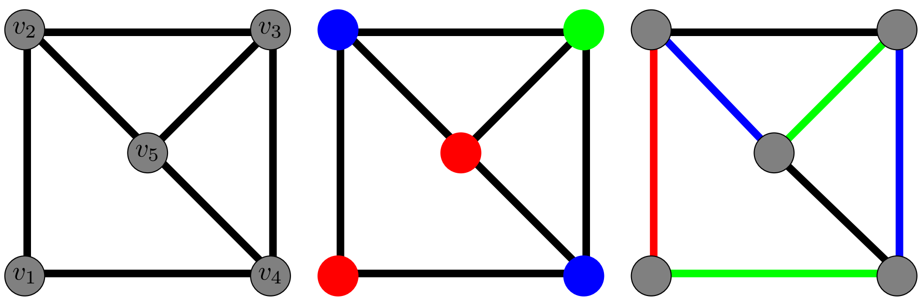


Figure 12.15: Voorbeeld van een graaf G met mogelijke kleuringen

Het is belangrijk om enkele basisgrafen te kennen. Zo heb je het **pad** P_n, **cykel** C_n, de **ster** S_n en **complete graaf** of **kliek** K_n. Zie Figuur 12.16 waarbij de grafen 5 knopen ($n = 5$) hebben. Een pad P_n wordt meestal weergegeven met n knopen op een lijn, maar dat is dus niet noodzakelijk. Alle knopen op 2 na hebben hier graad 2. Bij een cykel hebben ze allen 2 buren, terwijl bij een ster $n - 1$ knopen verbonden zijn met de laatste knoop. De kliek bevat een zijde tussen elke 2 knopen. Deze grafen geven al vaak een idee voor de potentiële extremale grafen. Dat zijn grafen die het maximum of minimum bereiken voor een zekere grootheid onder alle grafen met bepaalde eigenschappen.

[2]De gestippelde lijnen hebben te maken met papers die nog niet officieel verwerkt zijn.

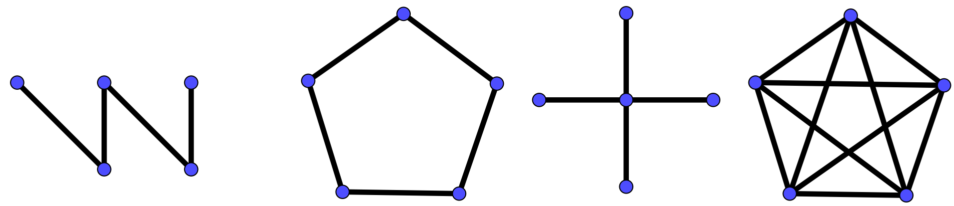


Figure 12.16: De grafen P_5, C_5, S_5 en K_5

Belangrijkste resultaten uit Deel I

Je kunt nooit een gunst te vroeg doen, omdat je nooit weet wanneer het te laat zal zijn.

–

In Deel I bekijken we afstanden tussen punten van grafen. De afstand $d(u, v)$ is zoals eerder gezegd gelijk aan de lengte (het aantal zijden) in een kortste pad van u tot v die een deelgraaf van de bestudeerde graaf G is. De **totale afstand** is dan gelijk aan de som (over alle paren knopen) van alle afstanden. De **diameter** van een graaf G, $\text{diam}(G)$, is gelijk aan de grootste afstand onder alle afstanden tussen de paren van knopen. Zo is de diameter van P_n gelijk aan $n - 1$ en de diameter van K_n gelijk aan 1. De totale afstand is ook maximaal respectievelijk minimaal onder de samenhangende grafen voor deze twee grafen. Dit is een folkloreresultaat die in Hoofdstuk 1 vermeld wordt. Een opmerkelijk voorbeeld van een resultaat uitgelegd in Hoofdstuk 1 is gerelateerd aan ongelijkheden. Zo was er een vermoeden door een andere groep wiskundigen, die vrij geloofwaardig leek, over een ongelijkheid waarbij enkele parameters van een graaf als input genomen werden. De eenvoudige versie van hun vermoeden bleek algemener waar te zijn als een gevolg van een algemene ongelijkheid (Karamata's ongelijkheid). De sterkere versie was echter foutief, aangezien oneindig veel tegenvoorbeelden geconstrueerd konden worden. Dit ondanks dat we bewezen dat het waar was voor 100% van de grafen (de fractie van tegenvoorbeelden is een onbepaalde breuk van de vorm $\frac{\infty}{\infty}$ die in dit geval gelijk is aan 0). Dit is uitgebreider uitgewerkt in [26].

In Hoofdstuk 2 werd vooruitgang gemaakt op een oud probleem van 1984, waarin gevraagd wordt voor de maximale gemiddelde (of totale) afstand gegeven de diameter (en aantal knopen) van een graaf. Voor vaste diameter en orde (aantal knopen) gaande naar oneindig, blijkt dat de gemiddelde afstand bijna gelijk kan zijn aan de diameter. Dit kan gezien worden via een constructie waarvan heel veel paren van knopen een afstand gelijk aan de diameter hebben. Hiervoor construeerden we **bomen** (grafen die geen cykel bevatten als deelgraaf) met veel takken vanuit het centrum, i.e. centraal was er een knoop van grote graad. Iedere tak splits zich met heel veel **bladeren** (knopen met maar 1 buur) op het einde. Een schets hiervan is gegeven in Figure 2.1a. Centraal wordt nog een extra kliek geplaatst indien de diameter oneven is. Bij het willekeurig nemen van 2 knopen, zal de kans dan groot zijn dat het beide bladeren zijn en dit van verschillende takken. Met kans 100% (asymptotisch resultaat, i.e. in de limiet) zal de onderlinge afstand gelijk zijn aan de diameter, ondanks dat dat niet zo is voor alle paren. We hebben met andere woorden gevonden hoe de oplossing zich gedraagt wanneer de orde n heel groot is.

