EQUIVARIANT HOMOLOGY AND REPRESENTATION THEORY
OF P-ADIC GROUPS

A Dissertation in
Mathematics
by
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Abstract

The two standard procedures for constructing representations of a reductive $p$-adic group $G$ are: parabolic induction from a Levi subgroup; and compact induction from a compact, open subgroup. Parabolic induction, along with its adjoint Jacquet restriction, underlie the theory of the Bernstein center. On the other hand, the representations of the compact open subgroups of $G$ are organized by chamber homology, which is a kind of equivariant homology for the Bruhat-Tits building. This dissertation studies the action of the parabolic/Jacquet functors and the Bernstein center on chamber homology.

We define an action of the Jacquet functors on chamber homology, and on the Hochschild homology of the Hecke algebra of $G$. Explicit descriptions of these actions are given: for general $G$, in the case of Jacquet restriction; and for $G = \text{SL}_2$, in the case of parabolic induction. Our computations extend earlier results of van Dijk, Nistor, and Dat.

We conjecture (for general $G$) and prove (for $\text{SL}_2$) that a formula of Clozel, relating the Jacquet functors in degree zero with a certain geometric partition of $G$, continues to hold for the action of the Jacquet functors on higher homology. Our result for $G = \text{SL}_2$ implies that the idempotents in the Bernstein center act on higher homology as diagonal operators, with respect to the decomposition of $G$ into its compact and non-compact parts; this extends an earlier result of Dat in degree zero.

We also construct a canonical-up-to-homotopy chain complex which computes the Bernstein components of chamber homology. The problem of computing these components was first raised by Baum, Higson and Plymen in their paper of 2000, and our result provides a new perspective on the conjectures made in that paper.
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Chapter 1

Introduction

Background

The idea of studying the representations of a group $G$ by relating them to the representations of subgroups of $G$ is a classical theme in representation theory, going back to the beginning of the subject (cf. [Cur99]). In the case of a reductive group $G$ over a $p$-adic field $F$, research since the 1950s has identified two classes of subgroups as being especially suited to this purpose:

(1) Parabolic/Levi subgroups: for example, the block-triangular/block-diagonal subgroups of the general linear group $\text{GL}_n(F)$.

(2) Compact open subgroups: for example, the general linear group $\text{GL}_n(\mathcal{O}) \subset \text{GL}_n(F)$ over the ring of integers $\mathcal{O} \subset F$.

These two families of subgroups correspond to two strands of research into the representations of reductive $p$-adic groups. Both strands are already recognizable in the earliest work in this area, Mautner’s papers [Mau58] and [Mau64] on the group $G = \text{PGL}_2(F)$. Mautner obtains the *principal series* of representations by induction from the diagonal Levi subgroup of $G$, following earlier constructions for real and complex groups. On the other hand, Mautner uses induction from the compact open subgroup $\text{PGL}_2(\mathcal{O})$ to construct *supercuspidal* representations of $G$, which are a particularity of $p$-adic groups.
The modern approach to strand (1) uses the *Jacquet functors* of parabolic induction and restriction to relate representations of $G$ to representations of its Levi subgroups. Work in this area has been heavily influenced by J. Bernstein and his collaborators (e.g., [BZ76], [BZ77], [Ber84], [BDK86], [BR92]), leading to what is now called the *Bernstein decomposition* of the category of smooth representations of $G$ (see below). Much of the theory can be organized around Bernstein’s *Second Adjoint theorem* ([Ber87], cf. [Bus01], [BK11]), an adjunction relation between the Jacquet functors.

As for strand (2), we note two prototypical results:

- Sally and Shalika show in [SS68] that all supercuspidal representations of $\text{SL}_2(F)$ are, like the supercuspidals of $\text{PGL}_2(F)$ discovered by Mautner, induced from a compact open subgroup. (This is the first of many results of this kind; we refer to [Kim09] for an historical survey.)

- Borel [Bor76] and Casselman [Cas80] characterize the unramified principal series, for a reductive group $G$, as consisting of precisely those irreducible representations which contain the trivial representation of a certain compact open subgroup, the *Iwahori* subgroup of $G$.

The study of representations by restriction to compact subgroups has been systematized by Bushnell and Kutzko in their theory of *types* [BK98].

These two strands, and the interaction between them, are the subject of much ongoing research. The collection [CN09] contains introductory surveys of these and other aspects of representation theory of reductive $p$-adic groups, including historical notes (which this author found very useful in preparing these remarks).

Work on the Baum-Connes conjecture¹ [BCH94] has suggested a novel framework in which to study the interplay between these two approaches to the representation theory of a $p$-adic group $G$: *chamber homology*, a kind of equivariant homology for the action of $G$ on its Bruhat-Tits building ([BCH94, Sections 5–6], [BHP93], [BHP00], [HN96], [Sch96]). Chamber homology is defined in terms of the representations of certain compact open

---

¹Note that for the groups under consideration here, this conjecture has been verified: by Baum, Higson and Plymen [BHP97] for $\text{GL}_n(F)$, and by V. Lafforgue [Laf02] for general reductive groups.
subgroups of $G$, and thus has an obvious affinity with strand (2). The connection with strand (1) is made via the following isomorphism, due to Higson and Nistor [HN96] and to Schneider [Sch96]:

$\bigoplus_{n \in \mathbb{Z}} H^G_{*+2n}(X) \cong HP_*(\mathcal{H}(G))$.  

The left-hand side of (1.1) is the periodized chamber homology for the action of $G$ on its Bruhat-Tits building $X$. The right-hand side is the periodic cyclic homology (a noncommutative generalization of de Rham cohomology) of the Hecke algebra $\mathcal{H}(G)$ of locally constant, compactly supported functions on $G$. As we shall outline below, the main constituents of strand (1)—the Jacquet functors and the Bernstein decomposition—have natural counterparts in the cyclic homology of the Hecke algebra, and so they may be transported into chamber homology via the isomorphism (1.1). This thesis is devoted to studying the Bernstein decomposition and the Jacquet functors in chamber homology.

The theory of the Bernstein decomposition associates to $G$ a certain commutative algebra $\mathfrak{Z}(G)$, the Bernstein center, which acts naturally on each $G$-module. Work of Bernstein and others identifies $\mathfrak{Z}(G)$ as the algebra of regular functions on a certain space $\Omega(G)$, a disjoint union of complex affine varieties, whose connected components are parametrized by equivalence classes of supercuspidal representations of Levi subgroups [Ber84]. The characteristic function of each such component is an idempotent in $\mathfrak{Z}(G)$, and these idempotents $E$ effect a natural direct-sum decomposition of every representation $V$ of $G$:

$V \cong \bigoplus_{E} EV$.  

This applies in particular to the representation of $G$ as left-translation operators on $\mathcal{H}(G)$, and the components $E\mathcal{H}(G)$ are in this case two-sided ideals in the algebra $\mathcal{H}(G)$. It follows almost immediately that

$HP_*(\mathcal{H}(G)) \cong \bigoplus_{E} HP_*(E\mathcal{H}(G))$.  

Baum, Higson and Plymen observed in [BHP00] that this decomposition yields, via a refinement of (1.1), a decomposition in chamber homology:

\[ H^G_*(X) \cong \bigoplus_E H^G_*(X)_E. \]

Those authors made a number of conjectures regarding the components \( H^G_*(X)_E \) (or, more accurately, regarding a certain chain complex which, conjecturally, computes \( H^G_*(X)_E \)). If verified, these conjectures would argue strongly in favor of the contention that (1.1) acts as a link between strands (1) and (2). Calculations in [BHP00] and [AHP06] provide some evidence for these conjectures.

To conclude these background remarks, let us mention another set of post-Baum-Connes conjectures connected to the Bernstein decomposition: those of Aubert, Baum and Plymen [ABP06]. Whereas the conjectures in [BHP00] aim to relate the Bernstein components to representations of compact subgroups (connecting strands (1) and (2)), the conjectures of [ABP06] focus on the geometric structure of these components, with applications to delicate questions about reducibility of parabolically induced representations (strand (1)). The two sets of conjectures overlap in predicting the homology groups \( H^G_*(X)_E \), and to that extent each may potentially be used to provide evidence for (or, perhaps, against) the other. The deeper connections between the two remain, as far as the author is aware, an intriguing question for future study.

**Main Results**

In Chapter 3, we consider the Bernstein components \( H^G_*(X)_E \) from a different point of view to that of [BHP00]. Our approach is based on the following isomorphism, due to Higson and Nistor [HN96, Proposition 3.6] and Schneider [Sch96, Proposition 2]:

\[ H^G_*(X) \cong \mathcal{H}_*(G, \mathcal{H}(G_c)_{Ad}). \]

The right-hand side denotes the *smooth homology* for the adjoint action of \( G \) on the subspace of functions in \( \mathcal{H}(G) \) supported on the union \( G_c \) of the compact subgroups of \( G \). The
Bernstein idempotents $E : \mathcal{H}(G) \to \mathcal{H}(G)$ are equivariant for the adjoint action of $G$, but they do not preserve the subspace $\mathcal{H}(G_c)_{\text{Ad}} \subset \mathcal{H}(G)_{\text{Ad}}$. Nonetheless, we can define a map $E_c : H_*(G, \mathcal{H}(G_c)_{\text{Ad}}) \to H_*(G, \mathcal{H}(G_c)_{\text{Ad}})$ as the composition

$$H_*(G, \mathcal{H}(G_c)_{\text{Ad}}) \xrightarrow{E} H_*(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{\text{restrict functions}} H_*(G, \mathcal{H}(G_c)_{\text{Ad}}).$$

The Bernstein component $H_*^G(X)_E$ may be characterized in terms of $E_c$, thanks to the following result (Lemma 3.2.11 and Corollary 3.2.8):

**Lemma.** The diagram

$$
\begin{array}{ccc}
H_*^G(X) & \xrightarrow{\text{projection}} & H_*^G(X) \\
\downarrow \cong & & \downarrow \cong \\
H_*(G, \mathcal{H}(G_c)_{\text{Ad}}) & \xrightarrow{E_c} & H_*(G, \mathcal{H}(G_c)_{\text{Ad}})
\end{array}
$$

is commutative. Thus $E_c$ is an idempotent, whose image is canonically isomorphic to $H_*^G(X)_E$.

Let $C_*(X)_G$ denote the canonical chain complex computing chamber homology; it is defined in terms of the building $X$ and the representation theory of the compact subgroups of $G$. Working with smooth homology allows us to import some basic tools from homological algebra, which we use to prove the following result (Theorem 3.2.19):

**Theorem.** The map $E_c$ induces a canonical-up-to-homotopy endomorphism of $C_*(X)_G$, and the Bernstein component $H_*^G(X)_E$ is canonically isomorphic to the homology of the chain complex

$$\lim \left( C_*(X)_G \xrightarrow{E_c} C_*(X)_G \xrightarrow{E_c} C_*(X)_G \xrightarrow{E_c} \cdots \right).$$

As applications of this theorem, we use chamber-homological methods to compute $H_*^G(X)_E$ in three examples: the rather straightforward case of $\text{GL}_1(F)$ (Section 3.2.6); the supercuspidal components of $\text{SL}_n(F)$ (Theorem 3.2.35); and the supercuspidal components
of $\text{GL}_2(F)$ (Theorem 3.2.48). We also obtain partial results for the generic principal-series components of $\text{SL}_2(F)$ (Proposition 3.4.23). In each of these cases, the main interest lies not in the!determination of the groups $H^G_*(X)_E$, which may be computed by other methods, but in the explicit form of the chain complexes appearing in the computation. Our results lend further support to the conjectures made in [BHP00]. See Remark 3.2.21 for some precise statements about the relationship between those conjectures and the above theorem.

A key role in the Bernstein decomposition (1.2) is played by the Jacquet functors of parabolic induction and restriction. A complete understanding of the Bernstein components $H^G_*(X)_E$ will, therefore, require a natural interpretation of these functors in the context of chamber homology. (For a specific example of a result along these lines, see the discussion of Clozel’s formula, below.)

The Jacquet functors relate smooth representations of $G$ to representations of the Levi subgroups of $G$. These functors induce maps in homology, thanks to the following result, proved in Chapter 2:

**Proposition.** Let $G$ be a totally disconnected, locally compact group. There are isomorphisms

\[
H_*(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{\cong} HH_*(\mathcal{H}(G)) \xrightarrow{\cong} HH_*(\mathcal{P}_G),
\]

where

- $H_*(G, \mathcal{H}(G)_{\text{Ad}})$ is the smooth homology for the adjoint action of $G$ on $\mathcal{H}(G)$;
- $HH_*(\mathcal{H}(G))$ is the Hochschild homology of the algebra $\mathcal{H}(G)$; and
- $HH_*(\mathcal{P}_G)$ is a Hochschild homology group associated to the category $\mathcal{P}_G$ of finitely generated, projective, smooth representations of $G$.

Suppose, in addition, that $G$ acts properly on an affine building $X$, such that for each $g \in G$ the set of points in $X$ minimizing the function $x \mapsto \text{dist}(x, gx)$ is a subcomplex of $X$. Then there is a further isomorphism

\[
H_*(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{\cong} H^G_*(X, \mathcal{G}^+),
\]
the right-hand side being a variant of chamber homology.

The isomorphism (a) is due to Blanc and Brylinski [BB92, Proposition 2.3]. The isomorphism (b) (Proposition 2.4.23) generalizes a result of McCarthy [McC94, Proposition 2.4.3]. The isomorphism (c) (Proposition 2.3.39) is an elaboration of (1.4).

Fundamental results about the Jacquet functors, notably the Second Adjoint theorem, imply that these functors preserve the subcategories of finitely generated and projective objects; the above proposition then implies that they induce maps in smooth and Hochschild homology:

**Corollary.** The Jacquet functors induce maps in Hochschild, chamber, and smooth homology:

\[
HH_*^G(\mathcal{H}(G)) \xrightarrow{i_M^G} HH_*^G(\mathcal{H}(M)), \quad H_*^G(X_G) \xrightarrow{i_M^G} H_*^M(X_M),
\]

and

\[
H_*^G(G, \mathcal{H}(G)_{Ad}) \xrightarrow{i_M^G} H_*^G(M, \mathcal{H}(M)_{Ad})
\]

(Here \(M\) is a Levi factor of some chosen parabolic subgroup of \(G\), and \(X_G\) and \(X_M\) denote the respective Bruhat-Tits buildings of \(G\) and \(M\).)

In Chapter 4, we compute the maps \(i_M^G\) induced in homology by Jacquet restriction, extending results of van Dijk [vD72, Theorem 2] and Nistor [Nis01, Section 6]. A corollary of these computations is (Proposition 4.4.19):

**Theorem.** Let \(M\) be the diagonal subgroup in \(G = \text{SL}_n(F)\). The Jacquet restriction map \(i_M^G : H_*^G(X_G) \to H_*^M(X_M)\) in chamber homology is given by applying the natural analog of Jacquet restriction to the representations of the compact subgroups of \(G\).

The determination of the maps induced in homology by the parabolic induction functors \(i_M^G\) is a more difficult matter. In Chapter 5, we study the simplest example: induction from the diagonal subgroup in \(\text{SL}_2(F)\) (Theorem 5.5.2):

**Theorem.** Let \(M\) be the diagonal subgroup of \(G = \text{SL}_2(F)\). The parabolic induction map \(i_M^G : H_*^M(X_M) \to H_*^G(X_G)\) in chamber homology is given by applying parahoric induction (as defined by Dat [Dat09]) to the representations of the compact subgroups of \(M\).
A complete account of parabolic induction in homology will require an explicit form of the Second Adjoint theorem. Recent work of Bezrukavnikov and Kazhdan [BK11] is relevant to this ongoing project.

Our main motive for studying the Jacquet functors as maps in homology is to understand the Bernstein decomposition in chamber homology. As an example of such an application, we formulate (for general groups) and prove (for SL₂(F)) an analog, in higher homology, of a result due to Clozel in degree zero [Clo89, Proposition 1]:

**Conjecture.** The following is an equality between operators on the Hochschild homology group $HH_*(\mathcal{H}(G))$:

$$
\sum_{M \leq G} r^G_M \circ 1_{M_{cz}} \circ \chi_M \circ r^G_M = \text{id}.
$$

The meaning of the notation is as follows:

- The sum is over a set of representatives for the conjugacy classes of Levi subgroups of $G$.
- $r^G_M$ is Jacquet restriction with respect to a fixed parabolic subgroup containing $M$, and $r^G_M$ is parabolic induction with respect to the opposite parabolic subgroup.
- $1_{M_{cz}}$ is operator of restriction of functions from $M$ to the union $M_{cz}$ of the compact-mod-center subgroups of $M$.
- $\chi$ is the operator of restriction from $M$ to a certain subset $M^+$ of “strictly positive elements” (see Section 4.1 for details).

We prove the conjecture for SL₂(F) in Theorem 4.5.2: the precise statement in this case is as follows:

**Theorem.** Let $M$ be the diagonal subgroup of $G = \text{SL}_2(F)$. The following is an equality between operators on $HH_*(\mathcal{H}(G))$:

$$
1_{G_c} + (r^G_M \circ \chi_M \circ r^G_M) = \text{id}.
$$

The meaning of the notation is:
$1_{G_c}$ is the restriction of functions from $G$ to $G_c$.

$r^G_M$ is parabolic induction with respect to the subgroup of lower-triangular matrices in $G$.

$r^G_M$ is Jacquet restriction with respect to the subgroup of upper-triangular matrices.

$\chi_M$ is the operator of restriction of functions from $M$ to the subset

$$M^+ := \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in M \mid |a|_F < 1 \} .$$

The relevance of this result to the Bernstein decomposition is (Corollary 4.5.3):

**Corollary.** Let $G = SL_2(F)$, and let $E$ be an idempotent in the Bernstein center of $G$. Then, as operators on $HH_*(\mathcal{H}(G))$,

$$E \circ 1_{G_c} = 1_{G_c} \circ E.$$  

In degree zero (and for general $G$) this result was proved by Dat [Dat03, Proposition 2.8], using Clozel’s formula. Dat’s proof remains valid in higher degrees, once the higher Clozel formula is known.

We also prove an analog of the Clozel formula for the affine Coxeter group of type $\tilde{A}_n$ (Theorem 4.3.12):

**Theorem.** Let $G$ be the affine Coxeter group of type $\tilde{A}_n$. The following is an equality between operators on the Hochschild homology $HH_*(\mathcal{C}(G))$:

$$\sum_{\sigma \in \Delta} \text{ind}^G_{G_\sigma} \circ 1_{(G_\sigma)_c} \circ \chi_\sigma \circ \text{res}^G_{G_\sigma} = \text{id} .$$

The meaning of the notation is:

$\Delta$ is an $(n - 1)$-simplex, a fundamental domain for the action of $G$ on the spherical boundary of its affine Coxeter complex; the sum is over the faces of $\Delta$.

$\text{ind}^G_{G_\sigma}$ and $\text{res}^G_{G_\sigma}$ are the maps induced in homology by the usual induction and restriction functors for the isotropy subgroups $G_\sigma \subseteq G$. 

\( 1_{(G_\sigma)_{cz}} \) is the operator which restricts functions from \( G_\sigma \) to the subset \( (G_\sigma)_{cz} \) of elements having finite order modulo the center.

\( \chi_\sigma \) is the operator of restriction to a certain subset \( G_\sigma^+ \subset G_\sigma \), notable for the fact that it is invariant under translation by all torsion elements of \( G_\sigma \).

A similar result, with a similar proof, is valid for general affine Coxeter groups. Such groups are closely related to reductive \( p \)-adic groups—they appear, for instance, as the automorphism groups of apartments in Bruhat-Tits buildings—and so we consider the above theorem as evidence in support of the conjectural “higher Clozel formula” for \( p \)-adic groups.
Chapter 2

Homology and Representation

Theory for Totally Disconnected Groups

This chapter is mostly a review of prior results.

Sections 2.1 and 2.2 collect some standard facts about totally disconnected groups and their smooth representations, and the appropriate homology theory (smooth homology) for these representations.

Section 2.3 begins by recalling some basic facts about CAT(0) spaces and affine buildings. We then associate equivariant homology groups, called chamber homology, to the action of a totally disconnected group on an affine building; the definition goes back to [BCH94]. Chamber homology is related to smooth homology. We recall one result of this kind, due to Higson and Nistor [HN96] and to Schneider [Sch96], and prove two new results along similar lines (Proposition 2.3.39 and Corollary 2.3.46).

Section 2.4 concerns Hochschild and cyclic homology. We summarize the results of Higson-Nistor [HN96] and Schneider [Sch96] on cyclic homology for groups acting on affine buildings, and explain how Proposition 2.3.39 complements those results. Finally, in preparation for the following chapters, we show (Proposition 2.4.23) that the Hochschild and cyclic homology of a locally unital algebra may be defined in terms of its category of mod-
ules; this generalizes a theorem of McCarthy [McC94].

2.1 Totally Disconnected Groups and their Representations

This section collects some standard facts about totally disconnected groups and smooth representations. All of the results stated here are readily available in the literature, and most of the “proofs” here consist of references to one or more of [BR92], [BH06], [Car79], [Ren10], and [Sil79].

2.1.1 Totally disconnected groups

All topological spaces are assumed Hausdorff, unless otherwise indicated. A topological space $X$ is totally disconnected if every connected component in $X$ consists of a single point. A locally compact Hausdorff space is totally disconnected if and only if every point admits a neighborhood basis consisting of compact, open subsets. (See, for instance, [HR63, Theorem 3.5].)

Definition 2.1.1. Let $X$ be a locally compact, totally disconnected space, and $V$ any set. A function $f : X \to V$ is called smooth if it is locally constant. The set of all such functions is denoted $C^\infty(X, V)$.

When $V$ is a vector space, we write $C^\infty_c(X, V)$ for the space of smooth functions with compact support. If $V = \mathbb{C}$, we write $C^\infty_c(X, \mathbb{C}) = C^\infty_c(X) = \mathcal{H}(X)$. (The $\mathcal{H}$ stands for “Hecke”, since the most important example for us will be the Hecke algebra $\mathcal{H}(G)$ of a totally disconnected group $G$; see below.)

Lemma 2.1.2. Let $X$ and $Y$ be locally compact, totally disconnected spaces, and $V$ a complex vector space. Then the natural maps induce isomorphisms

(1) $\mathcal{H}(X) \otimes_\mathbb{C} \mathcal{H}(Y) \cong \mathcal{H}(X \times Y)$, and

(2) $\mathcal{H}(X) \otimes_\mathbb{C} V \cong C^\infty_c(X, V)$.

Proof. See [Ren10, II.1.2–3]
Lemma 2.1.3. Let $U$ be an open subset of a locally compact, totally disconnected space $X$. Consider the map $H(U) \to H(X)$ given by extending functions by zero, and the map $H(X) \to H(X \setminus U)$ given by restriction of functions. The sequence

$$0 \to H(U) \to H(X) \to H(X \setminus U) \to 0$$

is exact.

Proof. See [Ren10, II.1.2], [BR92, Proposition 2], or [Sil79, p.15]. □

Our main object of study in this thesis will be totally disconnected, locally compact topological groups. The following result of van Dantzig characterizes such groups in terms of their compact open subgroups:

Theorem 2.1.4. A locally compact group $G$ is totally disconnected if and only if the compact, open subgroups of $G$ constitute a neighborhood basis at the identity for the topology of $G$. If $G$ is compact, the same is true of the compact, open, normal subgroups.

Proof. See [HR63, Theorem 7.7] □

The next few examples lead to the notion of a $p$-adic field. We omit most of the details, which can be found in [Cas67] and [Ser73], among many other places.