Als we ons focussen op bomen, kan de schatting zelfs nog exacter gemaakt worden. Een voorbeeld van de extremale boom voor orde $n = 249$ en diameter 8, wordt weergegeven in Figuur 12.17. Hier is er dus een centrale knoop, waaruit 10 takken vertrekken. Drie takken hebben elk 19 bladeren, terwijl er 7 takken zijn die nogmaals splitsen in deeltakken die elk eindigen in 10 bladeren. Een groot deel van de knopen van een tak zijn dus bladeren. De afstand tussen bladeren van verschillende takken is hier gelijk aan 8. Dit concrete voorbeeld toont wel dat de extremale graaf niet helemaal regelmatig is, in de zin dat de takken nogal verschillend kunnen zijn. Dit is een reden waarom we voor het algemene geval enkel een schatting hebben gegeven. Bij die schatting kijken we vooral naar het verschil tussen de diameter en maximale afstand en dat verschil is nu beter bepaald dan ooit tevoren.

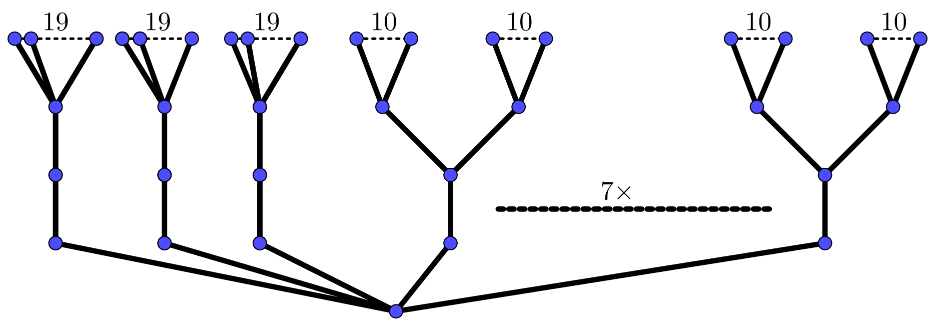


Figure 12.17: De boom met 249 knopen en diameter 8 met maximale totale afstand

Verder worden dan nog soortgelijke vragen behandeld. We vermelden nog 1 voorbeeld. De **eccentriciteit**, $\mathrm{ecc}(v)$ van een knoop v is gelijk aan de afstand tot de knoop die het verst verwijderd is van v. De diameter van een graaf was dan gelijk aan $\max_v \mathrm{ecc}(v)$. Zo ook bestaat de notie van de **straal** van een graaf, deze is gelijk aan $\min_v \mathrm{ecc}(v)$. Voor een boom is de diameter ongeveer gelijk aan twee keer de straal, maar voor een cykel zijn beide gelijk. De maximale gemiddelde afstand gegeven straal en orde hebben we goed afgeschat. Voor het minimum was er een vermoeden van Chen-Wu-An over de structuur van de extremale grafen. Zij dachten dat de extremale grafen met minimale totale afstand gegeven orde n en straal r altijd zouden bestaan uit een cykel C_{2r} waar enkele knopen vervangen waren door klieken. Dat vermoeden bleek foutief te zijn, aangezien de graaf van de kubus, Q_3, ook extremaal is voor $r = 3$ en $n = 8$. Echter is hun vermoeden wel bewezen voor n voldoende groot voor vaste r groter dan 3. Zij die willen kunnen de totale afstand nog even narekenenen voor de 3 extremale grafen, die weergegeven zijn in Figuur 12.18.

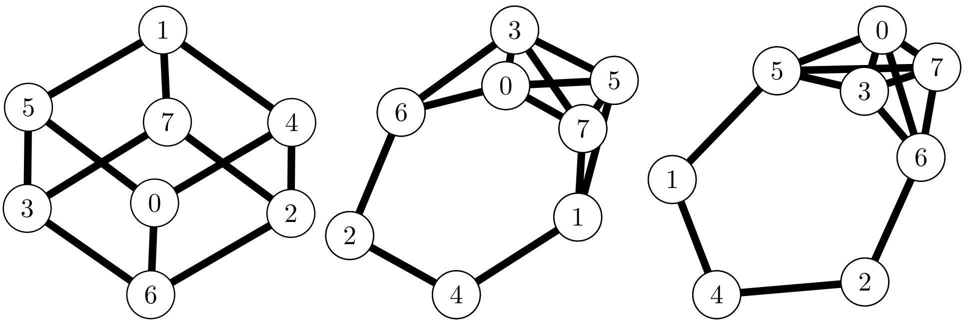


Figure 12.18: De grafen $Q_3, G_{8,3,2}$ en $G_{8,3,1}$

Belangrijkste resultaten uit Deel II

Welke koers je ook bepaalt, er is altijd iemand die je verteld dat je het verkeerd hebt.

–

Deel II bevat enkele overgangshoofdstukken, waarbij verbanden tussen verschillende begrippen voor grafen worden gegeven tussen achtereenvolgens afstand, size (aantal lijnen), klieken en kleuringen. In Hoofdstuk 4 gaan we even in op het mogelijke verband tussen totale afstand en aantal lijnen. Over het algemeen kun je dan ook vermoeden dat hoe meer lijnen er zijn, hoe kleiner afstanden worden. Hier is iets van aan, maar een exact verband is lastiger. Zo merken we op dat voor gegeven straal $r = 3$ en orde $n = 8$, de graaf Q_3 een lijn/ zijde minder heeft dan de grafen $G_{8,3,2}$ en $G_{8,3,1}$. Hier bepalen we opnieuw extremale grafen in enkele speciale gevallen.

In Hoofdstuk 5 kijken we naar enkele vragen binnen de extremale combinatoriek die gaan over het verband tussen het aantal zijden en het aantal klieken van een bepaalde grootte. De beroemde stelling van Mantel [54] uit 1907 zegt dat onder alle grafen zonder driehoek op n knopen, de gebalanceerde complete bipartiete graaf $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ de meeste zijden heeft. Dit is de graaf bestaande uit twee verzamelingen A en B van grootte $\lfloor \frac{n}{2} \rfloor$ en $\lceil \frac{n}{2} \rceil$ dewelke zijden heeft tussen twee knopen als en slechts als (dan en slechts dan)[3] de ene in A zit en de andere in B. In Figuur 12.19 zie je bijvoorbeeld $K_{3,3}$. Het aantal zijden van de graaf $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is dan $\left\lfloor \frac{n^2}{4} \right\rfloor$. Als we echter naar **reguliere grafen** kijken, dat zijn grafen waarbij elk punt gelijke **graad** (hetzelfde aantal buren) heeft, zien we dat wanneer n oneven is, het maximum mogelijk aantal zijden ongeveer $\frac{n^2}{5}$ is, hetgeen een opmerkelijk verschil is met de $\frac{n^2}{4}$ mogelijke zijden indien n even is.