Example 2.1.5. Let $G = \{0,1\} \times \{0,1\} \times \ldots$ be the set-theoretic product of countably many copies of the two-element group, equipped with the product topology. Thus $G$ is a compact, totally disconnected space, homeomorphic to the Cantor set. Define a group structure on $G$ as follows. View each sequence $(a_0, a_1, a_2, \ldots) \in G$ as a formal power series in the “indeterminate” $2$:

$$(a_0, a_1, a_2, \ldots) = \sum_{i=0}^{\infty} a_i 2^i.$$ 

Each finite series can be thought of as the base-2 expansion of a non-negative integer, and the addition of integers extends by continuity to an associative addition on $G$. This turns out to be a group law: for example, one has

$$(1, 0, 0, 0, \ldots) + (1, 1, 1, 1, \ldots) = (0, 0, 0, 0, \ldots).$$
In this way, \( G \) becomes a compact, abelian topological group, containing the additive group of integers \( \mathbb{Z} \) as a dense subgroup.

The multiplication of integers also extends to an associative multiplication on \( G \), and this turns \( G \) into a compact topological ring. The subsets

\[
G_n := \mathbb{Z}_2^\infty = \left\{ \sum_{i=n}^{\infty} a_i 2^i \in G \right\}
\]

are compact, open ideals in \( G \), constituting a neighborhood basis for the topology at 0.

The ring \( G \) is usually denoted \( \mathbb{Z}_2 \), and is called the ring of \( 2 \)-adic integers. The role of 2 can be played by any other integer, although here we shall only consider the case of a prime, \( p \). The resulting ring \( \mathbb{Z}_p \) of \( p \)-adic integers is, as before, a compact topological ring, containing the usual integers \( \mathbb{Z} \) as a dense subring. The ideal \( G_1 = p\mathbb{Z} \) is usually denoted by \( p \); it is the unique maximal ideal in \( \mathbb{Z}_p \). The quotient \( \mathbb{Z}_p / p \) is the field \( \mathbb{F}_p \) with \( p \) elements. The chain of compact, open ideals

\[
\mathbb{Z}_p \supset p \supset p^2 \supset \ldots
\]

is a neighborhood basis at 0.

The subset \( \mathbb{Z}_p^\times \subset \mathbb{Z}_p \), consisting of those elements possessing a multiplicative inverse, is equal to the complement \( \mathbb{Z}_p \setminus p \) of \( p \) in \( \mathbb{Z}_p \). \( \mathbb{Z}_p^\times \) is a compact, totally disconnected group in its own right; the chain of compact, open, normal subgroups

\[
\mathbb{Z}_p^\times \supset 1 + p \supset 1 + p^2 \supset \ldots
\]

is a neighborhood basis at 1.

**Example 2.1.6.** Instead of considering “power series” in \( p \), one can consider “Laurent series” \( \sum_{i=N}^{\infty} a_i p^i \), where \( N \) is possibly negative (but definitely finite). The algebraic operations on \( \mathbb{Z}_p \) extend in a natural way to this larger set, and the resulting locally compact, totally disconnected topological ring turns out to be a field, called the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. This field contains \( \mathbb{Q} \) as a dense subfield, and also contains \( \mathbb{Z}_p \) as a compact,
open subring. The multiplicative group $\mathbb{Q}_p^\times$ contains $\mathbb{Z}_p^\times$ as a compact, open subgroup. For each nonzero element $a = \sum_{i=N}^{\infty} a_i p^i$, with $a_N \neq 0$, one has $p^{-N}a \in \mathbb{Z}_p^\times$, this shows that $\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times \mathbb{Z}$.

There is another useful characterization of $\mathbb{Q}_p$, which we will now describe. Given a non-zero element $x = \sum_{i=N}^{\infty} a_i p^i \in \mathbb{Q}_p$, where $a_N \neq 0$, we define the valuation and the absolute value of $x$ by

$$\text{val}_p(x) = N \quad \text{and} \quad |x|_p = p^{-\text{val}_p(x)},$$

respectively. (The subscript $p$ will often be omitted.) These functions are extended to all of $\mathbb{Q}_p$ by declaring $\text{val}_p(0) = \infty$, and $|0|_p = 0$. The absolute value defines a metric on $\mathbb{Q}_p$, and $\mathbb{Q}_p$ is complete in this metric. Moreover, the metric topology agrees with the topology defined above. Since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, we obtain a second definition of $\mathbb{Q}_p$: it is the completion of $\mathbb{Q}$ in the metric determined by the absolute value $|\cdot|_p$. The subsets $\mathbb{Z}_p$, $\mathfrak{p}$ and $\mathbb{Z}_p^\times$ have the following characterizations in terms of the absolute value:

$$\mathbb{Z}_p = \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \}$$

$$\mathfrak{p} = \{ x \in \mathbb{Q}_p \mid |x|_p < 1 \}$$

$$\mathbb{Z}_p^\times = \{ x \in \mathbb{Q}_p \mid |x|_p = 1 \}.$$

**Example 2.1.7.** A $p$-adic field is, by definition, a finite extension $F$ of $\mathbb{Q}_p$. Each structural feature of $\mathbb{Q}_p$ described above has its counterpart in $F$. The following dictionary lists some of the notation that will be used in the rest of the thesis:

\[
\begin{array}{ll}
\mathbb{Q}_p & F \\
\mathbb{Z}_p & \mathcal{O} \quad \text{(elements of } F \text{ that are integral over } \mathbb{Z}_p) \\
\mathbb{Z}_p^\times & \mathcal{O}^\times \quad \text{(invertible elements in } \mathcal{O}) \\
\mathfrak{p} & \mathfrak{p} \quad \text{(the unique maximal ideal in } \mathcal{O}) \\
p \in \mathfrak{p} & \varpi \in \mathfrak{p} \quad \text{(a generator of this ideal).}
\end{array}
\]

Note that the element $\varpi \in \mathfrak{p}$ is defined only up to multiplication by $\mathcal{O}^\times$, but nothing will
depend on the choice of such a $\varpi$. Each choice of $\varpi$ induces an isomorphism $F^\times \cong \mathcal{O}^\times \times \mathbb{Z}$.

The residue field $\mathcal{O}/p$ is a finite field of cardinality $p^{[F:\mathbb{Q}]}$. This field will be denoted $\mathfrak{f}$, and its cardinality will usually be denoted by $q$.

**Example 2.1.8.** For each $p$-adic field $F$, the general linear group $\text{GL}_n(F)$, topologized as an open subset of $F^{n^2}$, is totally disconnected and locally compact. The subgroup $\text{GL}_n(\mathcal{O})$ is compact and open. For each $n$, the quotient map $\mathcal{O} \to \mathcal{O}/p^n$ induces a group homomorphism $\text{GL}_n(\mathcal{O}) \to \text{GL}_n(\mathcal{O}/p^n)$. Since the target is a finite group, the kernel of this map is a compact, open, normal subgroup of $\text{GL}_n(\mathcal{O})$. These congruence subgroups constitute a neighborhood basis at the identity in $\text{GL}_n(F)$.

**Example 2.1.9.** Similar considerations apply to $\text{SL}_n(F)$. More generally, if $G$ is any algebraic subgroup of $\text{GL}_n$ defined over $F$ (for instance, $G$ might be defined over $\mathbb{Q}$), then the group $G(F)$ of $F$-points inherits a topology from $\text{GL}_n(F)$, and is locally compact and totally disconnected in this topology. The intersections of $G(\mathbb{Q}_p)$ with the congruence subgroups constitute a neighborhood basis at the identity consisting of compact, open subgroups.

The main focus of this thesis will be groups of the above kind. They will simultaneously be groups of the following kind:

**Example 2.1.10.** Let $X$ be a tree (i.e., a connected, simply connected, one-dimensional simplicial complex), and consider the group $G = \text{Aut}(X)$ of simplicial automorphisms of $X$. The set of vertices of $X$ can be viewed as a discrete metric space, the distance between any two vertices being defined as the number of edges in the (unique) shortest path between them. The group $G$ is then the group of isometries of this metric space. The compact-open topology turns $G$ into a topological group.

If $Y \subset X$ is any finite set of vertices, then the subgroup $G_Y$ of automorphisms that fix $Y$ (pointwise) is open in $G$, and the collection of all such subgroups forms a neighborhood basis at the identity in $G$. If $X$ is locally finite (meaning that every vertex emits only finitely many edges), then these $G_Y$ are also compact, and so $G$ is a locally compact, totally disconnected group. See [FTN91, Chapter I.4] for details.
Every locally compact group possesses a left-invariant Haar measure, unique up to a positive scalar multiple, which allows one to integrate continuous functions (for example). In the case of totally disconnected groups, and smooth functions, the construction of Haar measure is a purely algebraic matter: see [Ren10, II.3.6] or [BH06, 3.1].

2.1.2 Smooth representations of totally disconnected groups

We now turn to representations, by which we always mean representations on complex vector spaces.

Definition 2.1.11. A representation of a group $G$ is a homomorphism of groups

$$\pi : G \to \text{GL}(V)$$

from $G$ to the group of invertible linear transformations of some complex vector space $V$. A $G$-module is a vector space $V$ equipped with a representation of $G$.

Such a representation will variously be referred to as $(\pi, V)$, $\pi$, or $V$. The notation $gv$ will often be used in place of $\pi(g)(v)$. It will sometimes be necessary to distinguish the above situation as a left representation, since we shall also consider anti-homomorphisms $G \to \text{GL}(V)$, which we view as right actions of $G$ on $V$ and call right representations.

We are primarily interested in the following class of representations, which reflect the total-disconnectedness of the groups under consideration.

Definition 2.1.12. Let $G$ be a totally disconnected, locally compact group, and let $\pi : G \to \text{GL}(V)$ be a representation of $G$. A vector $v \in V$ is smooth (with respect to $G$) if the isotropy subgroup

$$G_v = \{ g \in G \mid gv = v \}$$

is open in $G$. The space of smooth vectors is denoted $V^\infty$, and the representation $V$ is called smooth if $V = V^\infty$.

The category of smooth representations of $G$ (with equivariant linear maps as morphisms) is denoted $\text{Mod}(G)$. The set of isomorphism classes of irreducible smooth repre-
sentations is denoted $\hat{G}$.

**Remark 2.1.13.** The category $\text{Mod}(G)$ is an abelian category; the only nontrivial content to this assertion is the fact that quotients and subrepresentations of smooth representations are, themselves, smooth. In particular, we may talk about exact sequences in $\text{Mod}(G)$, exact functors from $\text{Mod}(G)$ to other abelian categories, projective and injective objects in $\text{Mod}(G)$, and so on. We refer to [Wei94, Chapters 1–3] for the basic facts and terminology from category theory and homological algebra that will be used below.

**Definition 2.1.14.** Let $V$ be a representation of $G$, and let $H \subset G$ be a subgroup. Define

$$V^H := \{ v \in V \mid hv = v \text{ for all } h \in H \}.$$  

We then have

$$V^\infty = \bigcup_K V^K,$$  

the union being taken over the set of compact, open subgroups of $G$.

**Remark 2.1.15.** The assignment $V \mapsto V^\infty$ determines a functor, from the category of all $G$-modules to the category $\text{Mod}(G)$. This functor is left-exact, meaning that each short exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

of $G$-modules gives an exact sequence

$$0 \rightarrow V_1^\infty \rightarrow V_2^\infty \rightarrow V_3^\infty.$$  

The functor $V \mapsto V^\infty$ is not right-exact: that is, a surjective map $V_2 \rightarrow V_3$ need not restrict to a surjective map $V_2^\infty \rightarrow V_3^\infty$. For example, take $V = \mathbb{C}[G]$, the space of finitely supported functions on $G$, with $G$ acting by left translations. The map $V \rightarrow \mathbb{C}$ given by $f \mapsto \sum_{g \in G} f(g)$ is an equivariant surjection from $V$ to the trivial representation. If $G$ is not discrete, then $V^\infty = \{0\}$, while $\mathbb{C}^\infty = \mathbb{C}$.

If $V$ and $W$ are smooth representations of $G$, then the space of linear maps $\text{Hom}_\mathbb{C}(V, W)$
carries a representation $\text{Ad}$ of $G$, defined by $\text{Ad}_g(T) = gTg^{-1}$. This representation will not, in general, be smooth; consider its smooth part, $\text{Hom}_C(V, W)^\infty$.

**Lemma 2.1.16.** For each fixed smooth representation $W$, the functor

$$V \mapsto \text{Hom}_C(V, W)^\infty$$
onumber

on $\text{Mod}(G)$ is exact. The same is true if we fix $V$ and vary $W$.

**Proof.** By definition, $\text{Hom}_C(V, W)^\infty = \bigcup_K \text{Hom}_C(V, W)^K$, and this decomposition is natural in $V$. For each compact, open subgroup $K$, the functor $V \mapsto \text{Hom}_C(V, W)^K = \text{Hom}_K(V, W)$ is exact (via the standard averaging argument). So $V \mapsto \text{Hom}_C(V, W)^\infty$ is a direct limit of exact functors, and therefore is exact. The same argument applies when $W$ is fixed and $V$ varies. \hfill $\square$

**Definition 2.1.17.** The contragredient of $V$ is the smooth representation of $G$ defined by $\tilde{V} := \text{Hom}_C(V, \mathbb{C})^\infty$.

**Lemma 2.1.18.** Let $V$ be a smooth representation of $G$. The following are equivalent:

1. For each compact open subgroup $K \subset G$, the space $V^K$ is finite-dimensional.
2. The natural injection $V \hookrightarrow \tilde{V}$ is an isomorphism.

When these conditions are satisfied, we say that $V$ is admissible.

**Proof.** See [BH06, 2.9 Proposition]. \hfill $\square$

There is a version of Schur’s lemma for smooth representations:

**Lemma 2.1.19** (Schur’s lemma). Suppose that $G$ is countable at infinity: i.e., that $G/K$ is countable for every (equivalently, for one) compact open subgroup $K$. If $V$ is an irreducible smooth representation of $G$, then $\text{End}_G(V) = \mathbb{C}$.

**Proof.** See [BR92, 4.2], [BH06, 2.6], or [Ren10, III.1.8]. \hfill $\square$
2.1.3 The Hecke algebra

If \( G \) is a discrete group, then every representation is smooth, and the category \( \text{Mod}(G) \) is equivalent to the category of (unital) modules over the group algebra \( \mathbb{C}[G] \). For general totally disconnected groups, the appropriate replacement for \( \mathbb{C}[G] \) is the Hecke algebra.

**Definition 2.1.20.** Fix a left Haar measure \( dg \) on \( G \). The formula

\[
f_1 * f_2(g) = \int_{G} f_1(h) f_2(h^{-1} g) \, dg
\]

defines an associative product on the space \( \mathcal{H}(G) \) of smooth, compactly supported, complex-valued functions on \( G \). This algebra is called the *Hecke algebra* of \( G \).

Different choices of Haar measure yield canonically isomorphic algebra structures, and the choice can in fact be avoided by instead defining \( \mathcal{H}(G) \) to be the convolution algebra of locally constant, compactly supported *distributions* on \( G \); see [BR92, 2.1] or [Ren10, II.3.12].

Let \( K \) be a compact open subgroup of \( G \). The space \( \mathcal{H}(K \setminus G/K) \) of finitely supported functions on the double-coset space \( K \setminus G/K \) may be identified with the subspace of \( \mathcal{H}(G) \) comprising the \( K \)-bi-invariant functions; this is a subalgebra of \( \mathcal{H}(G) \), often denoted by \( \mathcal{H}(G,K) \). Similarly, \( \mathcal{H}(G/K) \) and \( \mathcal{H}(K \setminus G) \) identify with (one-sided) ideals in \( \mathcal{H}(G) \). One has

\[
\mathcal{H}(G) = \bigcup_K \mathcal{H}(G,K),
\]

the union being taken over the set of compact open subgroups \( K \subset G \); analogous formulae hold for \( \mathcal{H}(G/K) \) and \( \mathcal{H}(K \setminus G) \).

**Definition 2.1.21.** For each compact open subgroup \( K \subset G \), define a function \( e_K \in \mathcal{H}(G) \) by

\[
e_K(g) = \begin{cases} 
\frac{1}{\text{vol}(K)} & \text{if } g \in K, \\
0 & \text{if } g \notin K.
\end{cases}
\]

**Lemma 2.1.22.** (1) \( e_K * e_K = e_K \).
(2) \( e_K \mathcal{H}(G) = \mathcal{H}(K \backslash G), \quad \mathcal{H}(G)e_K = \mathcal{H}(G/K), \quad \text{and} \quad e_K \mathcal{H}(G)e_K = \mathcal{H}(G,K). \)

**Proof.** See [BH06, 4.1 Proposition]. \hfill \Box

**Remark 2.1.23.** More generally, if \( \pi \in \hat{K} \) is any smooth, irreducible representation of \( K \), then the function

\[
e_\pi(g) = \begin{cases} 
\frac{\dim(\pi)}{\text{vol}(K)} \text{Trace}(\pi(g^{-1})) & \text{if } g \in K, \\
0 & \text{if } g \notin K
\end{cases}
\]

is an idempotent in \( \mathcal{H}(G) \) [BH06, 4.4]. If \( \pi \) is any finite-dimensional representation of \( K \), and if \( \pi_1, \ldots, \pi_n \) are the distinct irreducible representations of \( K \) contained in \( \pi \), then we let

\[
e_\pi = e_{\pi_1} + \ldots + e_{\pi_n}.
\]

This element is also an idempotent; see Example 2.1.24 below.

Lemma 2.1.22 implies that \( \mathcal{H}(G,K) \) is a unital algebra, with unit \( e_K \), and that the algebra \( \mathcal{H}(G) \) is locally unital: for every finite subset \( S \subset \mathcal{H}(G) \), there is an idempotent \( e \in \mathcal{H}(G) \) such that \( es = se = s \) for each \( s \in S \). Locally unital algebras are also known as idempotented algebras. We refer to [Ren10, Chapitre I] for a general discussion of these algebras and their modules.

**Example 2.1.24.** Let \( G \) be a compact totally disconnected group. Write

\[
\mathcal{H}(G) = \bigcup_K \mathcal{H}(G/K),
\]

the union being taken over the open, normal subgroups of \( G \). Each \( G/K \) is a finite group, and so

\[
(2.1.25) \quad \mathcal{H}(G/K) \cong \bigoplus_{\pi \in \mathcal{L}(G/K)} \text{End}_\mathbb{C}(\pi)
\]

(see [Ser77, Proposition 10], for example). Each inclusion \( K' \subset K \) gives an inclusion of algebras \( \mathcal{H}(G/K) \hookrightarrow \mathcal{H}(G/K') \), and the corresponding map on the right-hand side of
(2.1.25) is the one induced by the inclusion $G/K \to \overline{G/K}$ (viewing representations of $G/K$ as representations of $G/K'$ that are trivial on $K/K'$). Now, every irreducible smooth representation of $G$ factors through some $G/K$: take $K$ to be an open normal subgroup of the isotropy group of a nonzero vector. Taking limits on both sides of (2.1.25) along the set of open normal subgroups $K$, we find that

$$\mathcal{H}(G) \equiv \bigoplus_{\pi \in \hat{G}} \text{End}_C(\pi).$$

For each $\pi$, the central idempotent $e_\pi \in \mathcal{H}(K)$ of Remark 2.1.23 corresponds to the identity operator in $\text{End}_C(\pi)$.

**Definition 2.1.26.** A (left) module $V$ over the algebra $\mathcal{H}(G)$ is nondegenerate if $\mathcal{H}(G)V = V$. The category of nondegenerate $\mathcal{H}(G)$-modules will be denoted by $\text{Mod}(\mathcal{H}(G))$.

Lemma 2.1.22 implies that $V$ is nondegenerate if and only if $V = \bigcup_K e_K V$, the union being taken over the set of compact, open subgroups of $G$.

**Example 2.1.27.** $\mathcal{H}(G)$, considered as a module over itself, is nondegenerate (by Lemma 2.1.22).

Let $V$ be a smooth representation of $G$. For each $f \in \mathcal{H}(G)$ and each $v \in V$, define

$$fv := \int_G f(g)v \ dg \in V.$$  

(2.1.28)

(Since $f$ is locally constant and compactly supported, the integral in fact reduces to a finite sum.)

**Proposition 2.1.29.** The formula (2.1.28) gives $V$ the structure of a nondegenerate module over $\mathcal{H}(G)$, and implements an equivalence between the categories $\text{Mod}(G)$ and $\text{Mod}(\mathcal{H}(G))$.

**Proof.** See [BR92, Theorem 2] or [Ren10, III.1.4], for example.

We shall very often apply Proposition 2.1.29 without comment. All of the above applies equally well to the categories of right modules.
Lemma 2.1.30. Suppose that $G$ is countable at infinity. For each $f \in \mathcal{H}(G)$ there is an irreducible representation $\pi$ of $G$ such that $\pi(f) \neq 0$.

Proof. See [BR92, I.4.2] or [Ren10, III.1.11].

Recall the functor $V \mapsto V^\infty$, sending each $G$-module to its subspace of smooth vectors. There is an analogous functor for $\mathcal{H}(G)$-modules:

Definition 2.1.31. Let $V$ be a (left) module over $\mathcal{H}(G)$. Denote by $V^{ndg}$ the submodule $\mathcal{H}(G)V \subset V$.

Thus $V^{ndg}$ is a nondegenerate $\mathcal{H}(G)$-module, and $V$ is nondegenerate if and only if $V = V^{ndg}$. The map

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V \to V^{ndg}, \quad f \otimes v \mapsto fv$$

is an isomorphism (see [Ren10, I.3.2 Lemme] or [BH06, 4.1 Proposition 2]). The assignment $V \mapsto V^{ndg}$ is a functor, from the category of all $\mathcal{H}(G)$-modules to the category of nondegenerate modules. This functor is exact, unlike the analogous functor $V \mapsto V^\infty$ for representations of $G$; see [Ren10, I.1.2 Proposition].

Recall that an admissible representation of $G$ is one in which $\dim(V^K) < \infty$ for every compact open subgroup $K \subset G$. Here is the corresponding notion for $\mathcal{H}(G)$-modules:

Lemma 2.1.32. Let $V$ be a nondegenerate $\mathcal{H}(G)$-module. Then $V$ is admissible as a representation of $G$ if and only if every $f \in \mathcal{H}(G)$ acts as a finite-rank operator on $V$.

Proof. The smooth representation $V$ is admissible if and only if each $e_K$ has finite rank, and $\mathcal{H}(G) = \bigcup_K e_K \mathcal{H}(G) e_K$.

Definition 2.1.33. The character of an admissible representation $\pi : G \to GL(V)$ is the linear functional on $\mathcal{H}(G)$ defined by

$$\text{ch}_\pi(f) = \text{Trace} \left( V \pi(f) : V \to V \right).$$
2.1.4 Change of groups and Frobenius reciprocity

This section reviews various methods for constructing representations of one group from representations of a related group. Our main references are [BR92, I.3], [Ren10, III.2], [BH06, 2.4], and [Car79, Section 1.7] (from which the title of this section is borrowed).

Let $G$ be a locally compact, totally disconnected group, and let $H \subset G$ be a closed subgroup. (For example, $G$ and $H$ might be discrete groups.) If $\pi : G \to \text{GL}(V)$ is a smooth representation of $G$, then we may consider $\pi$ as a representation of $H$, by restriction; this representation will be denoted $\pi|_H$ or $\text{res}^G_H \pi$. Any $G$-equivariant map is certainly $H$-equivariant, so $\text{res}^G_H$ is a functor $\text{Mod}(G) \to \text{Mod}(H)$.

As in the case of finite groups (see [Ser77, 3.3 Lemma 1]), the functor $\text{res}^G_H$ possesses an easily described right adjoint.