Ook is er een zogenaamd supersaturatieresultaat bewezen, die zegt dat als de graad van zo'n reguliere graaf $2 \left\lfloor \frac{n}{5} \right\rfloor + 2$ is, dit direct betekent dat er een kwadratisch aantal driehoeken aanwezig is.

In Hoofdstuk 6 tot slot, kijken we nog naar een analogon van een beroemd probleem. Daar kijken we naar de afstand tussen de zijden in plaats van tussen de knopen. Omdat het kliekgetal (grootte van maximale kliek gevat in de graaf) altijd een ondergrens is voor het kleurgetal (chromatisch getal)[4], zijn er verbanden tussen de resultaten over klieken en die over kleuringen. Als speciaal geval, werd bepaald dat de graaf in Figuur 6.3 de unieke graaf is met 22 zijden en maximale graad 3 waarvoor elke 2 zijden met elkaar verbonden kunnen worden gebruikmakend van maximaal 2 zijden. In het algemeen werden betere afschattingen bepaald voor het probleem dan degene die voorheen gekend waren.

Belangrijkste resultaten uit Deel III

Het leven is niets dan een experiment, hoe meer je experimenteert, hoe beter.

–

In Deel III zijn er 5 hoofdstukken die allen gaan over het kleuren van grafen. Hierbij zijn er verschillende soorten kleuringen. Bekijken we bijvoorbeeld de graaf $K_{3,3}$ (zie Figuur 12.19), dan bestaat er een propere kleuring met 2 kleuren. Je kunt namelijk de linkse knopen in 1 kleur kleuren en de rechtse knopen in een ander kleur. Er bestaat dus een partitie in 2 (bi) **onafhankelijke** verzamelingen (tussen de knopen is er geen enkele zijde), vandaar de naam **bipartiete** graaf. Als er nu echter voor iedere knoop slechts 2 keuzes zijn voor de kleuren, is het niet altijd mogelijk om met een propere kleuring te eindigen. Die mogelijke keuzes van kleuren voor een knoop v, wordt de lijst van kleuren $L(v)$ genoemd. Als je voor elke knoop in Figuur 12.19 een kleur kiest van de 2 waarmee het gedeeltelijk gekleurd is, zul je zowel links als rechts minstens 2 kleuren nodig hebben, terwijl er maar 3 kleuren in totaal toegelaten waren. Met $\chi_\ell(G)$ wordt aangeduid wat de minimale lengte van alle lijsten moet zijn, omdat je steeds een propere kleuring kunt kiezen waarbij elke knoop een kleur krijgt uit diens lijst. In het geval van $K_{3,3}$ is die waarde gelijk aan 3. Enkele elegante resultaten

[3] Er is geen aparte samenvatting voor Vlaams en Nederlands.
[4] Uitleg komt zodadelijk

over lijstkleuringen van doorheen de geschiedenis zijn vermeld in Hoofdstuk 7. Ook enkele andere soorten chromatische getallen zijn daar gedefinieerd. Een **chromatisch getal** is net de waarde van hoeveel kleuren je minstens nodig hebt onder bepaalde voorwaarden om zeker te zijn dat een goeie kleuring mogelijk is.

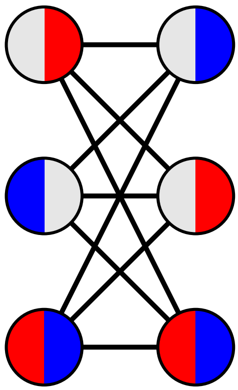


Figure 12.19: De graaf $G = K_{3,3}$ voldoet aan $\chi(G) = 2$ en $\chi_\ell(G) = 3$

In Hoofdstuk 8 gaan we verder met enkele voorbeelden en leggen we connecties met gekende resultaten of vermoedens in het deelgebied over kleuringen van grafen. Eén van de grootste problemen binnen de grafentheorie is Hadwiger's vermoeden. Dit vermoeden is na zo'n 80 jaar nog steeds niet bewezen, noch ontkracht. Dit ondanks de vele jaren dat er al gezocht is naar een oplossing door verschillende wiskundigen. In Hoofdstuk 8 is bewezen dat uit Hadwiger's vermoeden zou volgen dat een ander vermoeden, hetgeen ook nog steeds niet bewezen is, waar is. De auteurs van dat tweede vermoeden hadden zelf geschreven dat hun vermoeden riskant was. Een tegenvoorbeeld voor hun vermoeden zou echter betekenen dat Hadwiger's vermoeden ook niet waar is, wat dan een grote doorbraak zou zijn.

In de verdere hoofdstukken van Deel III focussen we op een ander gekend vermoeden, het Alon-Krivelevich vermoeden. Dat vermoeden zegt dat voor bipartiete grafen met maximale graad $\Delta \geq 2$, als de lijsten minstens $C \log \Delta$ kleuren hebben voor zekere C, we steeds een propere kleuring kunnen vinden waarbij enkel kleuren uit de corresponderende lijsten gebruikt wordt. Tot nog toe was de beste grens ongeveer gelijk aan $\frac{\Delta}{\log \Delta}$, hetgeen exponentieel ver verwijderd is van de grens uit het vermoeden. In Hoofdstuk 9 keken we naar een assymetrische variant en bewezen dat we dan dichter kunnen komen van het vermoeden van Alon-Krivelevich. Een bipartiete graaf heeft 2 partitieklassen A, B. Het is voldoende dat de lijsten in de ene partitieklas lengte $\log \Delta$ hebben en de andere kant ongeveer $\frac{\Delta}{\log \Delta}$. Met een roze bril op, zou je kunnen zeggen dat we halfweg zijn op die manier, maar dat is waarschijnlijk niet zo. Het bewijs gebruikt hier enige willekeurigheid. Als je de knopen van A willekeurig kleurt, is het voldoende om elke knoop in B te kunnen kleuren met een kleur die nog niet aan een buur geassocieerd is. Vervolgens konden de kansen daarop bepaald worden en zo de conclusie getrokken worden.