**Definition 2.1.34.** Let $\sigma : H \to \text{GL}(W)$ be a smooth representation. Consider the space

$$\{ f : G \to W \mid f(hg) = \sigma(h)f(g) \text{ for all } h \in H, g \in G \}.$$

The group $G$ acts on this space by right translation; denote by $\text{Ind}^G_H W$ the space of smooth vectors for this action. The representation $\text{Ind}^G_H \sigma$ of $G$ on $\text{Ind}^G_H W$ by right translations is called the representation *induced* by $\sigma$.

The subspace

$$\text{ind}^G_H W := \{ f \in \text{Ind}^G_H W \mid \text{supp}(f) \text{ is compact modulo } H \}$$

is $G$-invariant. The representation $\text{ind}^G_H \sigma$ of $G$ on this subspace is called the representation *compactly induced* by $\sigma$.

Every $H$-equivariant map $T : W \to W'$ determines a $G$-equivariant map $\text{Ind}^G_H W \to \text{Ind}^G_H W'$, by $(Tf)(g) := T(f(g))$. This map sends $\text{ind}^G_H W$ to $\text{ind}^G_H W'$. Thus induction and compact induction are both functors from $\text{Mod}(H)$ to $\text{Mod}(G)$. Note that if $H$ is cocompact in $G$, then $\text{ind}^G_H = \text{Ind}^G_H$. 
Theorem 2.1.35 (Frobenius reciprocity). The functor $\text{Ind}^G_H$ is right-adjoint to $\text{res}^G_H$:

$$\text{Hom}_G(\pi, \text{Ind}^G_H \sigma) \cong \text{Hom}_H(\text{res}^G_H \pi, \sigma).$$

If $H$ is an open subgroup of $G$, then $\text{ind}^G_H$ is left-adjoint to $\text{res}^G_H$:

$$\text{Hom}_G(\text{ind}^G_H \sigma, \pi) \cong \text{Hom}_H(\sigma, \text{res}^G_H \pi).$$

Proof. We refer to [BH06, 2.4], for example, for the details of the proof. It will nonetheless be useful to recall the explicit transformations implementing the adjunctions.

If $H$ is any closed subgroup of $G$, and $W$ a smooth representation of $H$, then the map

$$\text{ev}_1: \text{Ind}^G_H W \to W, \ f \mapsto f(1)$$

is $H$-equivariant. One shows that the map

$$\text{Hom}_G(V, \text{Ind}^G_H W) \to \text{Hom}_H(\text{res}^G_H V, W), \quad T \mapsto \text{ev}_1 \circ T$$

is an isomorphism, natural in $V$ and $W$.

If $H$ is now an open subgroup of $G$, then for any smooth representation $W$ of $H$ there is an embedding

$$\alpha: W \to \text{ind}^G_H W, \quad \alpha(w) = \begin{cases} \ g w \ & \text{if } g \in H, \\ 0 & \text{otherwise}. \end{cases}$$

One shows that the map

$$\text{Hom}_G(\text{ind}^G_H W, V) \to \text{Hom}_H(W, \text{res}^G_H), \quad T \mapsto T \circ \alpha$$

is a natural isomorphism. □

Lemma 2.1.36. Both $\text{Ind}^G_H$ and $\text{ind}^G_H$ are exact functors.
Proof. See [BH06, 2.4].

In view of Proposition 2.1.29, we may also consider restriction and induction as functors between categories of modules over Hecke algebras. These functors admit very straightforward descriptions, as we shall now recall.

Here it becomes important to account for the fact that the groups in question might not be unimodular. Motivated by the applications to come, we will always assume that $G$ is unimodular, but $H$ might not be. Let $\delta = \delta_H$ be the modular function of $H$. Recall (e.g., from [Wei40, II§8]) that this function is characterized by the property that

$$\delta(q) \int_H f(hq) \, dh = \int_H f(h) \, dh$$

for all $f \in \mathcal{H}(H)$ (here and elsewhere, $dh$ denotes a left Haar measure on $H$). Another way of writing this equation is

$$d(hq) = \delta(q)dh.$$

We also have

$$\int_H f(h^{-1}) \, dh = \int_H f(h)\delta(h)^{-1} \, dh,$$

or in other words

$$d(h^{-1}) = \delta(h)^{-1}dh.$$

**Warning 2.1.37.** To avoid possible confusion when consulting the references, let us point out that some authors denote by $\delta$ (or $\Delta$) what we would call $\delta^{-1}$. Among our most often-cited references, [Ren10] and [BH06] follow the same convention as above, while [BR92] uses the opposite convention.

Since $\delta$ is a multiplicative function $H \to \mathbb{C}^\times$, and is equal to 1 on the union of the compact subgroups of $H$, it defines a smooth, one-dimensional representation of $H$.

**Definition 2.1.38.** For each smooth representation $W$ of $H$, we let

$$W_\delta := W \otimes_{\mathbb{C}} \delta$$
be the representation obtained by twisting $W$ by $\delta$.

The space $H$ carries actions of $G$ by left and by right translations. Either (or both) of these actions may be restricted to $H$, and so $H$ may be viewed as an $H$-bimodule, or as an $H$-bimodule.

**Proposition 2.1.39.** Let $G$ be totally disconnected and unimodular, and let $H \subseteq G$ be a closed subgroup with modular character $\delta$. There are natural isomorphisms

$$\text{ind}_H^G W \cong H \otimes_{H(H)} (W_\delta), \quad \text{Ind}_H^G(W) \cong (\text{Hom}_{H(H)}(H, W))^\text{ndg},$$

and

$$\text{res}_H^G(V) \cong H \otimes_{H(G)} V.$$

**Proof.** See [Ren10, III.2.5–6]. For later reference, let us record the map effecting the isomorphism $H(G) \otimes_{H(H)} (W_\delta) \cong \text{ind}_H^G W$: it is

(2.1.40)  \[ \eta : H(G) \otimes_{H(H)} (W_\delta) \to \text{ind}_H^G W \quad \eta(f \otimes w)(g) = \int_H f(g^{-1}h)hw \, dh. \]

The induction and restriction functors are associated to subgroups of $G$. There is another family of functors associated to quotients, as we shall now recall.

Let $N \subseteq P$ be a closed, normal subgroup of a locally compact totally disconnected group $P$. The quotient $M := P/N$ is again a totally disconnected locally compact group [Ren10, II.3.4]. Each smooth representation $\sigma$ of $M$ can be “inflated” to a representation $\text{infl}_M^P \sigma$ of $P$, by precomposing $\sigma$ with the quotient map $P \to M$. This representation of $P$ will be denoted $\text{infl}_M^P \sigma$, or just $\sigma$. The functor $\text{infl}_M^P : \text{Mod}(M) \to \text{Mod}(P)$ is exact.

On the other hand, let $\pi : P \to \text{GL}(V)$ be a smooth representation of $P$. The quotient $M = P/N$ acts on the quotient vector space

$$V_N := V/\text{span}\{\pi(n)v - v \mid n \in N, \ v \in V\},$$

and we denote this representation of $M$ by $\text{coinv}_M^P \pi$ or by $\pi_N$.

In all of our applications, the extension $0 \to N \to P \to M \to 0$ will be split, and we henceforth assume this to be the case.
Lemma 2.1.41. Let $P$ be the semidirect product $M \rtimes N$, where $M$ and $N$ are totally disconnected groups.

(1) The functor $\text{infl}^P_M$ is right-adjoint to $\text{coinv}^P_M$:

$$\text{Hom}_M(\pi_N, \sigma) \cong \text{Hom}_P(\pi, \text{infl}^P_M \sigma).$$

(2) $\text{coinv}^P_M \circ \text{infl}^P_M \cong \text{id}_M$.

(3) There is a natural isomorphism

$$\text{coinv}^P_M V \cong \mathcal{H}(M) \otimes_{\mathcal{H}(P)} V,$$

where $\mathcal{H}(M)$ carries the usual left action of $\mathcal{H}(M)$, and the right action of $\mathcal{H}(P)$ inflated from the right-translation action of $M$.

(4) $\text{coinv}^M_1 \circ \text{coinv}^P_M \cong \text{coinv}^P_1$, where $1$ denotes the trivial subgroup.

(5) The functor $\text{coinv}^P_M$ is right-exact.

(6) If $N$ is compact, then $\pi_N \cong \pi^N$.

(7) If $N$ is an increasing union of compact subgroups, then $\text{coinv}^P_M$ is exact.

Proof. Parts (1) and (2) are clear from the definitions. Part (3) is proved in [Ren10, III.2.10 Proposition], for example. Part (4) follows from part (3), by transitivity of tensor products. Part (5) follows from part (1), since functors with left adjoints are right-exact. For part (6), the map $v \mapsto \int_N n v \ dn$ descends to an isomorphism $V_N \xrightarrow{\cong} V^N$. If $N = \bigcup N_k$ is an increasing union of compact subgroups, then $\pi_N \cong \lim_{\longrightarrow} \pi_{N_k}$. Each of the functors $\pi \mapsto \pi_{N_k}$ is exact, by part (6), and so their limit is also exact. \hfill \Box

Example 2.1.42 (Parabolic induction). Let $G = \text{SL}_2(F)$, $F$ a $p$-adic field, and let $M \subset G$ be the diagonal subgroup. The subgroup $P \subset G$ of upper-triangular matrices is isomorphic
to the semidirect product \( P = M \ltimes N \), where \( N \) is the subgroup of unipotent upper-triangular matrices. The functor

\[
i^G_M := \text{ind}^G_P \circ \text{infl}_M^P : \text{Mod}(M) \to \text{Mod}(G)
\]

is an example of parabolic induction. Functors of this kind play a central role in the study of the representations of \( G \), and in the rest of this thesis. (Note that we will use a slightly modified definition of \( i^G_M \) in later chapters; see Definition 3.3.3.)

**Example 2.1.43.** Let \( G \) be a compact totally disconnected group, and consider the adjoint representation of \( G \) on its Hecke algebra \( \mathcal{H}(G) \): \((\text{Ad}_g f)(g_0) = f(g^{-1}g_0g)\). This smooth representation will be denoted \( \mathcal{H}(G)_{\text{Ad}} \). Part (6) of Lemma 2.1.41 shows that the map

\[
f \mapsto \int_G \text{Ad}_g f \, dg
\]

descends to an isomorphism from the space of coinvariants \( (\mathcal{H}(G)_{\text{Ad}})_G \) to the space \( \text{Cl}^\infty(G) \) of locally constant class functions (i.e., conjugation-invariant functions) on \( G \). Note that the map depends on the choice of Haar measure.

As in Example 2.1.24, the usual arguments for finite groups generalize to this case, and one finds that the map sending each irreducible \( \pi \in \hat{G} \) to its character \( \text{ch}_{\pi}(g) := \text{Trace}(\pi(g)) \) establishes an isomorphism of vector spaces between \( \text{Cl}^\infty(G) \) and the vector space \( R_{\mathbb{C}}(G) \) with basis \( \hat{G} \). The composite isomorphism

\[
R_{\mathbb{C}}(G) \to \text{Cl}^\infty(G) \to \mathcal{H}(G)_{\text{Ad}}_G
\]

sends \( \pi \in \hat{G} \) to the class in \( (\mathcal{H}(G)_{\text{Ad}})_G \) of the function \( \frac{1}{\text{vol}(G)} \text{ch}_{\pi} \).

Now suppose that \( H \subset G \) is an open subgroup. For each irreducible \( \pi \in \hat{H} \), the induced representation \( \text{ind}_H^G \pi \) is a (finite) direct sum of irreducible representations of \( G \). The functor \( \text{ind}_H^G \) thus induces a linear map \( \text{ind}_H : R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G) \). On the other hand, the inclusion of \( H \) as an open subset of \( G \) induces an inclusion \( \mathcal{H}(H) \hookrightarrow \mathcal{H}(G) \), which descends to a linear map \( i : \mathcal{H}(H)_{\text{Ad}}_H \to \mathcal{H}(G)_{\text{Ad}}_G \). Frobenius’s formula for the character of an induced
representation [Ser77, Theorem 12] ensures that the diagram

\[
\begin{array}{c}
\left( \mathcal{H}(H)_{Ad} \right)_H \\
\uparrow \text{ch} \downarrow \text{vol}_H \\
R_c(H) \end{array} \xrightarrow{i} \begin{array}{c}
\left( \mathcal{H}(G)_{Ad} \right)_G \\
\uparrow \text{ch} \downarrow \text{vol}_G \\
R_c(G)
\end{array}
\]

is commutative, provided the Haar measures on \(G\) and \(H\) agree on \(H\).

As a final example of a change-of-groups functor, we consider the action of a group \(G\) on representations of its subgroups, by conjugation.

**Definition 2.1.44.** Let \(H \subseteq G\) be a closed subgroup of \(G\), and let \(\pi : H \to \text{GL}(V)\) be a smooth representation of \(H\). Define \(H^g := g^{-1}Hg\), and

\[
\pi^g : H^g \to \text{GL}(V), \quad \pi^g(g^{-1}hg) := \pi(h).
\]

### 2.2 Smooth Homology

In this section, we study the homological algebra of the functor \(V \mapsto V_G\), where \(G\) is a locally compact, totally disconnected group. The purely algebraic approach described here goes back to work of Casselman [Cas81, Appendix], and was continued by Blanc and Brylinski in their paper [BB92]

#### 2.2.1 Projective modules

We begin by observing that the category \(\text{Mod}(G)\) is a suitable venue for homological algebra. We refer to [Wei94, Chapter 2] for the basic theory and terminology of derived functors; in particular, we recall that a \(G\)-module \(V\) is called *projective* if \(W \mapsto \text{Hom}_G(V, W)\) is an exact functor from \(\text{Mod}(G)\) to vector spaces. The category \(\text{Mod}(G)\) is said to have *enough projectives* if every \(V\) is a quotient of a projective module.

**Lemma 2.2.1.** The category \(\text{Mod}(G)\) is abelian with enough projectives.

**Proof.** See [BR92, Theorem 3] or [Ren10, I.5.2]. The basic observation is that \(\mathcal{H}(G/K)\) is projective for every compact open subgroup \(K \subset G\).
Remark 2.2.2. We will work almost exclusively with homology, as opposed to cohomology. But we note here that the category Mod\(_pGq\) also contains enough injectives: for any \(_G\)-module \(V\), the module Ind\(_G^1(V)\) is injective, and Frobenius reciprocity gives an embedding \(V \rightarrow \text{Ind}_G^1(V)\). (This proof is Casselman’s [Cas81, p.923].) See [BR92, Theorem 4] for another proof in the case of reductive \(p\)-adic groups.

Before moving on to homology, let us pause to gather some examples of projective modules.

Lemma 2.2.3. The \(G\)-module \(\mathcal{H}(G)\), with \(G\) acting by left (or right) translation, is projective.

(Note that, since \(\mathcal{H}(G)\) is not unital, the lemma is not entirely trivial.)

Proof. Let \(H_1 \supseteq H_2 \supseteq H_3 \supseteq \ldots\) be a neighborhood basis at \(1 \in G\) consisting of compact, open subgroups. Then we have an isomorphism of left \(G\)-modules,

\[
\mathcal{H}(G) \cong \bigoplus_{i=1}^{\infty} \mathcal{H}(G)(e_{H_{i+1}} - e_{H_i}),
\]

and each of the summands on the right-hand side is projective. (This argument was pointed out to me by Nigel Higson. See Casselman’s paper [Cas81, Theorem A.4] for another proof.)

The following simple observation is very useful in constructing projective objects.

Lemma 2.2.4. If \(L : \mathcal{A} \rightarrow \mathcal{B}\) is a functor between abelian categories possessing an exact right adjoint \(R : \mathcal{B} \rightarrow \mathcal{A}\), then \(L\) sends projective objects in \(\mathcal{A}\) to projective objects in \(\mathcal{B}\).

Proof. If \(P\) is projective in \(\mathcal{A}\), then the functor

\[
B \mapsto \text{Hom}_\mathcal{B}(LP,B) \cong \text{Hom}_\mathcal{A}(P,RB)
\]

is a composition of two exact functors, and is therefore exact.

Lemma 2.2.5. (1) If \(H\) is a closed subgroup of \(G\), then the functor \(\text{res}_H^G\) sends projective \(G\)-modules to projective \(H\)-modules.
(2) If $H$ is an open subgroup of $G$, then the functor $\text{ind}^G_H$ sends projective $H$-modules to projective $G$-modules.

(3) If $H$ is open and compact, then $\text{ind}^G_H \pi$ is projective for every smooth representation $\pi$ of $H$.

Proof. Theorem 2.1.35 implies that the functor $\text{res}^G_H$ has an exact right-adjoint—namely, $\text{Ind}^G_H$—and so it preserves projectivity. The same argument applies to $\text{ind}^G_H$ when $H$ is an open subgroup of $G$. If $H$ is compact, then every smooth representation of $H$ is isomorphic to a direct sum of irreducibles (Example 2.1.24), and is therefore projective.

Lemma 2.2.6. If $P$ is projective over $G$, and $V$ is any smooth $G$-module, then the module $P \otimes_{\mathbb{C}} V$ (with the diagonal $G$-action) is projective.

Proof. The functor $W \mapsto W \otimes_{\mathbb{C}} V$ admits $W \mapsto \text{Hom}_{\mathbb{C}}(V,W)^{\otimes}$ as a right adjoint (see [Ren10, III.1.15 Corollaire]). The latter functor being exact (Lemma 2.1.16), Lemma 2.2.4 applies.

2.2.2 Smooth homology

Definition 2.2.7. The left-derived functors $V \mapsto H_s(G,V)$ of the functor $V \mapsto V_G$ on $\text{Mod}(G)$ are called smooth homology.

By Part (3) of Lemma 2.1.41, we may alternatively define $H_s(G,V)$ as the derived functor $\text{Tor}_s^{\text{Mod}(\mathcal{H}(G))}(\mathbb{C},V)$.

Example 2.2.8. If $G$ is compact, or an increasing union of compact subgroups, then $H_0(G,V) \cong V_G$, while $H_i(G,V) = 0$ for all $i > 0$ (because in this case the functor $V \mapsto V_G$ is exact, by Lemma 2.1.41).

Example 2.2.9. If $G$ is discrete, then smooth homology is usually called group homology (working over $\mathbb{C}$); see [Bro94].

Definition 2.2.10 (The standard resolution). Define a chain complex $B_s(G)$ of $G$-modules by

$$B_n(G) = \mathcal{H}(G^{n+1}),$$
with $G$ acting diagonally by left-translation, and with differential

$$df(g_0, \ldots, g_{n-1}) = \sum_{i=0}^{n} (-1)^i \int_G f(g_0, \ldots, g_{i-1}, g, g_i, \ldots, g_{n-1}) \, dg.$$ 

Define an augmentation $B_0(G) \to \mathbb{C}$ by $f \mapsto \int_G f(g) \, dg$.

For each smooth $G$-module $V$, define $B_s(G, V) := B_s(G) \otimes_{\mathbb{C}} V$, with diagonal $G$-action.

**Lemma 2.2.11.** $B_s(G, V)$ is a projective resolution of $V$ in $\text{Mod}(G)$.

**Proof.** It is sufficient to prove this for $V = \mathbb{C}$. Indeed, if such is the case, then for arbitrary $V$ one knows that $B_s(G, V)$ is exact (because the functor $X \mapsto X \otimes_{\mathbb{C}} V$ is exact) and projective (Lemma 2.2.6).

Since $B_n(G) \cong \mathcal{H}(G)^{\otimes (n+1)}$, with the diagonal $G$-action, Lemmas 2.2.3 and 2.2.6 ensure that each $B_n(G)$ is projective. To show that the complex is exact, one constructs a contracting homotopy by analogy with the case of finite groups; see [BD08, Lemme 2] for the details.

It follows that $H_s(G, V)$ is the homology of the complex $B_s(G, V)_G$. This complex can be described explicitly, and such a description is sometimes useful when doing calculations.

**Definition 2.2.12.** For each smooth $G$-module $V$, let $\overline{B}_s(G, V)$ denote the chain complex of vector spaces

$$\overline{B}_n(G, V) = C^c_\infty(G^n, V)$$

with differential

$$df(g_1, \ldots, g_n) = \int_G g^{-1} f(g, g_1, \ldots, g_n) dg + \sum_{i=1}^{n} (-1)^i \int_G f(g_1, \ldots, g_{i-1}, g, g^{-1}g_i, g_i, \ldots, g_n) \, dg + (-1)^{n+1} \int_G f(g_1, \ldots, g_n, g) \, dg.$$
Lemma 2.2.13. The map $\Phi : B_s(G, V)_G \to B_s(G, V)$ defined by

$$\Phi f(g_1, \ldots, g_n) = \int_G g^{-1} f(g, gg_1g_2, \ldots, gg_1 \cdots g_n) \, dg,$$

for $f \in C^\infty_c(G^{n+1}, V)$, is an isomorphism of chain complexes.

Proof. See [BD08, Lemme 3]

For the kind of groups that will be considered in the rest of this thesis, geometry provides another source of projective resolutions. The next section develops some geometric preliminaries, after which we shall return to the subject of smooth homology.

2.3 Chamber Homology

In this section, we consider totally disconnected groups acting on affine buildings. An affine building is a complete CAT(0) metric space, equipped with additional combinatorial structure. This section begins, accordingly, with a review of some basic terminology and properties of CAT(0) spaces, followed by a brief discussion of affine buildings. The metric and combinatorial features of buildings are then used to define a kind of equivariant homology called chamber homology. We will see that chamber homology is an example of smooth homology. Later, in Section 2.4, chamber homology will also be related to Hochschild and cyclic homology.

Our main references are: for CAT(0) spaces, [BH99]; for buildings, [AB08]; and for chamber homology, [BCH94, Sections 5–6], [HN96] and [Sch96]. The exposition in this section is strongly influenced by that of [HN96].

2.3.1 CAT(0) spaces and their isometries

Definition 2.3.1. Let $X$ be a metric space. A geodesic segment (respectively, ray, line) in $X$ is an isometric mapping, into $X$, of a Euclidean interval $[0, l]$ (respectively, $[0, \infty)$, $(-\infty, \infty)$). A geodesic segment $\gamma : [0, l] \to X$ is referred to as a geodesic from $\gamma(0)$ to $\gamma(l)$. 
We will not be overly fastidious about the distinction between the map $\gamma$ and its image in $X$.

The CAT(0) property is one way of capturing the idea of negative curvature in a metric space: roughly speaking, $X$ is CAT(0) if any two geodesics issuing from the same point diverge at least as quickly as in Euclidean space. A precise definition is given below.

**Definition 2.3.2.** A geodesic triangle $\Delta$ in $X$ consists of three points ("vertices") $v_1, v_2, v_3 \in X$, and three geodesic segments ("edges") $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_1]$ (where $[v_i, v_j]$ is a geodesic from $v_i$ to $v_j$). A comparison triangle for $\Delta$ is a geodesic triangle $\overline{\Delta}$ in the Euclidean plane $E$, with vertices $\overline{v}_1, \overline{v}_2, \overline{v}_3$, such that $d_E(\overline{v}_i, \overline{v}_j) = d_X(v_i, v_j)$.

Given a geodesic triangle $\Delta$, choose a comparison triangle $\overline{\Delta}$. Let $f: \overline{\Delta} \to \Delta$ be the unique map which sends each edge $[\overline{v}_i, \overline{v}_j]$ isometrically onto the corresponding edge $[v_i, v_j]$. The triangle $\Delta$ is said to satisfy the CAT(0) inequality if $d_X(f(x), f(y)) \leq d_E(x, y)$ for all $x, y \in \overline{\Delta}$. The space $X$ is called a CAT(0) space if every geodesic triangle satisfies the CAT(0) inequality.

**Examples 2.3.3.** Every Euclidean space is CAT(0). The upper half-plane with its hyperbolic metric is CAT(0). The geometric realization of any tree is CAT(0).

**Proposition 2.3.4.** (1) Every CAT(0) space $X$ is uniquely geodesic: for every two points $x, y \in X$, there is a unique geodesic segment $[x, y]$ from $x$ to $y$.

(2) In particular, $X$ is contractible.

(3) If $\gamma_1$ and $\gamma_2$ are geodesic lines in $X$, then the function $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex.

**Proof.** See [BH99, II.1.4, II.1.5, and II.2.2].

We now consider isometries of CAT(0) spaces, following [BH99, II.6].

**Definition 2.3.5.** For each isometry $f: X \cong X$ of a CAT(0) space $X$, define

$$d_f := \inf\{d(x, fx) \mid x \in X\} \quad \text{and} \quad \min(f) := \{x \in X \mid d(x, fx) = d_f\}.$$
We say that \( f \) is \textit{semisimple} if \( \min(f) \) is non-empty. A semisimple isometry \( f \) is called \textit{elliptic} if \( d_f = 0 \), and \textit{hyperbolic} otherwise.

If \( X \) is a (finite-dimensional) Euclidean space, then every isometry is semisimple [BH99, II.6.5]. The same is true of affine buildings; see Theorem 2.3.11 below.

**Proposition 2.3.6.** If \( f \) is an isometry of a \textit{CAT}(0) space \( X \), then \( \min(f) \) is a closed, convex subset of \( X \).

**Proof.** [BH99, II.6.2]. \qed

Note that \( f \) is elliptic if and only if it fixes a point in \( X \); the set \( \min(f) \) is then the fixed-point set of \( f \). In the case of a hyperbolic isometry, the set \( \min(f) \) also admits a simple description. Recall that geodesics \( \gamma \) and \( \gamma' \) are \textit{parallel} if the function \( t \mapsto d(\gamma(t), \gamma'(t)) \) is bounded—in which case it is in fact constant, by convexity of the metric [BH99, II.2.13].

**Theorem 2.3.7.** (1) An isometry \( f \) of \( X \) is hyperbolic if and only if there is a geodesic line \( \gamma : \mathbb{R} \to X \) which is translated non-trivially by \( f \), meaning that \( f \cdot \gamma(t) = \gamma(t + d) \) for some \( d > 0 \). In this case, we in fact have \( d = d_f \).

(2) If \( f \) is hyperbolic, then for each \( x \in \min(f) \) there is a unique geodesic \( \gamma_{f,x} \) which is translated by \( f \) and has \( \gamma_{f,x}(0) = x \).

(3) If \( f \) is hyperbolic and \( x, y \in \min(f) \), then the geodesics \( \gamma_{f,x} \) and \( \gamma_{f,y} \) are parallel. The space \( \min(f) \) is isometric to a product \( Y \times \mathbb{R} \), and the restriction of \( f \) to \( \min(f) \) has the form \( f(y, t) = (y, t + d_f) \).

**Proof.** [BH99, II.6.8]. \qed

**2.3.2 Affine buildings and their automorphisms**

One can approach the subject of affine buildings (also known as Euclidean buildings) from a number of different directions. Many of these approaches are discussed in the book of Abramenko and Brown [AB08]; other perspectives may be found in the books of Ronan [Ron89] and Bridson-Haefliger [BH99, II.10A], for example. Affine buildings first arose in
the work of Bruhat and Tits, on the structure of reductive $p$-adic groups: see [Tit79] for an introduction, and [BT72] for the details. This section briefly introduces affine buildings, from a point of view suitable for the applications in the rest of the thesis.

An affine building is a space built from affine Coxeter complexes. Roughly speaking, an affine (or Euclidean) Coxeter complex is a Euclidean space $E$, in which one specifies a highly symmetrical family of affine hyperplanes; these hyperplanes carve out a simplicial structure on $E$. We refer to [AB08, Sections 10.1–10.2] for the precise definition. Examples include the line $E = \mathbb{R}$, with a vertex at each integer point; and the plane $E = \mathbb{R}^2$, tessellated by equilateral triangles.

For applications to $p$-adic groups, it is most natural to consider buildings as polysimplicial complexes. A polysimplex is a product of simplices. A face of a polysimplex $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ is a subset of the form $\tau_1 \times \tau_2 \times \cdots \times \tau_n$, where each $\tau_i$ is a face of the simplex $\sigma_i$. A polysimplicial complex is a space constructed by attaching polysimplices to one another along their faces. For example, a finite product of simplicial complexes is a polysimplicial complex.

**Definition 2.3.8.** Let $X$ be polysimplicial complex. A collection $\mathcal{A}$ of subcomplexes of $X$ is called a **system of affine apartments** if:

1. Each $A \in \mathcal{A}$ is isomorphic to a finite product of affine Coxeter complexes;
2. Every pair of polysimplices of $X$ is contained in some $A \in \mathcal{A}$; and
3. For every pair $\sigma, \tau$ of polysimplices in $X$, and every pair $A_1, A_2$ of apartments containing both $\sigma$ and $\tau$, there is a polysimplicial isomorphism $A_1 \cong A_2$ that fixes $\sigma$ and $\tau$ pointwise.

An **affine building** is a polysimplicial complex admitting a system of affine apartments. A **chamber** in an affine building is a polysimplex of maximal dimension.

If $X$ is an affine building, then the union $\mathcal{A}_{\text{max}}$ of all of the systems of affine apartments for $X$ is again a system of apartments [AB08, Theorem 4.54]. Thus $X$ possesses a canonical system of apartments.
Examples 2.3.9. (1) Every affine Coxeter complex $E$ is a building, with $A = \{E\}$.

(2) Let $X$ be a tree, and let $A$ be the collection of all bi-infinite paths in $X$. Then $A$ is a system of affine apartments for $X$; in fact, $A = A_{\text{max}}$.

(3) Let $A_1$ and $A_2$ be systems of affine apartments for buildings $X_1$ and $X_2$, respectively. Then

$$A_1 \times A_2 := \{A_1 \times A_2 \subset X_1 \times X_2 \mid A_i \in A_i\}$$

is a system of affine apartments for $X_1 \times X_2$.

The Euclidean metric on an affine Coxeter complex is determined, up to a scalar factor, by the simplicial structure [AB08, Lemma 10.36]. Thus each apartment of an affine building $X$ carries a canonical metric, up to a scalar. If we normalize these metrics so that the isomorphisms in part (3) of Definition 2.3.8 become isometries, we obtain a metric on $X$:

Theorem 2.3.10. Let $X$ be an affine building. For each pair of points $x, y \in X$, choose an apartment $A$ containing both $x$ and $y$, and let $d_X(x, y) := d_A(x, y)$. Then $d_X$ is a well-defined metric on $X$, and $X$ with this metric is complete and CAT(0).

Proof. See [AB08, Theorem 11.16] or [BH99, II.10A.4 Theorem].

We now consider automorphisms of an affine building $X$, by which we mean isometries which preserve the polysimplicial structure.

Theorem 2.3.11. Every automorphism of an affine building is semisimple (in the terminology of Definition 2.3.5).

Proof. See [HN96, Theorem 5.6], where the theorem is deduced from the corresponding fact about Euclidean space.

The following hypotheses will be referred to frequently in the rest of the thesis. (“GAB” stands for “Group Acting on a Building”.)

Hypotheses 2.3.12. Let $G$ be a locally compact, totally disconnected group, and let $X$ be a locally finite affine building, equipped with an action of $G$ by automorphisms. Assume:
(GAB1) The polysimplicial complex $X$ is oriented, and $G$ preserves the orientation.

(GAB2) For each point $x \in X$, the isotropy group $G_x$ is a compact, open subgroup of $G$.

(GAB3) For each $g \in G$, the fixed-point set $X^g = \{x \in X \mid gx = x\}$ is a subcomplex of $X$.

**Remarks 2.3.13.**

1. By definition, an orientation of a polysimplex $\sigma = \sigma_1 \times \cdots \times \sigma_n$ is an orientation of each $\sigma_i$, with two orientations of $\sigma$ being considered equivalent if they differ on an even number of $\sigma_i$. An orientation of $X$ is then a choice of orientation for each polysimplex.

2. The hypothesis (GAB3) implies that if $g$ stabilizes a polysimplex $\sigma$, then $g$ fixes $\sigma$ pointwise.

**Lemma 2.3.14.** Assume the hypotheses (GAB1–3). The action of $G$ on $X$ is proper: for each compact subset $C \subset X$, the set

$$\{g \in G \mid gC \cap C \text{ is nonempty}\}$$

is precompact in $G$.

**Proof.** Since $X$ is locally finite, we may find a finite subcomplex $C' \subset X$ with the property that

$$gC \cap C \neq \emptyset \Rightarrow gC \subset C'.$$

For each point $x \in X$, the set $Gx \cap C'$ is finite, because the orbit $Gx$ intersects each polysimplex in at most one point (by (GAB3)). Since $G_x$ is compact, the set $\{g \in G \mid gx \in C'\}$ is compact in $G$.

We will occasionally impose the following strengthening of (GAB3):

(GAB3+) For each $g \in G$, the set $\text{min}(g) \subset X$ is a subcomplex of $X$.

**Example 2.3.15.** Let $F$ be a $p$-adic field, and consider $G = \text{SL}_n(F)$. There is an affine building canonically associated to $G$, the Bruhat-Tits building, equipped with an action of $G$ satisfying the conditions (GAB1–3+). In this case, the building is a simplicial complex.
Example 2.3.16. The action of $\text{SL}_n(F)$ on its Bruhat-Tits building $X$ extends in a natural way to an action of $G = \text{GL}_n(F)$. This action does not satisfy the hypothesis (GAB2), because the (noncompact) center of $G$ acts trivially. To rectify this problem, we consider the line $L \cong \mathbb{R}$, triangulated with a vertex at each integer point, on which $G$ acts via 

$$g(l) = l + \text{val}(|\det(g)|).$$
Figure 2.3.2: An apartment in the Bruhat-Tits building of $\text{SL}_3(F)$

(Here $\text{val} : F^* \to \{0, 1, 2, \ldots\}$ is the valuation on $F$.) Then the product $X \times L$ is an affine building, called the (extended) Bruhat-Tits building of $G$.

The hypotheses (GAB1–3) are all satisfied for the action of $G$ on $X \times L$. The hypothesis (GAB3+) does not hold for the canonical polysimplicial structure. For example, consider $\text{GL}_2(F)$, for which each apartment in $X \times L$ is a plane, tessellated by squares, as pictured in gray in Figure 2.3.3. The element $g = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ stabilizes one such apartment, where it acts as the affine transformation $(x, l) \mapsto (1 - x, l + 1)$, as indicated by the dashed arrows in Figure 2.3.3. The intersection of $\text{min}(g)$ with this apartment is the black vertical line $(1/2, l)$, which is not a subcomplex. Note that the hypothesis (GAB3+) is satisfied once we pass to the barycentric subdivision of $X$.

For the rest of this section, we assume the hypotheses (GAB1–3). For each $g \in G$, we define $d_g$ and $\text{min}(g)$, and classify $g$ as being either elliptic or hyperbolic, as in Definition 2.3.5.

**Lemma 2.3.17.** The function $G \to \mathbb{R}$, $g \mapsto d_g$, is locally constant.

**Proof.** See [HN96, Lemma 5.9].

**Theorem 2.3.18.** Let $H$ be a subgroup of $G$. The closure $\overline{H}$ is compact if and only if $H$ fixes a point in $X$. 
Proof. If \( H \subseteq G_x \), then \( H \) is precompact because \( G_x \) is compact. The converse is the Bruhat-Tits fixed-point theorem; see [AB08, Theorem 11.23] or [BH99, II.2.8 Corollary]. 

**Corollary 2.3.19.** Let \( G_c \) denote the union of the compact subgroups of \( G \), and let \( G_{nc} \) denote the complement of \( G_c \) in \( G \).

(1) \( G_c \) and \( G_{nc} \) are, respectively, the sets of elliptic and of hyperbolic elements of \( G \).

(2) Both \( G_c \) and \( G_{nc} \) are open, closed, conjugation-invariant subsets of \( G \).

Proof. See [HN96, Theorem 5.10].

### 2.3.3 Chamber homology and smooth homology

We continue to assume the hypotheses (GAB1–3): \( G \) is a locally compact, totally disconnected group, acting by automorphisms on a locally finite, oriented affine building \( X \), in such a way that the isotropy group \( G_x \) of each point \( x \in X \) is compact and open in \( G \), and
each fixed-point set $X^g$ is a subcomplex of $X$. We will use the geometry of $X$ to construct certain homology groups, which turn out to be the smooth homology groups of a certain $G$-module. Most of the results stated in this section appear in at least one of [BCH94, Sections 5–6], [HN96], and [Sch96].

**Definition 2.3.20.** (cf. [BCH94, Section 5], [SS97, Chapter V]) An *equivariant coefficient system* $\mathcal{F}$ on $X$ consists of:

- A complex vector space $\mathcal{F}(\sigma)$ for each polysimplex $\sigma$ of $X$.
- For each face $\tau$ of $\sigma$, a linear map
  
  $$\mathcal{F}(\tau, \sigma) : \mathcal{F}(\sigma) \to \mathcal{F}(\tau),$$

  such that $\mathcal{F}(\tau, \rho) = \mathcal{F}(\tau, \sigma) \circ \mathcal{F}(\sigma, \rho)$ whenever $\sigma$ is a face of $\rho$.
- For each $g \in G$ and each polysimplex $\sigma$, a linear map
  
  $$g(\sigma) : \mathcal{F}(\sigma) \to \mathcal{F}(g\sigma),$$

  such that

  - The map $g \mapsto g(\sigma)$ restricts to a smooth representation of $G_\sigma$ on $\mathcal{F}(\sigma)$,
  - $(g_1 g_2)(\sigma) = g_1(g_2\sigma) \circ g_2(\sigma)$ for all $g_1, g_2 \in G$, and
  - for each face $\tau \subset \sigma$ the diagram

    $$\begin{array}{ccc}
    \mathcal{F}(\tau) & \xrightarrow{g(\tau)} & \mathcal{F}(g\tau) \\
    \mathcal{F}(\tau, \sigma) \downarrow & & \downarrow \mathcal{F}(g\tau, g\sigma) \\
    \mathcal{F}(\sigma) & \xrightarrow{g(\sigma)} & \mathcal{F}(g\sigma)
    \end{array}$$

    commutes.

**Definition 2.3.21.** Let $\mathcal{F}$ be an equivariant coefficient system on $X$. For each $n \geq 0$, let
$X^{(n)}$ denote the set of $n$-dimensional polysimplices in $X$, and define

$$C_n(X, \mathcal{F}) := \bigoplus_{\sigma \in X^{(n)}} \mathcal{F}(\sigma).$$

This complex vector space is a smooth representation of $G$, via the maps $g(\sigma) : \mathcal{F}(\sigma) \rightarrow \mathcal{F}(g\sigma)$.

Define a differential $\partial : C_n(X, \mathcal{F}) \rightarrow C_{n-1}(X, \mathcal{F})$ as follows. If $\sigma_i$ is a simplex, and $\tau_i \subset \sigma_i$ a codimension-one face, define $[\tau_i : \sigma_i] = +1$ or $-1$, according to whether the orientation that $\tau_i$ inherits from $\sigma_i$ does or does not agree with the chosen orientation on $\tau_i$. Each codimension-one face of a polysimplex $\sigma = \sigma_1 \times \cdots \times \sigma_l$ is of the form $\tau = \sigma_1 \times \cdots \times \tau_i \times \cdots \times \sigma_l$, where $\tau_i$ is a codimension-one face of $\sigma_i$, and we let $[\tau : \sigma] := [\tau_i : \sigma_i]$. Now define

$$\partial : \mathcal{F}(\sigma) \rightarrow \bigoplus_{\tau \subset \sigma \text{ codim-1}} \mathcal{F}(\tau), \quad \partial := \sum_{\tau} [\tau : \sigma] \mathcal{F}(\tau, \sigma).$$

**Lemma 2.3.22.** $C_\ast(X, \mathcal{F})$ is a chain complex of projectives in $\text{Mod}(G)$.

**Proof.** The standard computation (e.g., [GM03, I.4.2]) shows that $C_\ast(X, \mathcal{F})$ is a chain complex. Equivariance of $\partial$ follows from the assumption that $G$ preserves the orientation. To see that each $C_n(X, \mathcal{F})$ is projective over $G$, one notes that for each polysimplex $\sigma$ there is an isomorphism of $G$-modules,

$$\bigoplus_{\sigma' \in G: \sigma} \mathcal{F}(\sigma') \cong \text{ind}_{G: \sigma}^G \mathcal{F}(\sigma)$$

(cf. [Sch96, Remark 1]). Since $G_\sigma$ is a compact open subgroup of $G$, Lemma 2.2.5 implies that $\text{ind}_{G_\sigma}^G \mathcal{F}(\sigma)$ is projective. Thus $C_n(X, \mathcal{F})$ is a direct sum of projectives.

We let $H_\ast^G(X, \mathcal{F})$ denote the homology of the complex $C_\ast(X, \mathcal{F})_G$ of $G$-coinvariants.

**Lemma 2.3.23.** Let $\mathcal{F}$ be a $G$-equivariant coefficient system on $X$. Define a new coefficient system $\mathcal{F}^G$ on $X$ by putting

$$\mathcal{F}^G(\sigma) := \mathcal{F}(\sigma)^{G_\sigma},$$
and

\[ F^G(\tau, \sigma) : F(\sigma)^{G_\sigma} \to F(\tau)^{G_\tau}, \quad v \mapsto \sum_{g \in G_\tau / G_\sigma} g \cdot F(\tau, \sigma)(v). \]

This coefficient system \( F^G \) carries a \( G \)-action, restricted from the \( G \)-action on \( F \). The maps

\[ F(\sigma)^{G_\sigma} \to F(\sigma), \quad v \mapsto \frac{1}{\text{vol}(G_\sigma)} v \]

assemble into an isomorphism of chain complexes \( C_*(X, F^G)_G \to C_*(X, F)_G \). In particular, \( H^G_*(X, F^G) \) is naturally isomorphic to \( H^G_*(X, F) \).

**Proof.** See [HN96, Lemma 3.3], whose proof is valid for arbitrary coefficient systems. \( \square \)

**Definition 2.3.24.** Define an equivariant coefficient system \( \mathcal{G} \) on \( X \) as follows. For each polysimplex \( \sigma \), let \( \mathcal{G}(\sigma) = \mathcal{H}(G_\sigma) \). If \( \tau \) is a face of \( \sigma \), then \( G_\sigma \) is an open subgroup of \( G_\tau \), and we define \( \mathcal{G}(\tau, \sigma) : \mathcal{H}(G_\sigma) \to \mathcal{H}(G_\tau) \) by extending functions by zero. For each \( g \in G \), define \( g(\sigma) : \mathcal{H}(G_\sigma) \to \mathcal{H}(G_{g\sigma}) \) by

\[ (g(\sigma)f)(g') = f(g^{-1}g'). \]

**Definition 2.3.25.** The group \( H^G_*(X, \mathcal{G}) \) is called the **chamber homology** of \( X \). We abbreviate the notation to \( H^G_*(X) \).

Passing from \( \mathcal{G} \) to \( \mathcal{G}^G \), as in Lemma 2.3.23, we obtain the following alternative description of chamber homology (this is the definition given in [BCH94]). Recall from Example 2.1.43 that for each of the compact groups \( G_\sigma \), we denote by \( R_\mathbb{C}(G_\sigma) \) the complex vector space having as a basis the set \( \widehat{G_\sigma} \) of isomorphism classes of irreducible smooth representations of \( G_\sigma \). If \( \tau \) is a face of \( \sigma \), so that \( G_\sigma \) is an open subgroup of \( G_\tau \), then the functor \( \text{ind}_{G_\sigma}^{G_\tau} \) determines a linear map \( R_\mathbb{C}(G_\sigma) \to R_\mathbb{C}(G_\tau) \).

**Lemma 2.3.26.** Define an equivariant coefficient system \( \mathcal{R} \) on \( X \) by

\[ \mathcal{R}(\sigma) = R_\mathbb{C}(G_\sigma), \]
with transition maps

\[ \mathcal{R}(\tau, \sigma) := \text{ind}_{G_\sigma}^{G_\tau} \]

for \( \tau \subseteq \sigma \), and with the \( G \)-action induced by the maps

\[ g(\sigma) : \widehat{G_\sigma} \to \widehat{G_{g\sigma}}, \quad \pi \mapsto \pi^{g^{-1}} = \pi \circ \text{Ad}_{g^{-1}}. \]

Then \( H^G_\ast(X) \cong H^G_\ast(X, \mathcal{R}) \).

**Proof.** The map sending each \( \pi \in \widehat{G_\sigma} \) to its character \( \text{ch}_\pi \) establishes an isomorphism of equivariant coefficient systems \( \mathcal{R} \cong \mathcal{G}^G \) (see Example 2.1.43). Therefore Lemma 2.3.23 applies. \( \square \)

**Example 2.3.27.** If \( G \) is the trivial group, then \( H^G_\ast(X) \) is just the (poly)simplicial homology, with complex coefficients, of the contractible space \( X \). More generally, if \( G \) acts trivially then one has

\[ H^G_n(X) = \begin{cases} \mathcal{H}(G) & (n = 0) \\ 0 & (n > 0). \end{cases} \]

At the other extreme, if \( G \) acts freely then \( H^G_\ast(X) \cong H_\ast(X/G, \mathbb{C}) \), the homology of the quotient space \( X/G \) with complex coefficients.

**Example 2.3.28.** Let \( F \) be a \( p \)-adic field, and let \( G = \text{GL}_1(F) \cong \mathbb{O}^\times \times \mathbb{Z} \) (see Example 2.1.7). Take \( X = \mathbb{R} \), triangulated with a vertex at each integer point. Let \( \mathbb{Z} \subset G \) act by integer translations, and let \( \mathbb{O}^\times \) act trivially. Then

\[ H^G_n(X) \cong H_n(X/\mathbb{Z}, \mathbb{C}) \otimes \mathcal{H}(\mathbb{O}^\times) \cong \begin{cases} \mathcal{H}(\mathbb{O}^\times) & (n = 0, 1) \\ 0 & (n > 1). \end{cases} \]

**Example 2.3.29.** [HN96, Lemma 3.3] Fix a chamber \( \Delta \subset X \), and suppose that \( \Delta \) is a fundamental domain for the action of \( G \) (i.e., that every polysimplex in \( X \) is conjugate under \( G \) to precisely one face of \( \Delta \)). Then there is an isomorphism of complexes \( C_\ast(X, \mathcal{G})_G \cong C_\ast(\Delta, \mathcal{R}) \) (cf. Lemma 2.3.26), so \( H^G_\ast(X) \cong H_\ast(\Delta, \mathcal{R}) \).
Example 2.3.30. [BCH94, Example 6.12] For $G = \text{SL}_2(F)$, let $X$ be the Bruhat-Tits tree (see Example 2.3.15). Each chamber (i.e., edge) in $X$ is a fundamental domain for the $G$-action; the standard choice is

$$
\textstyle \Delta = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
$$

where

$$
G_v = K = \text{SL}_2(\mathcal{O}), \quad G_{v'} = K' = \left[ \begin{array}{c} \mathcal{O} \backslash p^{-1} \\ \mathcal{O} \end{array} \right], \quad \text{and} \quad G_e = I = \left[ \begin{array}{c} \mathcal{O} \\ \mathcal{O} \end{array} \right].
$$

(The notation means that, for instance, $I$ is the subgroup of $\text{SL}_2(F)$ consisting of those matrices whose bottom-left entries lie in $p$, and whose other entries all lie in $\mathcal{O}$. This subgroup $I$ is called the (standard) Iwahori subgroup of $G$.)