In Hoofdstuk 10 tot slot kijken we naar een andere vorm, waarbij we niet slechts 1 kleuring willen vinden, maar net een partitie in kleuringen. Dat zijn verschillende kleuringen, die samen elke kleur uit een lijst eenmaal gebruiken. We illustreren dit aan de hand van een kleuring voor de cykel C_4, wat ook gelijk is aan de bipartitiete graaf $K_{2,2}$. Het is niet zo moeilijk in te zien dat C_4 steeds met 2 kleuren gekleurd kan zijn, ook in het geval dat iedere knoop een vooropgestelde lijst van 2 mogelijke kleuren krijgt. Onderscheid hiervoor het geval dat iedere lijst hetzelfde is en het geval dat er verschillende lijsten bestaan.

Als je kijkt naar Figuur 12.20, merk je echter op dat als je de knoop v in het blauw kleurt, het een rode en grijze buur zal moeten hebben. Maar u heeft dezelfde buren en kan nu niet zo gekleurd worden dat we eindigen met een propere kleuring. In het bijzonder zullen twee verschillende mensen geen 2 propere kleuringen kunnen kiezen waarbij elke knoop een verschillend kleur heeft gekregen van de 2 personen. We hebben hiervoor het list packing chromatic number $\chi_\ell^\star$ gedefinieerd en nu opgemerkt dat $\chi_\ell^\star(C_4) > 2$. Met iets meer werk zien we dat $\chi_\ell^\star(C_4) = 3$.

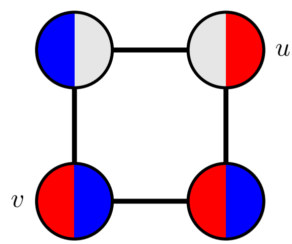


Figure 12.20: $G = K_{2,2}$ satisfies $\chi_\ell(G) = 2$ and $\chi_\ell^\star(G) = 3$

Het was een uitdaging om te bewijzen dat voor elke bipartiete graaf geldt dat $\chi_\ell^\star(G)$ begrensd is door $(1+o(1))\frac{\Delta}{\log \Delta}$. Hier is de $o(1)$ iets die verwaarloosbaar klein is (in vergelijking met 1) wanneer Δ erg groot is. Zo hebben we uiteindelijk een sterkere versie bewezen van enkele eerder gekende resultaten gerelateerd aan het vermoeden van Alon en Krivelevich.

Belangrijkste resultaten uit Deel IV

Als je nooit wat vindt, heb je te hard gezocht.

–

Hoofdstuk 11 voelt als een afsluiter van het voornaamste dat bestudeerd werd. Hierbij kijkt men naar de diameter van een zogenaamde reconfiguratiegraaf. De **reconfiguratiegraaf** $\mathcal{C}_k(G)$ heeft als knopen alle mogelijke propere kleuringen van graaf G met k (vaste) kleuren. De zijdes van $\mathcal{C}_k(G)$ bestaan uit paren van 2 kleuringen van G die overal gelijk zijn, met uitzondering van exact 1 knoop van G. Als voorbeeld is $\mathcal{C}_3(K_2)$ volledig weergegeven in Figuur 12.21. Dit is dan eigenlijk gewoon de cykel C_6 en heeft diameter 3. We werken hier dus wel degelijk met zowel kleuringen van grafen G als de diameter van de graaf $\mathcal{C}_k(G)$. Er werd hier, in Hoofdstuk 11, onder andere bewezen dat de diameter van $\mathcal{C}_k(K_n)$ exact gelijk is aan $\left\lfloor \frac{3n}{2} \right\rfloor$ voor k voldoende groot.

Het is echter wel geweten dat de grens niet altijd lineair is, bijvoorbeeld $\mathcal{C}_3(P_n)$ heeft een diameter die kwadratisch is als functie van n. Voor reguliere grafen is dit wel het geval wanneer $k \geq \Delta + 2$. Dat betekent dat een vermoeden van Cereceda waar is met een lineaire in plaats van een kwadratische grens in het geval van reguliere grafen.

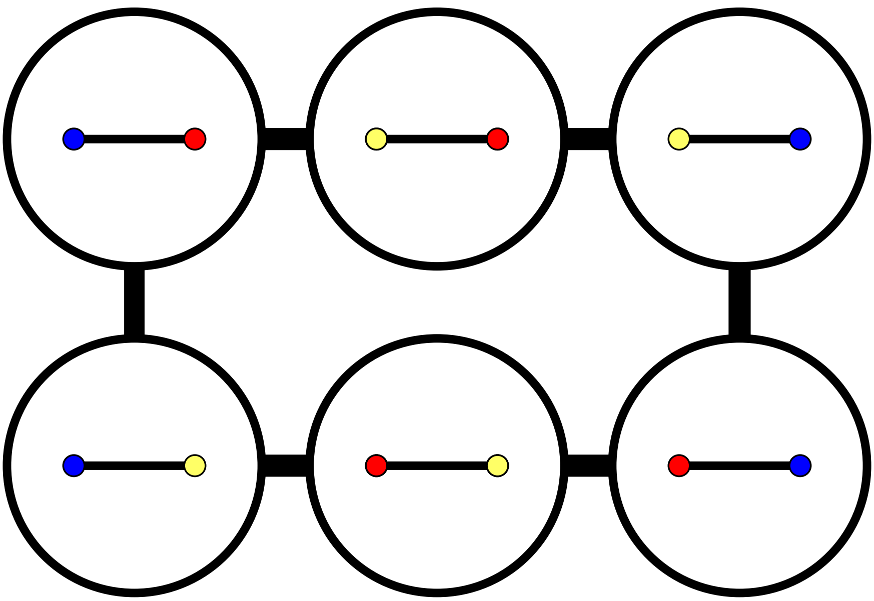


Figure 12.21: De configuratiegraaf $\mathcal{C}_3(K_2)$

Hoofdstuk 12 is een afsluitend hoofdstuk waarbij gepuzzeld wordt in hogere dimensies. In tegenstelling tot eerder onderzoek, werd hier gekeken naar niet-samenhangende tegels. In geval paren van rechthoekige tegels die niet verbonden zijn, maar wel een vaste oriëntatie hebben tegenover elkaar. Met behulp van elementaire getaltheorie wordt net bewezen dat verschillende patronen gevormd kunnen worden. Het idee is dat er eerst drie patronen, gedeeltelijke betegelingen in 2D, gevormd worden. Dit op zodanige wijze dat deze in een volgorde kunnen gezet worden waarbij alle onbedekte plaatsen gevuld kunnen worden met tegels die loodrecht op de vlakken staan. Ook voor een zogenaamde leek, zou het begin van dit hoofdstuk best wel aantrekkelijk kunnen zijn.

Belangrijkste resultaten niet vermeld in de thesis

Haal de moet eruit en hou de moed erin!

–

Bij de masterdiploma-uitreiking werd een verhaal voorgelezen door twee klasgenotes van me, Anne en Katelijne, waarbij ze de wiskundige resultaten vergeleken met sprookjes. Aangezien een promotietraject lang duurt, was er voldoende tijd voor heel wat onderzoek en dus veel sprookjes. Echter worden de sprookjes niet verzonnen, maar is het meer een ontdekkingsreis, waarbij verbanden en vreemde monsters (extremale grafen) ontdekt worden die men eerst niet voor mogelijk hield. Omdat die ontdekkingsreis best intrigerend was en er ondervonden werd dat het gewoon leuk is (dus niet van moeten), zijn er meer sprookjes geschreven dan noodzakelijk voor de thesis.

In deze thesis werden dan ook vooral resultaten vermeld die pasten om te bundelen binnen een geheel. Hierbij was Hoofdstuk 12 al een uitzondering, hetgeen voor zij die voldoende Engels kennen net een meer elementair deel is (of toch de introducties) en dus een aanrader. Echter zijn er nog leuke zaken waarop gewerkt is die ik hier kort wil beschrijven, omdat dit het moment is om voor jullie allen eens te vertellen wat ik zoal gedaan heb in die vier jaar.

Als eerste wil ik nog even verder gaan binnen de grafentheorie. Samen met Stephan Wagner en Hua Wang [29] hebben we ook gekeken naar de bomen waarvan de deelbomen gemiddeld het grootst zijn. Een boom T is een samenhangende graaf zonder cykel en een **deelboom** is een samenhangende deelgraaf van T. Voor kleine waarden van n (meer bepaald tot 8) is de ster S_n degene waarvoor die gemiddelde deelboom maximaal is. Merk op dat de deelbomen dan singletons zijn of bomen S_k met $k \leq n$. Hierbij zijn er $\binom{n-1}{k-1}$ mogelijke deelbomen S_k wanneer $k \geq 2$. Wanneer n heel groot zou zijn, zou de gemiddelde deelboom ongeveer de helft van de knopen hebben. Echter zijn er manieren om dat beter te doen. Jamison vermoedde dat de extremale graaf wel altijd een caterpillar (rups) zou zijn. Dit is een graaf die bestaat uit een pad waarbij enkele extra knopen verbonden zijn met knopen van het pad. Om aan te tonen dat dit mogelijk is, bewezen we vooreerst een lokale versie, waarbij enkel gekeken wordt naar deelbomen die een vaste knoop r bevat. De extremale graaf is dan een bezem (broom), een pad verbonden met een ster. Figuur 12.22 toont de oplossing wanneer $n = 16$.

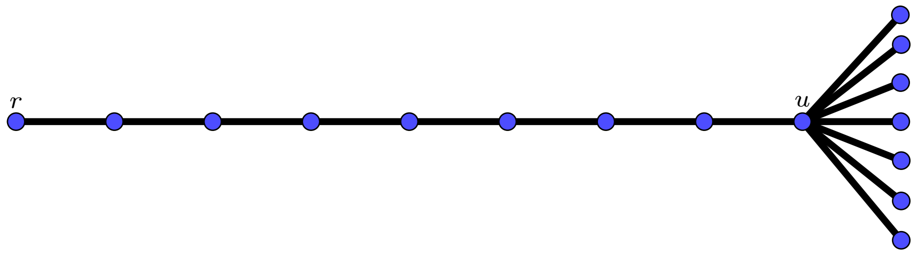


Figure 12.22: De bezem met lengte 9 en breedte 7 maximaliseert de lokale subboomgrootte.

Merk op dat er in dit geval 2^7 deelbomen zijn waarvoor het volledige pad van r tot en met u een deel is van de deelboom die r bevat. In het algemeen zal het aantal bladeren op het einde ongeveer gelijk zijn aan $2 \log_2 n$ bij het extremale geval. Het leek even aannemelijk dat de extremale graaf voor de globale versie van het probleem dan zou bestaan uit een dubbele bezem, zodat heel veel deelbomen het volledige centrale pad zouden bevatten. Echter blijkt dat niet het geval en hebben we zelfs bewezen dat er heel veel extra bladeren rond de stam (dat een pad is) van de boom ronddwarrelen. Dit is dan algemeen bewezen en als voorbeeld hebben we de waarschijnlijke extremale boom voor $n = 45$ geconstrueerd in Figuur 12.23. Net door die iets grilligere structuur lijkt het laatste open vermoeden van Jamison moeilijker te bewijzen. Als een gevolg van de lokale versie, hadden we echter wel een goeie schatting voor de maximale gemiddelde subboomgrootte.