Then $H^G_\bullet(X)$ is isomorphic to the homology of the complex

$$
0 \to R_{\mathcal{C}}(I) \xrightarrow{\partial} R_{\mathcal{C}}(K) \oplus R_{\mathcal{C}}(K') \to 0,
$$

where $\partial(\pi) = (\text{ind}^K_{\mathcal{O}} \pi, - \text{ind}^{K'}_{\mathcal{O}} \pi)$. Baum, Higson and Plymen have computed these homology groups [BHP93].

For $G = \text{SL}_n(F)$, $H^G_\bullet(X)$ similarly identifies with the homology $H_\bullet(\Delta, \mathcal{R})$ of an $(n-1)$-simplex, with coefficients in the spaces of representations of certain compact, open subgroups of $G$.

Example 2.3.31. [BCH94, Example 6.15] Let $G = \text{GL}_n(F)$, and consider the Bruhat-Tits building $X = X^0 \times L$, where $X^0$ is the building for $\text{SL}_n(F)$, and $L$ is a line (see Example 2.3.16). For each simplex $\sigma_2 \subset L$ we have

$$
G_{\sigma_2} = G^0 := \{ g \in G \mid \det(g) \in \mathcal{O}^X \}.
$$

For each polysimplex $\sigma = \sigma_1 \times \sigma_2$ in $X$, we thus have $G_\sigma = G^0_{\sigma_1}$. It follows from this that

$$
C_*(X, G) \cong C_*(X^0, G^0) \otimes \mathbb{C} C_*(L, \mathbb{C}),
$$
where $G^\sigma$ is the $G$-equivariant coefficient system $G^\sigma(\sigma_1) = \mathcal{H}(G^\sigma_{\sigma_1})$. The complex of coinvariants $C_\sigma(X,\mathcal{G})_G$ may be identified explicitly on the basis of this decomposition; see [BCH94]. For $G = \text{GL}_2(F)$, for example, $H^G_\sigma(X)$ is isomorphic to the homology of the total complex of the bicomplex

$$
R_C(K) \oplus R_C(K^\gamma) \xleftarrow{\text{id}-\gamma} R_C(I) \xrightarrow{\text{ind}_{\gamma}^K - \text{ind}_{I}^{K^\gamma}} R_C(I)
$$

where $K = \text{GL}_2(O)$, $\gamma = [\begin{smallmatrix} 0 & 1 \\ \omega & 0 \end{smallmatrix}]$, and $I = K \cap K^\gamma$. (See Definition 2.1.44 for the meaning of the superscripts $\gamma$.)

Recall (Corollary 2.3.19) that the union $G_c$ of the compact subgroups of $G$ is equal to the set of elliptic elements, and is a conjugation-invariant, open subset of $G$. Consider the smooth $G$-module $\mathcal{H}(G_c)_{\text{Ad}}$ of smooth, compactly supported functions on $G_c$, with $G$ acting by conjugation. For each vertex $v \in X(0)$, the isotropy group $G_v$ is an open subset of $G_c$, and so $\mathcal{H}(G_v)$ embeds in $\mathcal{H}(G_c)$. These embeddings combine to give a $G$-equivariant map

$$
(2.3.32) \quad \alpha : C_0(X, \mathcal{G}) \to \mathcal{H}(G_c)_{\text{Ad}}.
$$

Our assumption that each fixed-point set $X^g$ be a subcomplex ensures that $\alpha$ is surjective.

**Proposition 2.3.33** ([HN96, Proposition 3.6], [Sch96, Proposition 2]). The augmentation (2.3.32) makes $C_\sigma(X, \mathcal{G})$ a projective resolution of $\mathcal{H}(G_c)_{\text{Ad}}$ in $\text{Mod}(G)$.

**Proof.** The following proof is taken from [Sch96]; we recall the argument for later use.

Lemma 2.3.22 ensures that each $C_n(X, \mathcal{G})$ is projective, so it remains to show that the augmented complex is exact.

Consider each $\mathcal{H}(G_\sigma)$ as the space of compactly supported sections of a sheaf on $G_c$ (namely, the direct image of the constant sheaf $\mathbb{C}$ under the embedding $i_\sigma : G_\sigma \hookrightarrow G_c$). Each of these sheaves is $c$-soft, by Lemma 2.1.3. Therefore the complex $C_\sigma(X, \mathcal{G}) \to \mathcal{H}(G_c)$
is exact if the complex of sheaves

$$0 \to \bigoplus_{\sigma \in X^{(\dim X)}} (i_{\sigma})_* \mathbb{C} \to \cdots \to \sum_{\sigma \in X^{(0)}} (i_{\sigma})_* \mathbb{C} \to \mathbb{C} \to 0$$

is exact [KS94, Proposition 2.5.8]. Exactness can be checked at each stalk [KS94, Remark 2.2.5]. The stalk over $g \in G_c$ identifies with the complex $C_\ast(X^g, \mathbb{C}) \to \mathbb{C}$ computing the reduced polysimplicial homology of the space $X^g$, and $X^g$ is contractible.

**Corollary 2.3.34.** $H_*^G(X) \cong H_*(G, \mathcal{H}(G_c)_\text{Ad})$.

In the next section, we exploit the geometry of $X$ to prove two variants of Corollary 2.3.34.

### 2.3.4 Variations on chamber homology

The $G$-module $\mathcal{H}(G_c)_\text{Ad}$ is a submodule (in fact, a direct-summand) of the module $\mathcal{H}(G)_\text{Ad}$. On the other hand, $\mathcal{H}(G_c)_\text{Ad}$ admits as a quotient the module $\mathcal{H}(G_p)_\text{Ad}$, where

$$G_p := \{ g \in G \mid \text{there is a geodesic ray } \gamma : [0, \infty) \to X \text{ fixed pointwise by } g \}.$$  

(Note that we refer to rays, not lines.) In this section, we describe the smooth homology of the modules $\mathcal{H}(G)_\text{Ad}$ and $\mathcal{H}(G_p)_\text{Ad}$ in terms of the building.

We begin with $\mathcal{H}(G)_\text{Ad}$. The method here is very close to that of [HN96] and [Sch96], although to our knowledge the result is not stated in the literature. Assume hypotheses (GAB1–3+) from Section 2.3.2: $X$ is an oriented, locally finite affine building; $G$ acts by automorphisms of $X$; each isotropy group $G_\sigma$ is compact and open in $G$; and for every $g \in G$, the set $\min(g)$ is a subcomplex of $X$.

**Definition 2.3.35.** For each polysimplex $\sigma$ in $X$, let

$$G_\sigma^+ := \{ g \in G \mid \sigma \subseteq \min(g) \}$$
and

\[ G^{++}_\sigma := \{g \in G^+_\sigma \mid d_g > 0\} = G^+_\sigma \setminus G_\sigma. \]

**Lemma 2.3.36.** (1) Both \( G^+_\sigma \) and \( G^{++}_\sigma \) are open subsets of \( G \).

(2) If \( \tau \) is a face of \( \sigma \), then \( G^+_\sigma \subseteq G^+_\tau \) and \( G^{++}_\sigma \subseteq G^{++}_\tau \).

(3) \( G^+_{g\sigma} = gG^+_\sigma g^{-1} \) and \( G^{++}_{g\sigma} = gG^{++}_\sigma g^{-1} \).

(4) \( G = \bigcup_{\sigma \in X^{(0)}} G^+_\sigma \) and \( G_{nc} = \bigcup_{\sigma \in X^{(0)}} G^{++}_\sigma \).

**Proof.**

(1) Fix \( g \in G^+_\sigma \), and find an open neighborhood \( U \) of \( g \) in \( G \) on which the function \( g \mapsto d_g \) is constant (Lemma 2.3.17). Then \( U \cap (gG_\sigma) \) is an open neighborhood of \( g \). For any \( g' \) in this set, and any \( x \in \sigma \), one has

\[ d(x, g'x) = d(x, gx) = d_g = d_{g'}, \]

so \( x \in \min(g') \). This proves that \( G^+_\sigma \) is open in \( G \), and it is clear from the definition that \( G^{++}_\sigma \) is open in \( G^+_\sigma \).

(2) This follows from the assumption that each \( \min(g) \) is a subcomplex of \( X \).

(3) Use the fact that \( \min(ghg^{-1}) = g \cdot \min(h) \).

(4) For every \( g \in G \), \( \min(g) \) is nonempty (Theorem 2.3.11). We have assumed that \( \min(g) \) is a subcomplex, so it contains a vertex. \[ \square \]

**Definition 2.3.37.** Define equivariant coefficient systems \( \mathcal{G}^+ \) and \( \mathcal{G}^{++} \) on \( X \), by replacing \( G_\sigma \) with \( G^+_\sigma \) and \( G^{++}_\sigma \), respectively, in Definition 2.3.24.

**Lemma 2.3.38.** The complex \( C_*(X, \mathcal{G}^+) \) is a projective resolution of \( \mathcal{H}(G)_{Ad} \), and the complex \( C_*(X, \mathcal{G}^{++}) \) is a projective resolution of \( \mathcal{H}(G_{nc})_{Ad} \).

**Proof.** Each \( \min(g) \) is convex, and therefore contractible, by Proposition 2.3.6. Repeat the proof of Proposition 2.3.33. \[ \square \]

Passing to coinvariants and taking homology, we obtain:
**Proposition 2.3.39.** There are isomorphisms

\[ H_*^G(X, G^+) \cong H_*(G, \mathcal{H}(G)_{Ad}) \quad \text{and} \quad H_*^G(X, G^{++}) \cong H_*(G, \mathcal{H}(G_{nc})_{Ad}). \]

**Example 2.3.40.** Take \( G = \text{SL}_2(F) \), acting on its tree \( X \) as in Example 2.3.30. We previously noted that for elliptic \( g \), the set \( \text{min}(g) = X^g \) is a subcomplex. For hyperbolic \( g \), \( \text{min}(g) \) is an apartment of \( X \), and in particular a subcomplex, so Proposition 2.3.39 applies. The homology of the coefficient system \( G \) — i.e., chamber homology — was computed by Baum, Higson and Plymen in [BHP93]. The homology of \( G^{++} \) is computed in Proposition 4.5.5.

We now relax the assumption (GAB3+) that each \( \text{min}(g) \) be a subcomplex, but continue to assume (GAB3) that each \( X^g \) is a subcomplex. Consider, as above, the set \( G_p \) of elements of \( g \) fixing (pointwise) a geodesic ray in \( X \). We will prove (Corollary 2.3.46) that the higher-degree chamber homology \( H_*^G(X) \) is determined by \( G_p \).

**Lemma 2.3.41.** \( G_p \) is a closed subset of \( G \).

The proof uses the boundary at infinity of \( X \), whose definition we now recall; see [BH99, II.8] or [AB08, 11.8] for more details.

**Definition 2.3.42.** Let \( X \) be a complete CAT(0) metric space. The boundary at infinity is the set \( \partial X \) of equivalence classes of geodesic rays \( \gamma : [0, \infty) \to X \), two rays being declared equivalent if they are parallel (see Section 2.3.1). The equivalence class of the ray \( \gamma \) will be denoted \( \gamma(\infty) \).

If we fix a base-point \( x_0 \), then each geodesic ray in \( X \) is parallel to a unique geodesic ray issuing from \( x_0 \) (see [BH99, II.8.2]).

Borel and Serre showed that when \( X \) is an affine building, \( \partial X \) inherits the structure of a spherical building; see [BS76] or [AB08, 11.8] for more information. Of more immediate use to us will be the cone topology on \( Y := X \cup \partial X \), which may be succinctly defined as follows; see [BH99, II.8.5] for details.
Definition 2.3.43. Fix a base-point $x_0 \in X$. For each $r \geq 0$, consider the closed ball $B(x_0, r)$ around $x_0$ of radius $r$. If $r' > r$, there is a retraction $B(x_0, r') \to B(x_0, r)$, defined by sending each point $x \in B(x_0, r') \setminus B(x_0, r)$ to the point $p_r(x) \in [x_0, x]$ having $d(x_0, p_r(x)) = r$. Then

$$\mathcal{X} \cong \varprojlim_r B(x_0, r)$$

as sets. The cone topology on $\mathcal{X}$ is defined as the topology induced on the projective limit by the metric topology on each $B(x_0, r)$.

If $X$ is proper—for example, if $X$ is a locally finite affine building—then $\mathcal{X}$ and its subspace $\partial X$ are compact in the cone topology, and the induced topology on the subspace $X$ coincides with the original metric topology. Every isometry of $X$ extends uniquely to a homeomorphism of $\mathcal{X}$. See [BH99, II.8] for proofs of these facts.

Proof of Lemma 2.3.41. First note that $G_p$ is equal to the set of those $g \in G_c$ which fix a point in $\partial X$: indeed, if $g$ fixes $x \in X$ and $\gamma(x) \in \partial X$, then $g$ fixes pointwise the geodesic ray parallel to $\gamma$ issuing from $x$.

Now take a sequence $\{g_n\} \subset G_p$, and suppose $g_n \to g$. Since $G_c$ is closed in $G$, we certainly have $g \in G_c$. Choose points $x_n \in \partial X$ with $g_n x_n = x_n$. By compactness of $\partial X$, we may assume that $x_n \to x$ for some point $x \in \partial X$. We will show that $gx = x$.

Since $g \in G_c$, there is a vertex $v \in X$ fixed by $g$. The stabilizer $G_v$ is open, and $g_n$ converges to $g$, so we may assume that all the $g_n$ fix $v$. Let $y$ be an arbitrary point on the geodesic ray $[v, x)$ from $v$ to $x$. We need to show that $gy = y$. For each $n$, let $y_n$ be the point on the ray $[v, x_n)$ with $d(v, y_n) = d(v, y)$. The fact that $x_n \to x$ implies that $y_n \to y$, by the definition of the cone topology. Now, since $g_n$ is an isometry that fixes $y_n$, we have

$$d(gy, y) \leq d(gy, g_n y) + d(g_n y, g_n y_n) + d(g_n y_n, y) = d(gy, g_n y) + 2d(y_n, y).$$

Since $g_n \to g$ and $y_n \to y$, the right-hand side can be made arbitrarily small. □

Lemma 2.3.41 implies that the restriction of functions from $G_c$ to $G_p$ gives an exact
sequence

\[(2.3.44) \quad 0 \to \mathcal{H}(G_c \backslash G_p)_{\text{Ad}} \to \mathcal{H}(G_c)_{\text{Ad}} \to \mathcal{H}(G_p)_{\text{Ad}} \to 0\]

of smooth $G$-modules.

**Proposition 2.3.45.** $H_n(G, \mathcal{H}(G_c \backslash G_p)_{\text{Ad}}) = 0$ for all $n \geq 1$.

The long exact sequence associated to (2.3.44) therefore degenerates, to give:

**Corollary 2.3.46.** There is a canonical isomorphism for each $n \geq 2$,

$$H_n^G(X) \cong H_n(G, \mathcal{H}(G_p)_{\text{Ad}}).$$

We will prove Proposition 2.3.45 by showing that $\mathcal{H}(G_c \backslash G_p)_{\text{Ad}}$ is projective as a $G$-module. The proof relies on some simple geometric lemmas.

**Lemma 2.3.47.** For each $g \in G_c \backslash G_p$, the fixed-point set $X^g$ is a finite, convex subcomplex of $X$. The stabilizer subgroup

$$G_{X^g} = \{ h \in G \mid hX^g = X^g \text{ (not necessarily pointwise)} \}$$

is a compact, open subgroup of $G$.

**Proof.** The fixed-point set of any automorphism of $X$ is convex, and $X^g$ is a subcomplex according to our standing assumption. The fixed-point set $\overline{X}^g$ is compact, because $\overline{X}$ is compact. Since $g \in G_c \backslash G_p$, $g$ fixes no point in $\partial X$, and so $\overline{X}^g = X^g$. Thus $X^g$ is compact, and therefore finite.

If $Y$ is any finite subcomplex of $X$, then the set of $g \in G$ which fix $Y$ pointwise is equal to the intersection of the stabilizers of the vertices of $Y$. This finite intersection of open subgroups is contained in the stabilizer subgroup $G_Y$, which is therefore open. $G_Y$ is compact, because it has a fixed point in $X$ (by the Bruhat-Tits fixed-point theorem).
Some temporary notation: if \( Y \) is a finite, connected subcomplex of \( X \), then let

\[
F_Y = \{ g \in G \mid X^g = Y \}, \quad \text{and} \quad \text{Ad}_G(F_Y) = \bigcup_{h \in G} hF_Yh^{-1}.
\]

Remark that the stabilizer \( G_Y \) is equal to the normalizer of \( N_G(F_Y) \) in \( G \).

**Lemma 2.3.48.** If \( Y \) is a finite, connected subcomplex of \( X \), then \( F_Y \) is open in \( G \) (and so, therefore, is \( \text{Ad}_G(F_Y) \)).

**Proof.** Since the fixed-point set in \( X \) of any \( g \in G \) is connected, we have

\[
F_Y = \left( \bigcap_{v \text{ a vertex in } Y} G_v \right) \setminus \left( \bigcup_{w \text{ a vertex adjacent to } Y} G_w \right).
\]

Both the intersection and the union are indexed by finite sets, and each \( G_u \) is both open and closed in \( G \), so we have expressed \( F_Y \) as the complement of a closed set in an open set. \( \square \)

**Lemma 2.3.49.** For each finite, connected subcomplex \( Y \) in \( X \), we have

\[
\mathcal{H}(\text{Ad}_G(F_Y))_{\text{Ad}} \cong \text{ind}^G_{G_Y}(\mathcal{H}(F_Y)_{\text{Ad}})
\]

as smooth \( G \)-modules.

**Proof.** Since \( G_Y \) is the normalizer, in \( G \), of the set \( F_Y \), we have

\[
\text{Ad}_G(F_Y) = \bigcup_{h \in G} hF_Yh^{-1} \cong G \times_{G_Y} F_Y,
\]

where \( G_Y \) acts on \( G \) by right multiplication, and on \( F_Y \) by conjugation. The result follows from Proposition 2.1.39. \( \square \)

**Proof of Proposition 2.3.45.** Choose a set \( S \) of representatives for the \( G \)-orbits of finite,
connected subcomplexes of $X$. By Lemma 2.3.48,

$$G_c \setminus G_p = \bigsqcup_{Y \in S} \text{Ad}_G(F_Y)$$

is a decomposition into disjoint, open, $G$-invariant subsets. We therefore have

$$\mathcal{H}(G_c \setminus G_p)_{\text{Ad}} \cong \bigoplus_{Y \in S} \mathcal{H}(\text{Ad}_G(F_Y))_{\text{Ad}}$$

as smooth $G$-modules. Lemmas 2.2.5 and 2.3.49 imply that each summand is projective. \qed

2.4 Hochschild and Cyclic Homology

This section begins by establishing the notation and terminology we shall use regarding Hochschild and cyclic homology, and recalling some key properties. We then review the computation by Higson and Nistor [HN96] and Schneider [Sch96] of the cyclic homology of $\mathcal{H}(G)$ when $G$ acts on an affine building. We finish by showing how the Hochschild and cyclic homology of $\mathcal{H}(G)$ may be defined in terms of the category Mod$(G)$; this generalizes a result of McCarthy [McC94, Proposition 2.4.3].

2.4.1 Hochschild and cyclic homology preliminaries

Our main reference for Hochschild and cyclic homology is the book of Loday [Lod92], although we shall follow Nistor [Nis93] in defining cyclic homology via precyclic modules.

**Definition 2.4.1.** (cf. [Nis93, Definition 2.2], [HN96, Section 2]) A *precyclic module* is a collection of complex vector spaces $C_0, C_1, C_2, \ldots$, together with linear maps

$$d_i : C_n \to C_{n-1} \quad (i = 0, \ldots, n), \quad \text{and} \quad t : C_n \to C_n,$$
satisfying
\[ d_i d_j = d_{j-1} d_i \quad (i < j) \]
\[
d_i t = \begin{cases} td_{i-1} : C_n \to C_{n-1} & (1 \leq i \leq n) \\ d_n : C_n \to C_{n-1} & (i = 0) \end{cases}
\]
\[ t^{n+1} = \text{id} : C_n \to C_n. \]

Given such an object \( C \), we define the following operators on \( \mathbb{C} \):
\[
\begin{align*}
\mu &= \sum_{i=0}^{n-1} (-1)^i d_i, & \beta &= \sum_{i=0}^{n} (-1)^i d_i, \\
\varepsilon &= 1 - (-1)^n t, & N &= \sum_{i=0}^{n} (-1)^n t^i.
\end{align*}
\]

Notice that \( \mu \) and \( \beta \) map \( \mathbb{C} \) to \( \mathbb{C}^{n-1} \), while \( \varepsilon \) and \( N \) map \( \mathbb{C} \) to \( \mathbb{C} \).

These operators assemble to form a periodic first-quadrant bicomplex:

The homology of the total complex of (2.4.2) is called the cyclic homology of \( C \), denoted \( HC_n(C) \). The homology of the total complex of the first two columns of (2.4.2) is called the Hochschild homology of \( C \), and is denoted \( HH_n(C) \). We will be exclusively concerned
with precyclic modules which are \( H\text{-unital} \), meaning that the \( U \) columns of (2.4.2) are contractible. In this case, \( HH_*(C) \) is the homology of the first column.

The inclusion \( I \) of the first two columns into (2.4.2) yields a short exact sequence of total complexes, and a long exact sequence in homology:

\[
\ldots \xrightarrow{I} HC_n(C) \xrightarrow{S} HC_{n-2}(C) \xrightarrow{B} HH_{n-1}(C) \xrightarrow{I} HC_{n-1}(C) \xrightarrow{S} \ldots
\]

We will refer to this long exact sequence as SBI\( _*(C) \).

One also defines the \emph{periodic cyclic homology} \( HP_*(C) \), by extending the bicomplex (2.4.2) infinitely to the left; see [Lod92, 5.1]. The two-fold periodicity in this extended bicomplex translates into a periodicity \( HP_* \cong HP_{*+2} \).

There is a natural notion of a morphism between two precyclic modules, and such a morphism induces morphisms in Hochschild, cyclic and periodic cyclic homology, compatible with the SBI sequences. Hochschild and cyclic homology are continuous with respect to direct limits. (This is not true, in general, of \( HP_* \), whose definition involves an infinite product.)

**Example 2.4.3.** Let \( A \) be an associative algebra, and set \( C_n = A^{\otimes n+1} \). Define

\[
d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} 
  a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{for } i = 0, \ldots, n-1, \\
  a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} & \text{for } i = n,
\end{cases}
\]

and

\[
t(a_0 \otimes \ldots \otimes a_n) = a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.
\]

In this way we obtain a precyclic module \( C(A) \), whose homology groups are usually denoted just by \( HH_*(A) \), \( HC_*(A) \) and \( HP_*(A) \). The construction \( A \mapsto C(A) \) is functorial with respect to algebra homomorphisms. If \( A \) is locally unital, then \( C(A) \) is \( H\)-unital.

**Example 2.4.4.** (cf. [BB92, Section 1] and [HN96, Example 2.1].) Take \( A = \mathcal{H}(G) \), the convolution algebra of \( G \). There are isomorphisms \( \mathcal{H}(G)^{\otimes n+1} \cong \mathcal{H}(G^{n+1}) \), and under these
isomorphisms the operators $d_i$ and $t$ become

$$d_i f(g_0, \ldots, g_{n-1}) = \int_G f(g_0, \ldots, g_{i-1}, h, h^{-1}g_i, \ldots, g_{n-1}) \, dh$$

for $i = 0, \ldots, n - 1$,

$$d_n f(g_0, \ldots, g_{n-1}) = \int_G f(h^{-1}g_0, g_1, \ldots, g_{n-1}, h) \, dh,$$

and

$$t f(g_0, \ldots, g_n) = f(g_1, \ldots, g_n, g_0).$$

Let $\chi \in \mathcal{H}(G)$ be a function with $\int_G \chi(g) \, dg = 1$. Then the operator

$$s f(g_0, \ldots, g_n) = \chi(g_0) f(g_0g_1, g_2, \ldots, g_n)$$

is a contracting homotopy for the $b'$ complex.