In een ander project, keken we naar een ander oud (uit de jaren '60) vermoeden, het vermoeden van Černý. Dit vermoeden gaat over **automaten**, hetgeen in computerwetenschappen iets is waarmee transities van toestanden gegeven wordt. Zonder de definitie te geven, willen we wel de Černý automaat C_n tonen als voorbeeld, weergegeven voor $n = 5$ in Figuur 12.24a. Merk op dat vanuit iedere toestand een pijl vertrekt met een operatie a, alsook een pijl met b. Het kan als een oefening gezien worden dat $w = ba^4ba^4ba^4b$ het kortste woord (dat een volgorde van operaties weergeeft) is waarvoor geldt dat men vanuit elke starttoestand in dezelfde toestand eindigt, namelijk 3. Het woord w wordt dan een synchroniserend woord van C_5 genoemd. Als zo'n automaat (afgekort een DFA) met n toestanden een synchroniserend

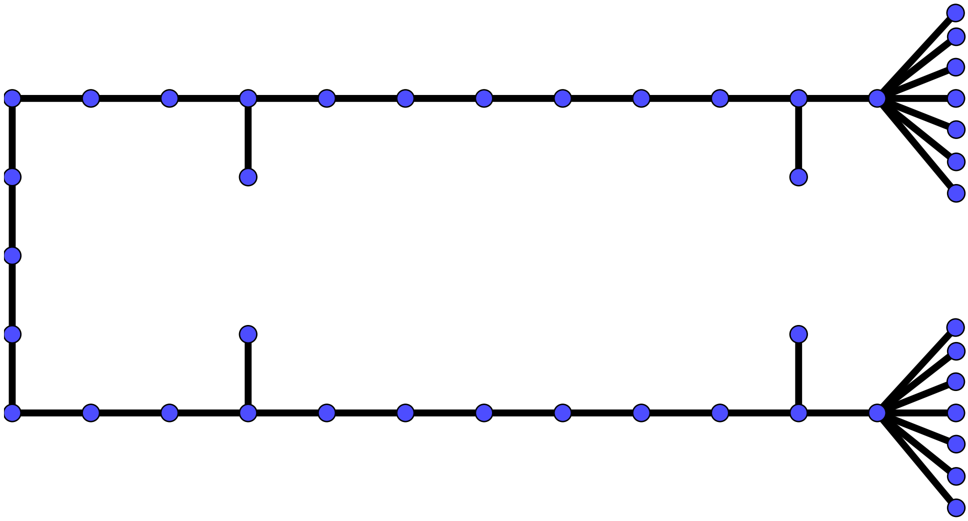


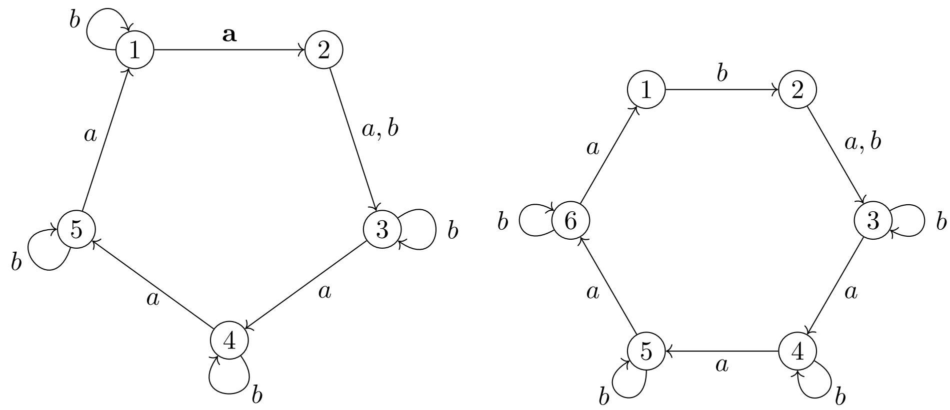
Figure 12.23: Boom met $n = 45$ en grote gemiddelde subboomgrootte

woord heeft, zegt het vermoeden van Černý dat het kortste synchroniserende woord niet meer dan $(n-1)^2$ letters heeft.

Wij keken echter naar een partiële variant (een zogenaamde PFA), waarbij niet vanuit elke toestand elke operatie mogelijk is. Wanneer we het niet toelaten dat een woord niet uitgevoerd kan worden, kan het kortste synchroniserende woord langer zijn dan $(n-1)^2$. Zo'n voorbeeld wordt weergegeven in Figuur 12.24b. Vanuit toestand 1 kan je operatie a niet uitvoeren.

In [22] merkten we nog opmerkelijke verbanden op met de rij van Fibonacci voor de automaten van de vorm P_n. Als je op alle toestanden pionnen zou plaatsen, blijkt dat bij een minimaal synchroniserend woord dat de pionnen zich samenvoegen met aantallen die te maken hebben met Fibonacci getallen. Hierbij werd een reductie gedaan naar een race met n opeenvolgende pionnen, waarbij in iedere beurt een pion naar voor kan gaan voor een prijs gelijk aan 2 of kan blijven staan met kost 1. Op die manier krijg je dat de rode lantaarn een inhaalrace moet doen op de gele trui. In Figuur 12.25 is dit weergegeven voor $n = 7$. Merk op dat $2 + 3 < 7 < 3 + 5$ en er daardoor meerdere optimale races waren.

Tot slot is er ook wat werk gedaan die zijn oorsprong vindt in de extremale verzamelingenleer. Om daar wat intuitie over te krijgen, hebben we 2 beroemde resultaten weergegeven. In Figuur 12.26a staan twee mogelijke grootste families weergegeven waarbij deze familie geen 2 verzamelingen bevat waarvan de een de ander bevat. Sperner's lemma zegt net dat voor een grondset $[n] = \{1, 2, \dots, n\}$ de



(a) De automaat (DFA) C_5

(b) De PFA P_6

Figure 12.24: Voorbeelden van automaten uit een Černý familie

maximale grootte altijd gelijk is aan $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Merk op dat in de tekening met 12 de verzameling $\{1, 2\}$ bedoeld is en analoog voor andere afkortingen. In het geval met $m = 3$, zijn $\binom{[3]}{1}$ and $\binom{[3]}{2}$ de enige extremale families. Dit zijn de families van alle singletons, respectievelijk alle verzamelingen met 2 elementen.

Een ander klassiek voorbeeld gaat over families $\mathcal{F} \subset 2^{[n]}$ die **snijdend** zijn, i.e. voor elke $A, B \in \mathcal{F}$ geldt $A \cap B \neq \varnothing$. De stelling van Erdős-Ko-Rado zegt dat zo'n familie niet meer dan 2^{n-1} elementen kan hebben. Dit kan je inzien door op te merken dat A en zijn complement $A^c = [n] \backslash A$ nooit beide in $\mathcal{F}$ kunnen zitten. Voor $n = 3$, zijn er 4 extremale snijdende families, de drie triviale families met een vast element $\mathcal{F}_i = \{A \subset [3] \mid i \in A\}$ voor $1 \leq i \leq 3$ en de familie $\binom{[3]}{\geq 2} = \{S \subset [3] \mid |S| \geq 2\}$. Deze zijn weergegeven in Figuur 12.26b (op $\mathcal{F}_2$ na).