Let $\text{Cl}^\infty(G)$ denote the space of locally constant, conjugation-invariant functions on $G$. Under pointwise multiplication, $\text{Cl}^\infty(G)$ is a commutative, unital algebra. This algebra acts on the precyclic module $C(\mathcal{H}(G))$: given $F \in \text{Cl}^\infty(G)$, and $f \in \mathcal{H}(G^{n+1})$, we set

$$(Ff)(g_0, \ldots, g_n) = F(g_0 \cdots g_n) f(g_0, \ldots, g_n).$$

The action commutes with the operators $d_i$ and with $t$, and so induces an action on Hochschild and on cyclic homology.

There is another way of understanding the Hochschild homology of an algebra, which will be important in our study of Hecke algebras. Let $A$ be a locally unital algebra, and consider the locally unital algebra $A \otimes_{\mathbb{C}} A^{\text{opp}}$, where $A^{\text{opp}}$ is $A$ with the opposite multiplication. The category $\text{Mod}(A \otimes A^{\text{opp}})$ of nondegenerate $A \otimes A^{\text{opp}}$-modules is an abelian category with enough projectives (Lemma 2.2.1). Note that a nondegenerate $A \otimes A^{\text{opp}}$-module is the same thing as an $A$-$A$-bimodule whose underlying left and right modules are both nondegenerate; we will accordingly refer to the objects in $\text{Mod}(A \otimes A^{\text{opp}})$ as nondegenerate $A$-bimodules.
The algebra $A$ acts on itself by left and right multiplications, so $A$ may be viewed as an object in $\text{Mod}(A \otimes A^{\text{opp}})$. Exchanging left and right, $A$ may also be viewed as a right module over $A \otimes A^{\text{opp}}$.

**Definition 2.4.5.** The left-derived functor of the functor

$$V \mapsto A \otimes_{A^{\text{opp}}} A^{\text{opp}} V$$

on $\text{Mod}(A \otimes A^{\text{opp}})$ is called the Hochschild homology of $A$ with coefficients in $V$, and denoted $H_{a}(A, V)$. In other words,

$$H_{a}(A, V) := \text{Tor}_{a}^{A \otimes A^{\text{opp}}}(A, V).$$

For any $V$, there is a standard complex computing $H_{a}(A, V)$: one sets

$$C_{n}(A, V) = V \otimes_{C} A^{\otimes n},$$

and defines the differential $b$ as in Example 2.4.3. See [Lod92, Proposition 1.1.13] for the proof that $H_{a}(A, V)$ is the homology of this complex. Putting $V = A$, one finds that

$$HH_{a}(A) \cong H_{a}(A, A).$$

### 2.4.2 Hochschild homology for $\mathcal{H}(G)$

When $A = \mathcal{H}(G)$ is the Hecke algebra of a locally compact, totally disconnected group, the Hochschild homology $H_{a}(\mathcal{H}(G), V)$ may be described in terms of smooth homology. This fact is due to Eilenberg and Mac Lane for discrete groups ([EM47, Section 5], cf. [Mac63, Theorem X.5.5]), and to Blanc and Brylinski for unimodular totally disconnected groups [BB92, Proposition 2.3]. Nistor showed how to adjust the result for non-unimodular groups in [Nis01, Lemma 3.1]; here we shall stick with the unimodular case.

By Proposition 2.1.29, a nondegenerate $\mathcal{H}(G)$-bimodule is the same thing as a vector space $V$ equipped with smooth representations of $G$ on the left and on the right. Restricting
to the diagonal in $G \times G$, we consider $V$ as a smooth $G$-module under the \textit{adjoint action}:

$$\text{Ad}_g(v) = gvg^{-1}.$$  

This smooth representation will be denoted $V_{\text{Ad}}$. The procedure $V \mapsto V_{\text{Ad}}$ is a functor from $\text{Mod}(\mathcal{H}(G) \otimes \mathcal{H}(G)^{\text{opp}})$ to $\text{Mod}(G)$.

**Proposition 2.4.6.** [BB92, Proposition 2.3] Let $G$ be a unimodular, locally compact, totally disconnected group. There is a natural isomorphism

$$H_*(\mathcal{H}(G), V) \cong H_*(G, V_{\text{Ad}}).$$

**Proof.** This is proved in [BB92] by showing that the standard complexes computing each side are in fact isomorphic. \hfill $\square$

**Remark 2.4.7.** Notice that every smooth left $G$-module $V$ may be turned into a $G$-bimodule $V_b$, by letting $G$ act trivially on the right. One then has $V \cong (V_b)_{\text{Ad}}$, and so Proposition 2.4.6 implies that smooth homology for $G$ and Hochschild homology for $\mathcal{H}(G)$ are, in a sense, interchangeable.

**Example 2.4.8.** The following results are due to Burghelea [Bur85]; cf. [Lod92, 7.4].

Let $G$ be a discrete group. The Hochschild homology $HH_*(\mathcal{H}(G))$ is isomorphic, by Proposition 2.4.6, to the group homology $H_*(G, \mathcal{H}(G)_{\text{Ad}})$. The $G$-module $\mathcal{H}(G)_{\text{Ad}}$ decomposes as a direct sum over the conjugacy classes $[g]$ in $G$:

$$H_*(G, \mathcal{H}(G)_{\text{Ad}}) \cong \bigoplus_{[g]} H_*(G, \mathcal{H}([g])_{\text{Ad}}).$$

Identifying the class $[g]$ with the quotient $N_g/G_g$ of the normalizer of $g$ by the centralizer of $g$, one finds that $\mathcal{H}([g])_{\text{Ad}} \cong \text{ind}_{G_g}^{N_g} \mathbb{C}$, and so $H_*(G, \mathcal{H}([g])_{\text{Ad}}) \cong H_*(G_g, \mathbb{C})$ by the Shapiro lemma (see [Bro94, Proposition 6.2]). Thus

$$HH_*(\mathcal{H}(G)) \cong \bigoplus_{[g]} H_*(G_g, \mathbb{C}).$$
The cyclic homology $HC_\ast(\mathcal{H}(G))$ likewise decomposes over the conjugacy classes in $G$, and one is led to distinguish the elements of finite order in $G$ from those of infinite order. We refer the reader to [Lod92, Theorem 7.4.10] for the computation and the result.

**Example 2.4.9.** Let $G$ be a locally compact, totally disconnected group. Proposition 2.4.6 implies that

$$HH_\ast(\mathcal{H}(G)) \cong H_\ast(\mathcal{H}(G), \mathcal{H}(G)) \cong H_\ast(G, \mathcal{H}(G)_{Ad}).$$

We noted above (Example 2.4.4) that the algebra $Cl^\infty(G)$ acts on $HH_\ast(\mathcal{H}(G))$. This algebra also acts, by pointwise multiplication, on the $G$-module $\mathcal{H}(G)_{Ad}$, and therefore on $H_\ast(G, \mathcal{H}(G)_{Ad})$. The isomorphism $HH_\ast(\mathcal{H}(G)) \rightarrow H_\ast(G, \mathcal{H}(G)_{Ad})$ is $Cl^\infty(G)$-linear [BB92, Proposition 2.4].

We have so far seen that the isomorphism $HH_\ast(\mathcal{H}(G)) \xrightarrow{\cong} H_\ast(G, \mathcal{H}(G)_{Ad})$ is natural with respect to $G$-bimodule endomorphisms of $\mathcal{H}(G)$ (Proposition 2.4.6), and also with respect to the action of $Cl^\infty(G)$ (Example 2.4.9). We shall need a third kind of functoriality, which we now explain.

**Definition 2.4.10.** Let $\alpha : G \rightarrow G$ be a group automorphism. Assume for simplicity that $\alpha$ preserves the Haar measure on $G$.

(1) The map

$$\alpha : \mathcal{H}(G) \rightarrow \mathcal{H}(G) \quad \alpha(f) \coloneqq f \circ \alpha^{-1},$$

being an algebra automorphism, induces an automorphism $\alpha_\ast$ of $HH_\ast(\mathcal{H}(G))$.

(2) For each smooth $G$-module $V$, let $V_\alpha$ be the $G$-module whose underlying vector space is equal to $V$, but with the $G$-action twisted by $\alpha$:

$$g \cdot V_\alpha \coloneqq \alpha(g) \cdot V.$$

The identity $V \rightarrow V_\alpha$ induces an isomorphism on coinvariants, $V_\alpha \xrightarrow{\cong} (V_\alpha)_G$, and so there is a natural isomorphism of derived functors $H_\ast(G, V) \xrightarrow{\cong} H_\ast(G, V_\alpha)$. Taking
\[ V = \mathcal{H}(G)_{\text{Ad}}, \text{the map } \alpha : \mathcal{H}(G)_{\text{Ad}} \to (\mathcal{H}(G)_{\text{Ad}})_{\alpha} \text{ defined as in (2.4.11) is } G\text{-equivariant,} \]

and we define an automorphism \( \alpha \) of \( H_\#(G, \mathcal{H}(G)_{\text{Ad}}) \) as the composition

\[
H_\#(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{\alpha} H_\#(G, (\mathcal{H}(G)_{\text{Ad}})_\alpha) \xrightarrow{\cong} H_\#(G, \mathcal{H}(G)_{\text{Ad}}).
\]

(See [Bro94, III.8] for more details on this construction.)

**Lemma 2.4.12.** The diagram

\[
\begin{array}{ccc}
HH_\#(\mathcal{H}(G)) & \xrightarrow{\alpha} & HH_\#(\mathcal{H}(G)) \\
\downarrow \cong & & \downarrow \cong \\
H_\#(G, \mathcal{H}(G)_{\text{Ad}}) & \xrightarrow{\alpha} & H_\#(G, \mathcal{H}(G)_{\text{Ad}})
\end{array}
\]

is commutative.

**Proof.** The map \( \alpha : H_\#(G, \mathcal{H}(G)_{\text{Ad}}) \to H_\#(G, \mathcal{H}(G)_{\text{Ad}}) \) may be described explicitly on the degree-\( n \) component \( B_n(G, \mathcal{H}(G)_{\text{Ad}}) \) of the standard resolution, as follows:

\[
\alpha : \mathcal{H}(G^{n+2}) \to \mathcal{H}(G^{n+2}), \quad \alpha(f)(g_0, \ldots, g_{n+1}) := f(\alpha^{-1}(g_0), \ldots, \alpha^{-1}(g_{n+1})).
\]

Comparing this formula with the explicit isomorphism \( HH_\#(\mathcal{H}(G)) \xrightarrow{\cong} H_\#(G, \mathcal{H}(G)_{\text{Ad}}) \) given in [BB92, Proposition 2.3], the commutativity of the diagram becomes clear. \( \square \)

**2.4.3 Cyclic homology for groups acting on buildings**

Return to the hypotheses (GAB1–3) of Section 2.3: \( G \) is a locally compact, totally disconnected group, acting by automorphisms of a locally finite, oriented affine building \( X \), such that the isotropy group \( G_\sigma \) of each polysimplex in \( X \) is compact and open in \( G \), and such that each fixed-point set \( X^g \) is a subcomplex of \( X \).

The cyclic homology of the Hecke algebra \( \mathcal{H}(G) \) was computed by Higson and Nistor [HN96], and by Schneider [Sch96], using the geometry of \( X \). In this section, we recall the results of those computations, and prove a complement to one of those results (Proposition 2.4.15).
The sets \(G_c\) and \(G_{nc}\), of elliptic and hyperbolic elements (respectively), are open, closed, and conjugation-invariant in \(G\) (Corollary 2.3.19). So the characteristic functions of these sets, \(1_{G_c}\) and \(1_{G_{nc}}\), define complementary idempotents in the algebra \(\text{Cl}^p(G)\) of locally constant class functions. As noted in Example 2.4.4, this algebra acts on the precyclic module \(C(\mathcal{H}(G))\), and so we have a decomposition

\[
C(\mathcal{H}(G)) = 1_{G_c} C(\mathcal{H}(G)) \oplus 1_{G_{nc}} C(\mathcal{H}(G))
\]

doing precyclic modules. This decomposition passes to Hochschild and to cyclic homology, where we will use the notation

\[
HH_*(\mathcal{H}(G)) = HH_*(\mathcal{H}(G))_c \oplus HH_*(\mathcal{H}(G))_{nc}
\]

and

\[
HC_*(\mathcal{H}(G)) = HC_*(\mathcal{H}(G))_c \oplus HC_*(\mathcal{H}(G))_{nc}.
\]

Here is a summary of the main results of [HN96] and [Sch96].

**Theorem 2.4.13.** [HN96], [Sch96]

1. \(HH_*(\mathcal{H}(G))_c \cong H^G_*(X)\).
2. The operator \(B : HC_*(\mathcal{H}(G))_c \rightarrow HH_{*+1}(\mathcal{H}(G))_c\) is equal to zero.
3. The operator \(S : HC_*(\mathcal{H}(G))_{nc} \rightarrow HC_{*+2}(\mathcal{H}(G))_{nc}\) is equal to zero.
4. \(HC_*(\mathcal{H}(G))_c \cong H^G_*(X) \oplus H^G_{*+2}(X) \oplus \cdots \oplus H^G_{*+1}(X)\).
5. \(HP_*(\mathcal{H}(G))_c \cong \bigoplus_{n \in \mathbb{Z}} H^G_{*+2n}(X)\), and \(HP_*(\mathcal{H}(G))_{nc} = 0\).

Note that (4) and (5) are immediate consequences of (1), (2) and (3). Statement (3) is the most substantial part of the theorem; it is proved using the structure of \(\text{min}(g)\) for hyperbolic \(g\), as described in Theorem 2.3.7. The vanishing of \(S\) on \(HC_*(\mathcal{H}(G))_{nc}\) implies the abstract Selberg principle of Blanc and Brylinski [BB92, Theorem 4.10].

Theorem 2.4.13 may be repackaged in the following form, which will be useful later:
Corollary 2.4.14. \( H^G_s(X) \) is isomorphic to the cohomology of the cochain complex

\[
\ldots \xrightarrow{B^\circ I} HH_{n-1}(\mathcal{H}(G)) \xrightarrow{B^\circ I} HH_n(\mathcal{H}(G)) \xrightarrow{B^\circ I} HH_{n+1}(\mathcal{H}(G)) \xrightarrow{B^\circ I} \ldots.
\]

Proof. Part (2) of Theorem 2.4.13 implies that the differential \( B \circ I \) is zero on \( HH_s(\mathcal{H}(G))_c \).

On \( HH_s(\mathcal{H}(G))_{nc} \), part (3) of the theorem implies that \( I \) is surjective and \( B \) injective, and so

\[
\ker(B \circ I) = \ker I = \text{image } B = \text{image}(B \circ I).
\]

Part (1) of Theorem 2.4.13 follows from Corollary 2.3.34 and Proposition 2.4.6. There is an analog of this result for the non-compact part, under an additional hypothesis:

**Proposition 2.4.15.** Suppose that the action of \( G \) on \( X \) satisfies the hypotheses \((GAB1–3+)\): i.e., in addition to the standing hypotheses on \( G \) and \( X \), assume that each \( \min(g) \subset X \) is a subcomplex of \( X \). Then

\[
HH_s(\mathcal{H}(G))_{nc} \cong H^G_s(X, G^{++})
\]

(notation of Proposition 2.3.39).

Proof. Proposition 2.4.6 gives \( HH_s(\mathcal{H}(G))_{nc} \cong H_s(G, \mathcal{H}(G_{nc})_{Ad}) \). The additional hypothesis on the \( G \)-action allows us to apply Proposition 2.3.39 to compute the homology of \( \mathcal{H}(G_{nc})_{Ad} \).

We conclude this section by remarking that Theorem 2.4.13 is closely related to the Baum-Connes conjecture. Here we shall be rather sketchy; details can be found in Solleveld’s paper [Sol09, 3.4].

Suppose that \( G \) is a reductive \( p \)-adic group, such as \( \text{GL}_n(F) \) or \( \text{SL}_n(F) \), and let \( X \) be the (extended) Bruhat-Tits building of \( G \). Then \( G \) and \( X \) satisfy the hypotheses of Theorem 2.4.13 (possibly after passing to a barycentric subdivision of \( X \)). There is an assembly map

\[
\mu : K^G_s(X) \to K_s(C^*_r(G)),
\]
from the equivariant $K$-homology of $X$ to the analytic $K$-theory of the reduced $C^*$-algebra of $G$, and the Baum-Connes conjecture for $G$ is the assertion that $\mu$ is an isomorphism [BCH94, Section 6]. The conjecture has been verified in this case: for $GL_n(F)$ by Baum, Higson and Plymen [BHP97], and for general reductive $p$-adic groups by Lafforgue [Laf02].

There is a Chern character

$$K_*^G(X) \to \bigoplus_{n \in \mathbb{Z}} H^G_{*+2n}(X),$$

which becomes an isomorphism upon tensoring over $\mathbb{Z}$ with $\mathbb{C}$; this was proved by Baum, Block and Higson in their unpublished paper [BBH]. Solleveld shows in [Sol09] that this Chern character may also be recovered from results of Voigt ([Voi08], [Voi07], [Voi09]) and of Baum and Schneider [BS02].

On the other hand, it is shown in [Sol09] that there is a Chern character

$$K_*(C^*_r(G)) \to HP_*(\mathcal{H}(G))$$

which likewise becomes an isomorphism over $\mathbb{C}$.

**Theorem 2.4.16.** [Sol09, Theorem 3.7] Suppose that $G$ is a reductive $p$-adic group, with extended Bruhat-Tits building $X$. The diagram

$$\begin{array}{ccc}
K_*^G(X) & \xrightarrow{\mu} & K_*(C^*_r(G)) \\
\bigoplus_{n \in \mathbb{Z}} H^G_{*+2n}(X) & \xrightarrow{\text{Theorem 2.4.13}} & HP_*(\mathcal{H}(G))
\end{array}$$

is commutative.

\[ \square \]

### 2.4.4 Cyclic homology via projective modules

In the previous section, we saw that for a group $G$ acting on an affine building, the Hochschild and cyclic homology of $\mathcal{H}(G)$ can be described via chamber homology. The main theme of this thesis is to compare this geometric computation with other, representation-theoretic pictures of these homology groups. This section lays the foundation for the
representation-theoretic side of the problem, by describing the Hochschild and cyclic homology groups of \( \mathcal{H}(G) \) in terms of the category \( \text{Mod}(G) \). We do this by extending a theorem of McCarthy [McC94, Proposition 2.4.3] from unital to locally unital algebras.

Let \( A \) be a locally unital algebra, and let \( \mathcal{P}_A \) denote the full subcategory of \( \text{Mod}(A) \) comprising the finitely generated, projective, nondegenerate modules. If \( P \) is an object in \( \mathcal{P}_A \), let \([P]\) denote its isomorphism class, and let \([\mathcal{P}_A]\) denote the set of all such classes. We choose a representative of each isomorphism class.

**Definition 2.4.17.** Define a precyclic module \( C(\mathcal{P}_A) \) as follows. \( C_n \) is the vector space

\[
\bigoplus_{([P_0],...,[P_n]) \in [\mathcal{P}_A]^{n+1}} \text{Hom}_A(P_0, P_1) \otimes_\mathbb{C} \text{Hom}_A(P_1, P_2) \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} \text{Hom}_A(P_n, P_0).
\]

The face maps are defined as

\[
d_i(T_0 \otimes \cdots \otimes T_n) = \begin{cases} 
T_0 \otimes \cdots \otimes T_{i+1} \circ T_i \otimes \cdots \otimes T_n & \text{if } 0 \leq i < n, \\
T_0 \circ T_n \otimes T_1 \otimes \cdots \otimes T_{n-1} & \text{if } i = n.
\end{cases}
\]

The cyclic operator is given by

\[
t(T_0 \otimes \cdots \otimes T_n) = T_n \otimes T_0 \otimes \cdots \otimes T_{n-1}.
\]

(The slight difference between this definition and the one in [McC94] comes from our preference for working with left modules.)

It is a straightforward matter to check that the axioms of a precyclic module are satisfied by \( C(\mathcal{P}_A) \). The fact that each \( \text{End}_A(P_i) \) is a unital algebra implies that \( C(\mathcal{P}_A) \) is \( H \)-unital. The homology groups of \( C(\mathcal{P}_A) \) will be denoted \( HH_*(\mathcal{P}_A) \) and \( HC_*(\mathcal{P}_A) \).

The passage from \( \mathcal{P}_A \) to \( C(\mathcal{P}_A) \) is natural, in the following sense: if \( B \) is another locally unital algebra, and if \( F : \mathcal{P}_A \to \mathcal{P}_B \) is a functor that is \( \mathbb{C} \)-linear on Hom sets, then there is an induced morphism of precyclic modules \( C(\mathcal{P}_A) \to C(\mathcal{P}_B) \).

**Remark 2.4.18.** One can, in fact, replace \( \mathcal{P}_A \) by any category in which

- The collection of isomorphism classes of objects forms a set, and
Each Hom set is a vector space over \( \mathbb{C} \), and composition is bilinear.

The above construction can then be seen as a functor, from the category of all such categories (with morphisms given by functors that are linear on morphisms) to the category of precyclic modules.

Suppose for a moment that \( A \) is unital. Then \( A \) can itself be considered as an object of \( \mathcal{P}_A \), and the map

\[
A \to \text{Hom}_{\mathcal{P}_A}(A, A), \quad a \mapsto (b \mapsto ba)
\]

induces an injective morphism of precyclic modules

\[
C(A) \hookrightarrow C(\mathcal{P}_A).
\]

(See Example 2.4.3 for the definition of \( C(\cdot) \).)

**Proposition 2.4.19.** [McC94, Proposition 2.4.3] Let \( A \) be a unital algebra. The inclusion \( C(A) \hookrightarrow C(\mathcal{P}_A) \) induces isomorphisms on Hochschild and cyclic homology.

**Proof.** We will not reproduce the proof of Proposition 2.4.19 here, but it will be useful to have an explicit formula for a homotopy-inverse to the inclusion \( C(A) \hookrightarrow C(\mathcal{P}_A) \).

For each \( P \in \mathcal{P}_A \), choose “coordinates” \( \alpha_i^P \in \text{Hom}_A(P, A) \) and \( \beta_i^P \in \text{Hom}_A(A, P) \) (for \( 1 \leq i \leq n_P \)) such that \( \sum_{i=1}^{n_P} \beta_i^P \alpha_i^P = \text{id}_P \). The existence of such \( \alpha \) and \( \beta \) is guaranteed by virtue of \( P \) being a direct-summand of a free module. Define

\[
\text{Trace} : \text{Hom}_A(P_0, P_1) \otimes_{\mathbb{C}} \text{Hom}_A(P_1, P_2) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \text{Hom}_A(P_n, P_0) \to A^{\otimes (n+1)}
\]

\[
T_0 \otimes T_1 \otimes \cdots \otimes T_n \mapsto \sum_i \alpha_{i_1}^{P_1} T_0 \beta_{i_0}^{P_0}(1_A) \otimes \alpha_{i_2}^{P_2} T_1 \beta_{i_1}^{P_1}(1_A) \otimes \cdots \otimes \alpha_{i_0}^{P_0} T_n \beta_{i_n}^{P_n}(1_A),
\]

the sum being taken over all \( i = (i_0, \ldots, i_n) \) with \( 1 \leq i_j \leq n_{P_j} \), and \( 1_A \) denoting the unit of \( A \). This gives a map of precyclic modules \( C(\mathcal{P}_A) \to C(A) \) that is left-inverse to the inclusion \( C(A) \hookrightarrow C(\mathcal{P}_A) \). McCarthy proves that it is also right-inverse to this inclusion, up to homotopy.

**Example 2.4.21.** Looking at \( HH_0(A) = A/[A, A] \), one recovers the following result of
Stallings (cited in [Bas76, 2.5]): Consider the vector space $V$ with basis consisting of the pairs $(P, T)$, where $P \in \mathcal{P}_A$ and $T \in \text{End}_A(P)$. Then $A/[A, A]$ is isomorphic to the quotient of $V$ by the relations

1. $(P, T) + (P, S) = (P, T + S)$,
2. $(P, \lambda T) = \lambda(P, T)$, and
3. $(P, ST) = (Q, TS)$ for $S \in \text{Hom}_A(Q, P)$ and $T \in \text{Hom}_A(P, Q)$.

The zeroth component of the Chern character $K_0(A) \to A/[A, A]$, also known as the Hattori-Stallings rank, has an easy definition in this picture: $P \mapsto (P, \text{id})$.

Now suppose that $A$ is only locally unital. We have $A \cong \varinjlim eAe$, the limit being taken over the set of idempotents $e \in A$ ordered by the relation $e \leq f \iff ef = fe = e$. The functor $B \mapsto C(B)$ commutes with direct limits, as does the formation of the cyclic bicomplex and the passage to Hochschild and cyclic homology; thus $HH_*(A) \cong \varinjlim HH_*(eAe)$, and likewise for cyclic homology.

Turning now to $A$-modules, we would like to approximate the category $\mathcal{P}_A$ by the categories $\mathcal{P}_{A,e}$. For each idempotent $e \in A$, let $\text{Mod}(A, e)$ denote the full subcategory of $\text{Mod}(A)$ consisting of those modules $V$ satisfying $V = AeV$. It is not necessarily true that $\text{Mod}(A, e)$ is equivalent to $\text{Mod}(eAe)$ (see [BK98, Proposition 3.3], which is valid for all locally unital algebras). The analogous statement about finitely generated projective modules is, however, true for all $e$:

**Lemma 2.4.22.** For each idempotent $e \in A$, let $\mathcal{P}_{A,e}$ be the full subcategory of $\text{Mod}(A, e)$ with objects $\text{Mod}(A, e) \cap \mathcal{P}_A$. The functors

$$\text{Mod}(A, e) \xrightarrow{r_e} \text{Mod}(eAe), \quad r_e(V) = eV$$

and

$$\text{Mod}(eAe) \xrightarrow{i_e} \text{Mod}(A, e), \quad i_e(W) = Ae \otimes_{eAe} W$$

restrict to mutually inverse equivalences between the categories $\mathcal{P}_{A,e}$ and $\mathcal{P}_{eAe}$. 

Proof. Since $eAe$ is a unital algebra, a module $W \in \text{Mod}(eAe)$ lies in $\mathcal{P}_{eAe}$ if and only if it is isomorphic to a direct-summand of $(eAe)^n$, for some $n$. Similarly, a module $V \in \text{Mod}(A,e)$ lies in the category $\mathcal{P}_{A,e}$ if and only if it is isomorphic to a direct-summand of $(Ae)^n$, for some $n$. Both $r_e$ and $i_e$ commute with direct sums, and there are natural isomorphisms

$$i_e r_e(Ae) = Ae \otimes_{eAe} eAe \cong Ae$$

and

$$r_e i_e(eAe) = e(Ae \otimes_{eAe} eAe) \cong eAe. \qed$$

Lemma 2.4.22 allows us to extend Proposition 2.4.19 to locally unital algebras. That this could be done in degree zero was observed by Dat [Dat00, 1.3].

**Proposition 2.4.23.** Let $A$ be a locally unital algebra. There are canonical isomorphisms

$$HH_* (A) \cong HH_* (\mathcal{P}_A) \quad \text{and} \quad HC_* (A) \cong HC_* (\mathcal{P}_A),$$

compatible with the SBI exact sequences.

**Corollary 2.4.24.** Each functor $\mathcal{P}_A \to \mathcal{P}_B$ that is $C$-linear on Hom sets induces maps $HH_* (A) \to HH_* (B)$ and $HC_* (A) \to HC_* (B)$, compatible with the SBI sequences. \qed

**Example 2.4.25.** Let $G = \text{SL}_2(F)$, $F$ a $p$-adic field, and let $M \subset G$ be the diagonal subgroup. The functor $i_M^G : \text{Mod}(M) \to \text{Mod}(G)$ of parabolic induction, defined in Example 2.1.42, restricts to a functor $\mathcal{P}_M \to \mathcal{P}_G$. (This is a consequence of Bernstein’s Second Adjoint theorem; see Theorem 3.3.5.) Therefore $i_M^G$ induces a map in homology, $i_M^G : HH_* (\mathcal{P}_M) \to HH_* (\mathcal{P}_G)$. This example, and others like it, are the subject of Chapters 4 and 5.

**Proof of Proposition 2.4.23.** For each idempotent $e$ in $A$, we have maps of precyclic modules

$$C(\mathcal{P}_{A,e}) \xrightarrow{i_e} C(\mathcal{P}_{eAe}) \xrightarrow{\text{Trace}} C(eAe),$$

where Trace is defined by choosing coordinates for each object in $\mathcal{P}_{eAe}$, as in (2.4.20). Lemma 2.4.22 implies that $r_e$ is an isomorphism. Proposition 2.4.19 implies that Trace...
induces isomorphisms in Hochschild and cyclic homology, and that these induced maps do not depend on the choice of coordinates (since their inverses are defined by canonical maps of precyclic modules).

Now let \( e \) and \( f \) be idempotents with \( e \leq f \). The inclusion of algebras \( eAe \subseteq fAf \) gives an inclusion of precyclic modules \( C(eAe) \subseteq C(fAf) \). On the other hand, \( \mathcal{P}_{A,e} \) embeds as a subcategory of \( \mathcal{P}_{A,f} \), giving \( C(\mathcal{P}_{A,e}) \subseteq C(\mathcal{P}_{A,f}) \). Choose coordinates for the objects in \( \mathcal{P}_{eAe} \) and \( \mathcal{P}_{fAf} \), such that the choices are compatible on the subcategory \( r_f i_e (\mathcal{P}_{eAe}) \subseteq \mathcal{P}_{fAf} \). We obtain a commuting diagram

\[
\begin{array}{ccc}
C(\mathcal{P}_{A,e}) & \xrightarrow{r_e} & C(\mathcal{P}_{eAe}) \\
\downarrow \leq & & \downarrow r_f i_e \\
C(\mathcal{P}_{A,f}) & \xrightarrow{r_f} & C(\mathcal{P}_{fAf})
\end{array}
\]

in which the horizontal arrows induce isomorphisms in Hochschild and cyclic homology; these induced maps do not depend on the choices of coordinates.

Now take the direct limit over the set of idempotents in \( A \). We have \( \varinjlim C(eAe) \cong C(A) \), because \( A \) is locally unital. On the other hand, every object in \( \mathcal{P}_A \) is finitely generated and nondegenerate, and so belongs to \( \mathcal{P}_{A,e} \) for some \( e \). It follows that \( \varinjlim C(\mathcal{P}_{A,e}) \cong C(\mathcal{P}_A) \). The commuting diagram (2.4.26) therefore induces a morphism of precyclic modules \( C(A) \to C(\mathcal{P}_A) \), which becomes an isomorphism on Hochschild and cyclic homology.

**Example 2.4.27.** Let \( G \) be a totally disconnected group, and let \( K \subset G \) be a compact open subgroup. If \( \rho : K \to \text{GL}(W) \) is an irreducible representation of \( K \), then the representation \( \text{ind}_K^G \rho \) is finitely generated and projective, and lies in \( \mathcal{P}_{G,e} \). Coordinates

\[
\begin{array}{ccc}
\text{ind}_K^G \rho & \xrightarrow{\alpha_i} & \mathcal{H}(G) e_\rho \\
\beta_i & & \\
\end{array}
\]

may be defined as follows. Let \( \{w_1, \ldots, w_n\} \) be a \( \mathbb{C} \)-basis for \( W \), and let \( \{\tilde{w}_1, \ldots, \tilde{w}_n\} \subset \tilde{W} \)
be a dual basis. Routine calculations show that
\[(\alpha_i f)(g) = \langle \tilde{w}_i, f(g^{-1}) \rangle \quad \text{and} \quad (\beta_i f)(g) = \frac{1}{\text{vol}(K)} \int_K f(g^{-1}k) k w_i \, dk\]
are $G$-equivariant maps with $\sum_{i=1}^n \beta_i\alpha_i = \text{id}_{\text{ind}_K^G \rho}$. Using these coordinates, one finds for example that the isomorphism
\[HH_0(\mathcal{P}_G) \rightarrow HH_0(\mathcal{H}(G))\]
sends the class of $(\text{ind}_K^G \rho, \text{id})$ to the class of
\[\sum_i \alpha_i\beta_i(e_\rho) = \frac{1}{\dim(\rho)} e_\rho \in \mathcal{H}(G)\].

We conclude this section with an observation of Dat [Dat00, 1.7]. Let $G$ be a totally disconnected, locally compact group, and assume for simplicity that $G$ is unimodular. To each admissible representation $\pi : G \rightarrow \text{GL}(V)$ is associated the character
\[\text{ch}_\pi : \mathcal{H}(G) \rightarrow \mathbb{C}, \quad \text{ch}_\pi(f) = \text{Trace}(f : V \rightarrow V).\]
Since the trace vanishes on commutators, $\text{ch}_\pi$ descends to a linear functional on $HH_0(\mathcal{H}(G))$. Using Proposition 2.4.23, we may view $\text{ch}_\pi$ as a linear functional on $HH_0(\mathcal{P}_G)$. For each finitely generated projective $G$-module $P$, the space $\text{Hom}_G(P, V)$ is a right module over $\text{End}_G(P)$ (via composition).

**Lemma 2.4.28.** [Dat00, 1.7] For each $P \in \mathcal{P}_G$ and $T \in \text{End}_G(P)$, one has
\[\text{ch}_\pi(P, T) = \text{Trace} \left( T \big|_{\text{Hom}_G(P, V)} \right),\]
the trace of $T$ as a linear operator on $\text{Hom}_G(P, V)$.

In particular, $\text{ch}_\pi(P, \text{id}) = \dim_\mathbb{C} \text{Hom}_G(P, V)$.

**Proof.** By Proposition 2.4.23, every class in $HH_0(\mathcal{P}_G)$ has a representative $(P, T)$ with $P = \mathcal{H}(G)e$ for some idempotent $e$, and $T \in e\mathcal{H}(G)e$ acting by convolution on the right.
The map
\[ \text{Hom}_G(\mathcal{H}(G) e, V) \to eV \quad S \mapsto S(e) \]
is an isomorphism, under which the induced action of \( T \) on \( eV \) is the usual action by convolution. We thus have
\[
\text{ch}_\pi(P, T) = \text{Trace}(T : V \to V) = \text{Trace}(T : eV \to eV) = \text{Trace} \left( T|_{\text{Hom}_G(P, V)} \right).
\]
\[ \square \]
Chapter 3

The Bernstein Decomposition and Chamber Homology

Let $F$ be a $p$-adic field, and consider $G = \text{GL}_n(F)$ or $\text{SL}_n(F)$, acting on its Bruhat-Tits building $X$. We saw in the previous chapter that the Hochschild homology $HH(\mathcal{H}(G))$ of the Hecke algebra of $G$ decomposes as a direct sum,

$$HH(\mathcal{H}(G)) = HH(\mathcal{H}(G))_c \oplus HH(\mathcal{H}(G))_{nc},$$

coming from the dichotomy between elliptic and hyperbolic isometries of $X$. The “compact part” $HH(\mathcal{H}(G))_c$ was identified with the chamber homology $H^G_*(X)$, which is defined in terms of the representations of compact, open subgroups of $G$, and the combinatorics of $X$.

Representation theory provides a second decomposition of $HH(\mathcal{H}(G))$. One way to describe this is via the isomorphism $HH(\mathcal{H}(G)) \cong HH(\mathcal{P}_G)$. Work of Bernstein and others has shown that every smooth representation of $G$ decomposes canonically as a direct sum, with each summand being associated to a family of representations of one of the block-diagonal subgroups of $G$. This applies in particular to the finitely generated and projective representations of $G$, giving a decomposition of $HH(\mathcal{P}_G)$ and therefore of $HH(\mathcal{H}(G))$:

$$HH(\mathcal{H}(G)) \cong \bigoplus_{\mathfrak{p} \in \mathfrak{P}(G)} HH(\mathcal{H}(G))_{\mathfrak{p}}.$$
This chapter studies the relationship between the decompositions (3.0.1) and (3.0.2), for $G = \text{SL}_n(F)$, $\text{GL}_n(F)$, and related groups. (The results apply in the general context of reductive $p$-adic groups, but we stick to the special and general linear groups in order to simplify the exposition. See [Ren10, Chapitre V], and the references therein, for the general theory of reductive $p$-adic groups.)

Following earlier work of Baum, Higson, and Plymen [BHP00], we show that the results of Higson, Nistor, and Schneider on the cyclic homology of $H_p^G$ ([HN96], [Sch96]; cf. Theorem 2.4.13) imply a degree of compatibility between the decompositions (3.0.1) and (3.0.2), leading to a decomposition

\begin{equation}
H_p^G(X) \cong \bigoplus_{\mathfrak{a} \in \mathcal{H}(G)} H_p^G(X)_{\mathfrak{a}}
\end{equation}

in chamber homology. We then consider the problem, first raised in [BHP00], of computing the direct summands $H_p^G(X)_{\mathfrak{a}}$ in terms of representations of compact open subgroups. This is done using the isomorphism $H_p^G(X) \cong H_*(G, \mathcal{H}(G)_c, \text{Ad})$, which allows us to apply standard tools from homological algebra. We construct a chain complex, uniquely defined up to a homotopy, to compute each of the components $H_p^G(X)_{\mathfrak{a}}$ (Theorem 3.2.19).

To illustrate the result, we consider several concrete examples: $\text{GL}_1(F)$ (Section 3.2.6), supercuspidal representations of $\text{SL}_n(F)$ (Section 3.2.7) and of $\text{GL}_2(F)$ (Section 3.2.8), and generic principal-series components of $\text{SL}_2(F)$ (Section 3.4). The results of these computations support the conjectures made by Baum, Higson and Plymen in [BHP00]; see the extended Remark 3.2.21 for the relationship between our results and those conjectures.

The decomposition (3.0.2) uses two essential ingredients:

- **Supercuspidal representations**, which are representations of $G$ that behave in some respects like representations of a compact group; and

- The **Jacquet functors**, which reduce the study of general representations of $G$ to that of supercuspidal representations of subgroups of $G$.

The methods developed in this chapter give a complete understanding of the supercuspidal
components of chamber homology for $\text{SL}_n(F)$ and $\text{GL}_2(F)$. Similar methods would likely apply to supercuspidals of other groups (contingent on the well-known and widely verified conjecture that every supercuspidal representation is compactly induced from a compact-mod-center subgroup). The problem of incorporating the Jacquet functors into chamber homology is taken up in Chapters 4 and 5.

The plan for this chapter is as follows. In Section 3.1, we recall the definition of the Bernstein center $\mathfrak{Z}(G)$, and give explicit constructions of some elements of $\mathfrak{Z}(G)$ for use in later sections. Section 3.2 studies the action of $\mathfrak{Z}(G)$ on chamber homology, including three examples of cuspidal components. In preparation for the study of non-cuspidal components, Section 3.3 gives an account of the work of Bernstein and others, leading to the decomposition (3.0.2). We then consider, in Section 3.4, the component in chamber homology associated to a generic principal-series component of $\text{SL}_2(F)$.

### 3.1 The Center of Mod($G$)

This section is mainly expository. References include [BR92, III.4.2], [Ber84, 1.2–9], [Ren10, I.1.7], and [Roc09].

#### 3.1.1 Definition of the center

We begin with two motivating examples.

**Example 3.1.1.** Let $G$ be a compact totally disconnected group. Every smooth representation $V$ of $G$ decomposes canonically into a direct sum of isotypical components:

$$V = \bigoplus_{\pi \in \hat{G}} e_{\pi} V.$$  

(See Example 2.1.24.) This decomposition is natural, meaning that every $G$-equivariant map $T : V \to W$ restricts to $e_{\pi} V \to e_{\pi} W$. Each $e_{\pi}$ therefore determines an idempotent in the commutative algebra $\mathfrak{Z}(G)$ of endomorphisms of the identity functor in the category Mod($G$). (Explicitly, an element $z \in \mathfrak{Z}(G)$ is a collection of maps $z(V) \in \text{End}_{\mathcal{O}}(V)$, one for
each smooth $G$-module $V$, such that for each $T \in \text{Hom}_G(V, W)$ the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{z(V)} & V \\
\downarrow T & & \downarrow T \\
W & \xrightarrow{z(W)} & W \\
\end{array}
$$

commutes.) Shur’s lemma implies that every element of $\mathfrak{Z}(G)$ is a (possibly infinite) linear combination of the minimal idempotents $e_\pi$:

$$
\mathfrak{Z}(G) \cong \prod_{\pi \in \hat{G}} \mathbb{C}e_\pi.
$$

**Example 3.1.2.** Now consider $G = \mathbb{Z}$. In contrast to the previous example, representations of $\mathbb{Z}$ do not decompose into isotypical components. In fact, the category $\text{Mod}(\mathbb{Z})$ is **indecomposable**: it cannot be expressed as the product of two proper subcategories. To understand why, we consider the algebra $\mathfrak{Z}(\mathbb{Z})$ of endomorphisms of the identity functor. Since $\mathbb{Z}$ is abelian, the usual action of $\mathcal{H}(\mathbb{Z})$ on each $\mathbb{Z}$-module gives an inclusion $\mathcal{H}(\mathbb{Z}) \hookrightarrow \mathfrak{Z}(\mathbb{Z})$.

On the other hand, the map

$$
\text{End}_\mathbb{Z}(\mathcal{H}(\mathbb{Z})) \to \mathcal{H}(\mathbb{Z}), \quad \varphi \mapsto \varphi(1)
$$

identifies the intertwining algebra of the regular representation with the algebra $\mathcal{H}(\mathbb{Z})$. Since every $\mathbb{Z}$-module is a quotient of some power of the regular representation, we in fact have $\mathcal{H}(\mathbb{Z}) = \mathfrak{Z}(\mathbb{Z})$. Now, the Fourier transform

$$
\mathcal{H}(\mathbb{Z}) \xrightarrow{\text{Fur.} f} \mathcal{O}(\mathbb{C}^\times), \quad \hat{f}(\lambda) = \sum_{n \in \mathbb{Z}} f(n)\lambda^n
$$

identifies $\mathcal{H}(\mathbb{Z})$ with the algebra $\mathcal{O}(\mathbb{C}^\times)$ of regular functions on the complex affine variety $\mathbb{C}^\times$. Since $\mathbb{C}^\times$ is connected, $\mathcal{H}(\mathbb{Z})$ contains no proper nonzero idempotent. Any decomposition of the category $\text{Mod}(\mathbb{Z})$ would give an idempotent in $\mathfrak{Z}(\mathbb{Z})$, and so $\text{Mod}(\mathbb{Z})$ is indecomposable.

**Definition 3.1.3.** Let $G$ be a locally compact, totally disconnected group. The *Bernstein
center $\mathcal{Z}(G)$ is the algebra of endomorphisms of the identity functor on $\text{Mod}(G)$.

The center $Z(G)$ of $G$, in the usual algebraic sense, lies in $\mathcal{Z}(G)$, as does the center of the Hecke algebra $\mathcal{H}(G)$. In most of the cases of interest here, the categorical center $\mathcal{Z}(G)$ will be larger than either of these algebraic centers.

**Lemma 3.1.4.** The action of $\mathcal{Z}(G)$ on $\mathcal{H}(G)$, induced by the left-translation action of $G$, identifies $\mathcal{Z}(G)$ with the algebra $\text{End}_{G \times G}(\mathcal{H}(G))$ of $G$-bimodule endomorphisms of $\mathcal{H}(G)$.

**Proof.** See [Ber84, 1.2], [BR92, III.4.2], or [Ren10, I.1.7].

**Example 3.1.5.** If $G$ is discrete, then $\mathcal{Z}(G) \cong Z(\mathcal{H}(G))$ identifies with the space of finitely supported, conjugation-invariant functions on $G$. In the non-discrete case, $\mathcal{Z}(G)$ is the center of a certain completion of the algebra $\mathcal{H}(G)$; see [Ber84, Lemme 1.2.1] or [Ren10, I.1.7].

The center $\mathcal{Z}(G)$ may also be described in terms of *distributions*. A distribution on $G$ is a linear map $D : \mathcal{H}(G) \to \mathbb{C}$. Each $f \in \mathcal{H}(G)$ defines a distribution $D_f$, via

$$\langle D_f, f' \rangle := (f * f')(1).$$

(Here and in the following we deviate a little from the usual conventions; cf. [Ren10, III.3.10].) If $D$ is a distribution and $f \in \mathcal{H}(G)$, define a locally constant function $D * f$ on $G$ by

$$(D * f)(g) := \langle D, f(-g) \rangle.$$

(Here $f(-g)$ is the function $h \mapsto f(hg)$.) Say that $D$ is *essentially compact* if $D * f \in \mathcal{H}(G)$ for every $f \in \mathcal{H}(G)$. The space $\mathcal{D}_{ec}(G)$ of all essentially compact distributions contains the distributions $D_f$, for $f \in \mathcal{H}(G)$, because $D_f * f' = f * f'$. The formula

$$\langle D_1 * D_2, f \rangle := \langle D_1, D_2 * f \rangle$$

defines an algebra structure on $\mathcal{D}_{ec}(G)$, which restricts to the usual multiplication on $\mathcal{H}(G)$. If $D$ is central in $\mathcal{D}_{ec}(G)$, then the operator $f \mapsto D * f$ lies in $\text{End}_{G \times G}(\mathcal{H}(G))$. 
Proposition 3.1.6. The above procedure defines an isomorphism of algebras, $\mathfrak{Z}(G) \cong Z(D_{cc}(G))$.

Proof. See [BR92, 1.4] or [Ren10, II.3.13].

3.1.2 Example: Compact representations

The most easily accessible elements of $\mathfrak{Z}(G)$ are those associated to compact representations of $G$. The purpose of this section is to establish notation and to recall the construction of the central idempotents associated to these representations, for use in Section 3.2.7 below. References for the background material recalled here include [BR92, I.5], [Ren10, IV.1], and [Roc09, 1.2]. We assume throughout this section that $G$ is unimodular and countable at infinity.

Definition 3.1.7. A smooth representation $\pi : G \to \text{GL}(V)$ is called compact if one of the following equivalent conditions is satisfied:

1. For every $v \in V$ and $\tilde{v} \in \tilde{V}$, the matrix coefficient

$$m_{v,\tilde{v}} : g \mapsto \langle \tilde{v}, g^{-1}v \rangle$$

is a compactly supported function on $G$.

2. For every $v \in V$ and every compact open subgroup $K \subset G$, the function

$$g \mapsto e_K g^{-1}v$$

has compact support in $G$.

See [BR92, Theorem I.6] or [Ren10, IV.1.3 Théorème] for the proof that the two conditions are equivalent. Compact representations are sometimes called finite representations.

Example 3.1.8. Let $K \subset G$ be a compact open subgroup, and let $\rho : K \to \text{GL}(W)$ be a smooth representation of $K$. If the compactly induced representation $\text{ind}_K^G \rho$ is irreducible, then it is compact. (See [BH06, 11.4].)
Compact representations behave like representations of compact groups: if \( \pi \) is an irreducible, compact representation of \( G \), then every other smooth representation \( \rho \) decomposes as a direct sum \( \rho \cong \rho_\pi \oplus \rho^\perp_\pi \), where \( \rho_\pi \) is a direct sum of copies of \( \pi \), and \( \rho^\perp_\pi \) admits no irreducible subquotient isomorphic to \( \pi \). This decomposition is natural, meaning that it comes from a decomposition of the category \( \text{Mod}(G) \):

\[
\text{Mod}(G) \cong \text{Mod}(G)_\pi \times \text{Mod}(G)^\perp_\pi.
\]

The projection \( E_\pi : \rho \to \rho_\pi \) therefore lies in the center \( Z(G) \). We now recall, in outline, a construction of \( E_\pi \).