Algemener kan men ook kijken naar families $\mathcal{F}$ die r-wijs t-snijdend zijn, dat betekent dat de doorsnede van elke r verzamelingen $F_1, \ldots, F_r$ minstens gelijk is aan t, i.e. $|F_1 \cap F_2 \cap \ldots \cap F_r| \geq t$. In [42] werd gekeken naar wat gebeurt als je twee zo'n eigenschappen tegelijk wil aannemen. Een 2-wijs 4-snijdende familie bijvoorbeeld bevat minder dan de helft van de verzamelingen omdat het een snijdende familie is. Een 3-wijs 2-snijdende familie bevat maximaal $\frac{1}{4}$ van de elementen (de verzamelingen die vaste 2 elementen hebben). De familie $\{S \subset [n] \colon |S \cap [8]| \geq 6\}$ bevat $37 \cdot 2^{n-8}$ verzamelingen (wanneer $n \geq 8$), wat beduidend meer is dan $\frac{1}{4} \cdot \frac{1}{2} \cdot 2^n$.

Samen met anderen [21, 24] hebben we gekeken naar iets minder elementaire vragen, deze gebruiken noties zoals de VC dimensie die we hier niet willen uitleg-

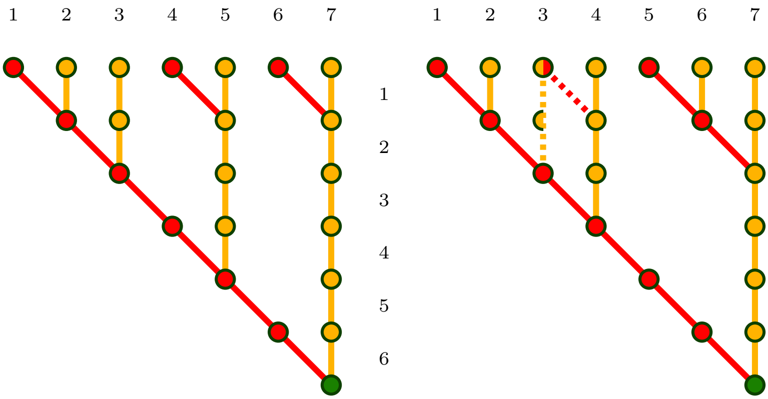
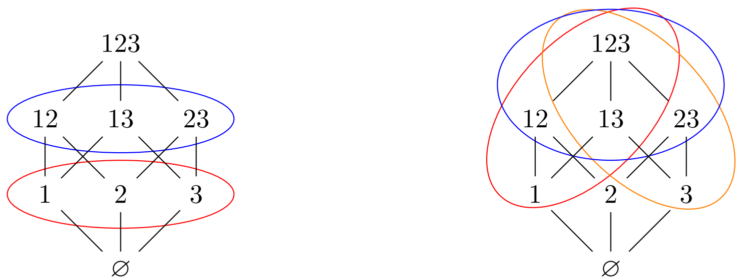


Figure 12.25: De drie optimale races voor $n = 7$, met pelotongroottes $4, 3$ en $5, 2$.



(a) Sperner's lemma

(b) ERK

Figure 12.26: Illustrations of some basic extremal set theory for $n = 3$

gen. Aangezien het wel gerelateerd was aan theoretische stellingen die betrekking hebben met machine learning, waren ook dit interessante projecten om over na te denken. In een latere periode werd ook nog gekeken naar andere vormen geïnspireerd door Erdős-Ko-Rado, zoals de resultaten in [28].

De beloning voor iets dat goed gedaan is, is het gedaan te hebben.

–

Acknowledgement

> *"When patients in their death bed were asked, what their number one regret in life was? It wasn't that they wish they spent more time at the office, or worked more overtime. The number one reason for regret was not spending more time with the people they love."*
>
> – Dhar Mann

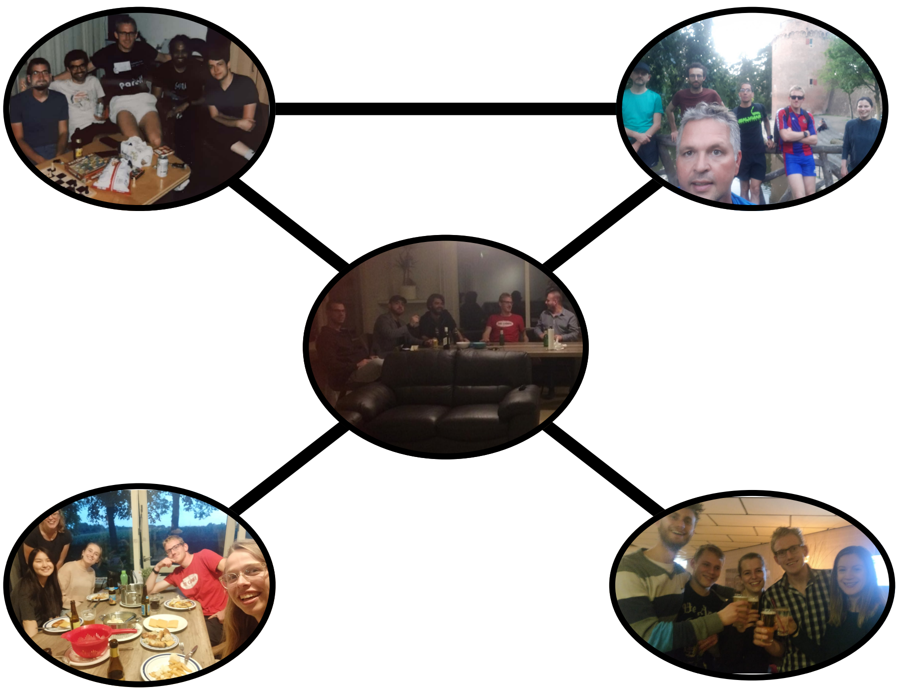
Based on this quote, I not only have to acknowledge the people who supported me and helped to obtain the PhD, but also apologise to all of the people I did not spend sufficient time with in the right way.

Still, I would like to start by thanking my supervisor, Ross. From the day of the interview for the position, he believed in my abilities to perform well as a PhD student. During the past four years, he guided me through the necessary processes, suggested many interesting problems and left me free to choose to work on the problems which were most appealing to me. I got the freedom to think about any paper and topic that interested me and to build my own belief of how much time it is worth spending on a problem. He made it possible to be flexible with my agenda to enjoy social - and sports activities. He also kept a critical view till the end of the writing process of the thesis and gave many suggestions for improvements. Furthermore, after all of this was done, I noted how much input he gave me for the long term.

For all of this, I have to thank Arne Smeets as well, who told me about the position here with a fitting subject and the nice environment of Nijmegen. I am grateful for the collaborations with Noga Alon, Bogdan Chornomaz, Rémi de Joannis de Verclos, Zeev Dvir, Yuval Filmus, Shay Moran, António Girão, N. R. Aravind, Wouter Cames van Batenburg, Viresh Patel, Stephan Wagner, Hua Wang, Henk Don, Michiel de Bondt, John Haslegrave, Ewan Davies and also Ross J. Kang himself. Some of them I have never met in real life, but still, we could add ideas and make the results even more powerful or finish a project faster. I would like to extends my thanks to Wouter in particular, for also helping me with finding accommodation for some time and sharing templates for the thesis. I am grateful to some people who were open to discuss nice problems with me, even while no joint paper came out of this.

Furthermore, I thank the members of the manuscript committee for their efforts and the time they spent on my work. Also thanks to Sarah Dresden, Lieneke Setton, Yolin van Kleef, my mother and sister for having a look at a part of the thesis.

Finishing a period of four years implies that many others made it an enjoyable and interesting time.



I like to thank my parents for letting me grow up carefree and letting me be free to choose my own life path. Thanks for taking care of me again during half of the lockdown(s) due to Covid-19. It is also nice to have a family that says they are proud and are there when forgetting e.g. an international passport.

I am blessed with the friends in Poperinge and Leuven, where I am always welcome to visit for a good time in the sporadic times being there. Once my PhD time was over, it turned out that I also had friends in Nijmegen offering to let me stay when necessary.

Thanks to the other PhD students, sharing their experiences, giving advice and organising board game evenings. I also recognise the different motivating teachers at the Radboud Sport Centrum and their student sports organisations, especially De Boosters, with whom I had some great social events and experienced nice comebacks due to the mental support during competition and tournaments. Many thanks likewise to the enthusiastic people at SSK, DKC, PON, ESN and DESDA, as well as other volunteers and different groups organising activities. Some lovely weeks with the other volunteers from SSK and the children are in my memories. Dear friends from DKC, it was a nice experience to learn the differences between consultancy and the academic world, and much more with such a great group. The end activities were among the best. People from different running groups, due to all of you, I kept running and (credits to Merlijn Smeele) I ran a marathon for the first time. Running together and supporting each other helps a lot. In the same way, completing a PhD can be seen as finishing a marathon. I got a lot of support on the side.

Special thanks to the housemates from the Scherpenkampweg for some enjoyable moments. Rob van Velp, thank you for the friendship and the long chats during our walks. Rob and Daphne, for the nice trips together and for being there during hard times and so much more. Thanks to my late housemate Tim Eppink, for always being open to help with any Crazy 88 challenge or other issues. Also, I like to thank Anissa Mouaddine and Linda Blaesing for sharing their different views on life and the start of thinking for the broader aim. These thoughts may be more important than the mathematics I studied. The inspirational quotes added to the thesis may be the most valuable insight for some of the readers as well.

Gratitude is a powerful catalyst for happiness.

–

Curriculum Vitae

> *"If you never take risks, you will never accomplish great things. Everybody dies, but not everyone lives."*
>
> – C.S. Lewis

Stijn Cambie was born the 28^{th} of April 1994 in Poperinge, Belgium. While growing up and working at a farm, he already asked many questions to his parents at a young age. As a young kid, he was full of energy and joy. During his high school studies at a technical school, he participated in many mathematical olympiads motivated by the fact that the best could go to an international event. By working hard to attain that goal, he was allowed to participate in the International Mathematical Olympiad 3 times. These opportunities to participate in the most prestigious maths olympiad, let him see some parts of the world that he otherwise would not have seen. In the period from 2012 to 2017, he did his bachelor and master at KULAK and KU Leuven with a few courses at the University of Ghent. He enjoyed student life as a praesidium member of WINA, and was enthusiastic about organising and participating in activities such as the famous 24-hour run. During his bachelor thesis, for the first time, he could do real research which lead to a publication with Wendy Goemans and Iris Van den Bussche [25]. In 2017, he got the opportunity to be a Ph.D. student at Radboud University, supervised by Dr. Ross Kang and by doing this, helping to add knowledge to the mathematical community in this period. In the past 4 years, he could investigate the problems that felt most appealing to him, leading to a bunch of diverse, but clustered publications. These were mainly in extremal combinatorics and graph theory.

In this period, he also joined the local student squash organisation De Boosters and the consultancy company De Kleine Consultant. Next to that, he did various volunteering jobs. Some of these volunteering jobs were related to mathematics such as being head of the problem selection committee of the BxMO in both 2020 and 2021 and organising the European Mathematical Cup in Belgium. Other jobs were not related to mathematics such as being a swimming buddy in preparations to benefits (e.g. the SingelSwim Utrecht and Amsterdam City Swim), or playing bridge with elderly people at ZZG Zorggroep.

In both 2020 and 2021, he was a finalist of the KWG PhD prize. At the end of 2020, he won the first IMAPP PhD Award for Scientific Communication with a talk entitled “Smiles from Tiles.”

After his PhD, he did a collaboration in Bordeaux, an internship as teacher in a high school and a 3 month postdoc at Warwick University in the group of Hong Liu.

> *“Realize deeply that the present moment is all you have. Make the NOW the primary focus of your life.”*
>
> – Eckhart Tolle