**Lemma 3.1.9.** Let \( \pi : G \to \text{GL}(V) \) be a compact representation of \( G \), and let \( \text{End}_C(V)^0 \) be the algebra of finite-rank operators on \( V \). For each \( T \in \text{End}_C(V)^0 \), the function

\[
m(T) : g \mapsto \text{Trace}(T \pi(g^{-1}))
\]

lies in \( \mathcal{H}(G) \).

**Proof.** The map \( V \otimes_C \tilde{V} \to \text{End}_C(V)^0 \) sending \( v \otimes \tilde{v} \) to the rank-one operator \( v' \mapsto \langle \tilde{v}, v' \rangle v \) is an isomorphism, making the diagram

\[
\begin{array}{ccc}
V \otimes_C \tilde{V} & \longrightarrow & \text{End}_C(V)^0 \\
\downarrow m & & \downarrow m \\
C^\infty(G) & & \end{array}
\]

commute. The range of \( m \) lies in \( \mathcal{H}(G) \), since \( V \) is compact.

**Proposition 3.1.10.** Every irreducible compact representation of \( G \) is admissible.

**Proof.** See [BR92, I.5.1 Proposition 11] or [Ren10, IV.1.3 Proposition].

It follows that the map \( \pi : \mathcal{H}(G) \to \text{End}_C(V) \) has image contained in \( \text{End}_C(V)^0 \).

**Theorem 3.1.11.** Let \( \pi : G \to \text{GL}(V) \) be an irreducible, compact representation of \( G \).
(1) There is a positive constant $d(\pi) = d(V) > 0$, dependent on the choice of Haar measure on $G$, such that the operator $E_\pi : \mathcal{H}(G) \to \mathcal{H}(G)$ defined as the composition

$$\mathcal{H}(G) \xrightarrow{\pi} \text{End}_\mathbb{C}(V)^0 \xrightarrow{d(\pi)m} \mathcal{H}(G)$$

is an idempotent in $\mathfrak{Z}(G)$.

(2) In terms of distributions, $E_\pi = d(\pi) \text{ch}_\pi$ (where $\text{ch}_\pi : f \mapsto \text{Trace}(\pi(f))$ is the character of $\pi$).

(3) If $W$ is any smooth representation of $G$, then

$$W \cong E_\pi W \oplus (1 - E_\pi)W,$$

where $E_\pi W$ is a direct sum of copies of $\pi$, and $(1 - E_\pi)W$ contains no subquotient equivalent to $\pi$.

Proof. See [BR92, I.5.1–3] or [Ren10, IV.1.4–5].

The number $d(\pi)$ is called the formal degree or formal dimension of $\pi$. Note that some authors, e.g., [BR92] and [Ren10], use the term “formal degree” for what we are calling $d(\pi)^{-1}$.

Example 3.1.12. If $G$ is compact, then every smooth representation of $G$ is compact. Letting $\pi$ be an irreducible representation, one has $d(\pi) = \frac{\dim(\pi)}{\text{vol}(G)}$.

Example 3.1.13. Suppose that $K$ is a compact open subgroup of $G$, and let $\rho$ be an irreducible representation of $K$ such that $\pi := \text{ind}_K^G \rho$ is irreducible. Then $\pi$ is a compact representation of $G$ (Example 3.1.8). The subspace $\pi_0 \subset \pi$ of functions supported on $K$ is a $K$-subrepresentation, isomorphic to $\rho$. Restricting the operator $E_\pi$ to this subspace, we find that $d(\pi) = d(\rho)$; that is,

$$d(\text{ind}_K^G \rho) = \frac{\dim(\rho)}{\text{vol}(K)}.$$  

(3.1.14)
3.1.3 Example: Compact-mod-center representations

Since the center $Z(G)$ acts by multiples of the identity on any irreducible representation, the support of any matrix coefficient is a union of $Z(G)$-cosets. Thus, if $Z(G)$ is not compact, no irreducible representation of $G$ is compact. A more useful notion in such cases is the following:

**Definition 3.1.15.** Say that a subset $S \subseteq G$ is compact-mod-center if the image of $S$ under the quotient map $G \to G/Z(G)$ is compact. A representation $\pi : G \to \text{GL}(V)$ is compact-mod-center if one of the following equivalent conditions is satisfied:

1. For every $v \in V$ and $\bar{v} \in \tilde{V}$, the support of the matrix coefficient $m_{v,\bar{v}}$ is compact-mod-center.

2. For every $v \in V$ and every compact open subgroup $K \subseteq G$, the support of the function $g \mapsto e_K g^{-1} v$

is compact-mod-center.

**Example 3.1.16.** Suppose that $K \subseteq G$ is an open, compact-mod-center subgroup of $G$. All irreducible representations of the form $\text{ind}_K^G \rho$ are compact-mod-center.

Let $G = \text{GL}_n(F)$, where $F$ is a $p$-adic field. The compact-mod-center representations of the group $G$ determine idempotents in $\mathcal{Z}(G)$, as we shall now explain. This material may be found in several places: for instance, [BR92, 1.12–1.20], [Ren10, VI.3], or [Roc09, 1.4–1.6]. (The same construction applies in the more general context of reductive $p$-adic groups.) The results and notation recalled here will be used in Section 3.2.8, below.

**Definition 3.1.17.** Consider the open, normal subgroup $G^o \subseteq G$,

$$G^o = \{ g \in G \mid \det(g) \in \mathcal{O}^\times \}.$$  

The quotient $G/G^o$ is isomorphic to $\mathbb{Z}$. Denote by $\Psi(G)$ the one-dimensional complex torus
of unramified characters of $G$,

$$\Psi(G) := \text{Hom}_{\mathbb{Z}}(G/G^o, \mathbb{C}^\times).$$

If $\pi$ is a representation of $G$ then we will write $\psi\pi$, instead of $\psi \otimes_G \pi$, for the “twisting” of $\pi$ by a character $\psi \in \Psi(G)$.

**Lemma 3.1.18.** A representation $\pi$ of $G$ is compact-mod-center if and only if $\pi|_{G^o}$ is a compact representation of $G^o$.

**Proof.** One direction follows from the fact that $Z(G) \cap G^o$ is compact; the other comes from the fact that $Z(G)G^o$ has finite index in $G$. \qed

Very little information is lost in passing from $\pi$ to $\pi|_{G^o}$:

**Lemma 3.1.19.** Let $\pi$ and $\pi'$ be irreducible representations of $G$. The following are equivalent:

1. $\pi|_{G^o} \cong \pi'|_{G^o}$
2. $\pi' \cong \psi\pi$ for some unramified character $\psi \in \Psi(G)$.

**Proof.** See [BR92, II Proposition 25], [Ren10, VI.3.2], or [Roc09, Proposition 1.4.1.1]. \qed

Let $\pi$ be an irreducible, compact-mod-center representation of $G$. The following construction depends only on the orbit $[\pi] = \Psi(G)\pi$.

The restriction $\pi|_{G^o}$ is a compact representation of $G^o$, of finite length ([BR92, II Proposition 25]; note that $\pi|_{G^o}$ need not be irreducible, cf. Lemma 3.2.43). By Theorem 3.1.11, $\pi|_{G^o}$ decomposes as a direct sum of irreducible, compact representations of $G^o$. Let $\pi_1, \ldots, \pi_l$ be the distinct irreducibles of $G^o$ appearing in $\pi|_{G^o}$. Define an idempotent $E_\pi \in \mathfrak{Z}(G^o)$ by

$$E_\pi := E_{\pi_1} + \ldots + E_{\pi_l},$$

where the $E_{\pi_i}$ are the mutually orthogonal idempotents in $\mathfrak{Z}(G^o)$ given by Theorem 3.1.11.
If $V$ is a representation of $G$, then the restriction $V|_{G^0}$ is a module over $\mathcal{Z}(G^0)$. Any $G$-equivariant map $V \rightarrow W$ restricts to a $G^0$-equivariant map, and so the action of $\mathcal{Z}(G^0)$ is natural with respect to $G$-equivariant maps. The action of $\mathcal{Z}(G^0)$ commutes with $G^0$, by definition, but not necessarily with $G$. However:

**Theorem 3.1.20.** (1) The action of $E_\pi \in \mathcal{Z}(G^0)$ on $V$ commutes with the action of $G$, and so $E_\pi$ defines an element of $\mathcal{Z}(G)$.

(2) Every irreducible subquotient of $E_\pi V$ has the form $\psi\pi$, for some unramified character $\psi \in \Psi(G)$. No irreducible subquotient of $(1 - E_\pi)V$ has this form.

**Proof.** See [Ren10, VI.3.3] or [Roc09, 1.4.3].

The idempotent $E_\pi \in \mathcal{Z}(G)$ thus induces a decomposition

$$\text{Mod}(G) \cong \text{Mod}(G)[\pi] \times \text{Mod}(G)^{\perp}[\pi]$$

of the category $\text{Mod}(G)$. Here $\text{Mod}(G)[\pi]$ is the category of those representations of $G$ whose irreducible subquotients are all of the form $\psi\pi$, and $\text{Mod}(G)^{\perp}[\pi]$ is the category of representations having no subquotient of this form.

The following theorem, due to Bernstein, identifies the algebra $\mathcal{Z}(G)[\pi] := E_\pi \mathcal{Z}(G)$.

**Theorem 3.1.21.** Let $\pi$ be an irreducible, compact-mod-center representation of $G = \text{GL}_n(F)$.

(1) The subgroup $\Psi(G)_\pi := \{\psi \in \Psi(G) \mid \psi\pi \cong \pi\}$ is finite, and the quotient $\Omega_\pi := \Psi(G)/\Psi(G)_\pi$ is a one-dimensional complex torus.

(2) The map

$$\mathcal{Z}(G) \rightarrow \{f : \Omega_\pi \rightarrow \mathbb{C}\}, \quad z \mapsto (\psi \mapsto z(\psi\pi))$$

restricts to an isomorphism between $\mathcal{Z}(G)[\pi]$ and the algebra $\mathcal{O}(\Omega_\pi)$ of regular functions on $\Omega_\pi$.

(3) The category $\text{Mod}(G)[\pi]$ is indecomposable.
Proof. See [Ber84, Proposition 1.15], [Ren10, VI.4.4], or [Roc09, 1.5–1.6].

We shall have more to say about \( \mathfrak{Z}(G) \) in Section 3.3. For now, though, we turn to chamber homology.

### 3.2 Action of Central Idempotents on Chamber Homology

We return to the hypotheses (GAB1–3) of Section 2.3: \( X \) is a locally finite, oriented, affine building, and \( G \) is a locally compact, totally disconnected group, acting by automorphisms of \( X \), in such a way that the isotropy group \( G_x \) of each point \( x \in X \) is compact and open in \( G \), and such that each fixed-point set \( X^g \) is a subcomplex of \( X \).

This section investigates the interaction between two families of linear maps on \( \mathcal{H}(G) \):

1. The operators \( 1_{G_c} \) and \( 1_{G_{nc}} \) of “restriction to the compact/noncompact parts”, and
2. The idempotents in \( \mathfrak{Z}(G) \).

We begin by explaining how these two kinds of maps act on the various homology groups that were associated to \( G \) in Chapter 2. We then show that each idempotent \( E \in \mathfrak{Z}(G) \) induces a direct-sum decomposition

\[
H^G_*(X) \cong H^G_*(X)_E \oplus H^G_*(X)^E
\]

in chamber homology. The definition of \( H^G_*(X)_E \) used here is different to the one given by Baum, Higson and Plymen in [BHP00], but we will show that the two groups in fact coincide.

Theorem 3.2.19 produces a chain complex whose homology is canonically isomorphic to \( H^G_*(X)_E \). The theorem will be applied to three examples: \( \text{GL}_1(F) \), compact representations of \( \text{SL}_n(F) \), and compact-mod-center representations of \( \text{GL}_2(F) \). Another example will be considered in Section 3.4.

#### 3.2.1 The action of \( 1_{G_c} \) on homology

The union \( G_c \) of the compact subgroups of \( G \) is an open, conjugation-invariant subset of \( G \), as is its complement \( G_{nc} = G \setminus G_c \) (Corollary 2.3.19). Thus the characteristic functions
$1_{G_c}$ and $1_{G_{nc}}$ of these sets lie in the algebra $\text{Cl}^\infty(G)$ of locally constant class functions on $G$. As observed in Examples 2.4.4 and 2.4.9, the algebra $\text{Cl}^\infty(G)$ acts on $H_*(G, \mathcal{H}(G)_{\text{Ad}})$, $HH_*(\mathcal{H}(G))$, and $HC_*(\mathcal{H}(G))$. So $1_{G_c}$ and $1_{G_{nc}}$ act as idempotent linear operators on all of these homology groups.

Corollary 2.3.34 asserts that

$$1_{G_c} \cdot H_*(G, \mathcal{H}(G)_{\text{Ad}}) = H_*(G, \mathcal{H}(G)_{\text{Ad}}) \cong H_*^G(X).$$

Theorem 2.4.13 describes how $1_{G_c}$ and $1_{G_{nc}}$ interact with the operators $S$, $B$ and $I$ on Hochschild and cyclic homology:

$$(3.2.1) \quad 1_{G_c} I = I 1_{G_c} \quad 1_{G_c} B = B 1_{G_c} = 0 \quad 1_{G_c} S = S 1_{G_c} = S.$$  

Writing $1_{G_{nc}} = \text{id} - 1_{G_c}$ yields similar formulas for $1_{G_{nc}}$. The image of $1_{G_c}$ on Hochschild homology will be denoted $HH_*(\mathcal{H}(G))_{c}$, as in Chapter 2. We use similar notation for $1_{G_{nc}}$ and for cyclic homology.

**Definition 3.2.2.** Let $G$ and $G'$ be two locally compact, totally disconnected groups acting on affine buildings, and let $T : HH_n(\mathcal{H}(G)) \to HH_m(\mathcal{H}(G'))$ be a linear map between Hochschild homology groups. We use the following notation for the matrix-entries of $T$ relative to the decomposition of Hochschild homology into compact and noncompact parts:

$$T : \begin{bmatrix} HH_n(\mathcal{H}(G))_c \\ HH_n(\mathcal{H}(G))_{nc} \end{bmatrix} \begin{bmatrix} T_c \\ T_{c, nc} \end{bmatrix} = \begin{bmatrix} HH_m(\mathcal{H}(G'))_c \\ HH_m(\mathcal{H}(G'))_{nc} \end{bmatrix}. $$

(So $T_c = 1_{G'_c} \circ T \circ 1_{G_c}$, $T_{c, nc} = 1_{G'_c} \circ T \circ 1_{G_{nc}}$, and so on.) Similar notation will be used for maps between cyclic homology groups, and for maps from Hochschild to cyclic homology (and vice versa).

**Example 3.2.3.** In this notation, the relations (3.2.1) become

$$I = \begin{bmatrix} I_c & 0 \\ 0 & I_{nc} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_{nc} \end{bmatrix}, \quad S = \begin{bmatrix} S_c & 0 \\ 0 & 0 \end{bmatrix}. $$
Using the isomorphisms of Proposition 2.4.23, the action of \(1_{G_c}\) and \(1_{G_{nc}}\) may be translated from \(HH_*(\mathcal{H}(G))\) and \(HC_*(\mathcal{H}(G))\) to \(HH_*(\mathcal{P}_G)\) and \(HC_*(\mathcal{P}_G)\). These operators do not appear to have a straightforward description in terms of projective modules.

### 3.2.2 The action of \(3(G)\) on homology

Let \(V\) be a nondegenerate \(\mathcal{H}(G)\)-bimodule. The left action of \(G\) induces an action of \(3(G)\) by bimodule endomorphisms of \(V\). The Hochschild homology \(H_*(\mathcal{H}(G), V)\) is thus a module over \(3(G)\). Since \(3(G)\) commutes with the adjoint action of \(G\) on \(V\), the smooth homology \(H_*(G, V_{Ad})\) is also a module over \(3(G)\). The natural isomorphism \(H_*(\mathcal{H}(G), V) \to H_*(G, V_{Ad})\) of Proposition 2.4.6 is \(3(G)\)-linear.

**Remark 3.2.4.** Note that there is another action of \(3(G)\) on \(V_{Ad}\), namely the one associated functorially to the adjoint action of \(G\). Let us emphasize that this is not the action we are using. (There is a third action, coming from the right action of \(G\); for \(V = \mathcal{H}(G)\), this action of \(3(G)\) is equal to the one coming from the left action of \(G\).)

Now put \(V = \mathcal{H}(G)\). Each idempotent \(E \in 3(G)\) acts on \(\mathcal{H}(G)\) as an algebra homomorphism:

\[
E(f_1f_2) = E^2(f_1f_2) = E(f_1)E(f_2).
\]

Since Hochschild and cyclic homology are functorial with respect to algebra homomorphisms, \(E\) induces maps on \(HH_*(\mathcal{H}(G))\) and \(HC_*(\mathcal{H}(G))\). The identification \(HH_*(\mathcal{H}(G)) = H_*(\mathcal{H}(G), \mathcal{H}(G))\) means that we now have two definitions for the action of \(E\) on \(HH_*(\mathcal{H}(G))\).

These two actions coincide, thanks to the following simple fact about Hochschild homology.

**Lemma 3.2.5.** The diagram

\[
\begin{array}{ccc}
HH_*(\mathcal{H}(G)) & \xrightarrow{\cong} & H_*(\mathcal{H}(G), \mathcal{H}(G)) \\
E \downarrow & & E \\
HH_*(\mathcal{H}(G)) & \xrightarrow{\cong} & H_*(\mathcal{H}(G), \mathcal{H}(G))
\end{array}
\]

is commutative. (The same is true for any idempotent endomorphism of a locally unital algebra.)
Proof. The left-hand vertical arrow is the one induced by the maps

\[ \varphi : \mathcal{H}(G)^{\otimes (n+1)} \to \mathcal{H}(G)^{\otimes (n+1)} \]

\[ \varphi(f_0 \otimes \cdots \otimes f_n) = E(f_0) \otimes E(f_1) \otimes \cdots \otimes E(f_n). \]

The right-hand vertical arrow is the one induced by

\[ \psi : \mathcal{H}(G)^{\otimes (n+1)} \to \mathcal{H}(G)^{\otimes (n+1)} \]

\[ \psi(f_0 \otimes \cdots \otimes f_n) = E(f_0) \otimes f_1 \otimes \cdots \otimes f_n. \]

We will construct a homotopy between \( \varphi \) and \( \psi \) on the subcomplex associated to the subalgebra \( e \mathcal{H}(G)e \), where \( e \) is an idempotent in \( \mathcal{H}(G) \). The lemma will follow upon taking the (direct) limit over \( e \).

Define maps

\[ h_i : e \mathcal{H}(G)e^{\otimes (n+1)} \to e \mathcal{H}(G)e^{\otimes (n+2)} \]

by

\[ h_i(f_0 \otimes \cdots \otimes f_n) = E(f_0) \otimes \cdots \otimes E(f_i) \otimes E(e) \otimes f_{i+1} \otimes \cdots \otimes f_n \]

for \( 0 \leq i \leq n - 1 \), and

\[ h_n(f_0 \otimes \cdots \otimes f_n) = E(f_0) \otimes \cdots \otimes E(f_n) \otimes E(e). \]

A routine verification shows that these maps define a presimplicial homotopy from \( \psi \) to \( \varphi \) (see [Lod92, 1.0.8] for the terminology). So \( \psi \) and \( \varphi \) induce equal maps in Hochschild homology [Lod92, Lemma 1.0.9]. \( \Box \)

The idempotents \( E \in \mathcal{Z}(G) \) also act on the precyclic module \( C(\mathcal{P}_G) \) in a natural way: if \( V \) and \( W \) are finitely generated, projective \( \mathcal{H}(G) \)-modules, then \( \mathcal{Z}(G) \) acts naturally on both \( V \) and \( W \), and so

\[ T \mapsto TE = ET = ETE \]
defines an endomorphism of $\text{Hom}_G(V, W)$, compatible with composition. Thus the idempotents $E \in \mathcal{Z}(G)$ act on $HH_*(P_G)$ and on $HC_*(P_G)$.

**Lemma 3.2.6.** The diagrams

$$
\begin{align*}
HH_*(\mathcal{H}(G)) & \xrightarrow{\approx} HH_*(P_G) \quad \text{and} \quad HC_*(\mathcal{H}(G)) \xrightarrow{\approx} HC_*(P_G) \\
& \quad \\
HH_*(\mathcal{H}(G)) & \xrightarrow{\approx} HH_*(P_G) \quad \text{and} \quad HC_*(\mathcal{H}(G)) \xrightarrow{\approx} HC_*(P_G)
\end{align*}
$$

are commutative. (The horizontal arrows are the isomorphisms of Proposition 2.4.23.)

**Proof.** By the definition of the horizontal arrows, the lemma reduces to proving that for each idempotent $e \in \mathcal{H}(G)$, the diagram

$$
\begin{align*}
\begin{array}{ccc}
\varepsilon \mathcal{H}(G)e & \rightarrow & \text{End}_{e\mathcal{H}(G)e}(\mathcal{H}(G)e) \\
E \downarrow & & \downarrow E \\
\varepsilon \mathcal{H}(G)e & \rightarrow & \text{End}_{e\mathcal{H}(G)e}(\mathcal{H}(G)e)
\end{array}
\end{align*}
$$

commutes. The upper composition sends $f \in \varepsilon \mathcal{H}(G)e$ to the endomorphism $v \mapsto fEv$, while the lower composition sends $f$ to the endomorphism $v \mapsto Ef v$. The two are equal, since $\mathcal{Z}(G)$ commutes with $\mathcal{H}(G)$.

\[ \square \]

### 3.2.3 The interaction of $1_{G_c}$ and $\mathcal{Z}(G)$ on homology

The interaction between $1_{G_c}$ and the idempotents $E \in \mathcal{Z}(G)$ is described by the next lemma. We first give a (somewhat technical) general statement, and then spell out its importance to the case at hand in Corollary 3.2.8. The general lemma will be applied in a different context in Chapters 4 and 5.

For locally unital algebras $A$ and $B$, we write $T : \text{SBI}(A) \rightarrow \text{SBI}(B)$ to indicate that $T$ is a collection of linear maps $HH_*(A) \rightarrow HH_*(B)$ and $HC_*(A) \rightarrow HC_*(B)$, commuting with the operators $S$, $B$ and $I$. For example, $T$ might be induced by an algebra homomorphism $A \rightarrow B$. 
**Lemma 3.2.7.** Let $G$, $G'$ and $G''$ be locally compact, totally disconnected groups, each acting on an affine building and satisfying the hypotheses (GAB1–3). Let $T : SBI(\mathcal{H}(G)) \to SBI(\mathcal{H}(G'))$ and $T' : SBI(\mathcal{H}(G')) \to SBI(\mathcal{H}(G''))$ be morphisms of SBI sequences. Then

1. $T_{nc,c} = 0$ on $HC_s(\mathcal{H}(G))$.

2. $T'_{c,nc} \circ T_{nc,c} = 0$ on $HH_s(\mathcal{H}(G))$ and $HC_s(\mathcal{H}(G))$.

3. $(T' \circ T)_c = T'_c \circ T_c$ on $HH_s(\mathcal{H}(G))$ and $HC_s(\mathcal{H}(G))$.

**Proof.** (1) The fact that $T$ commutes with $B$ gives

$$B_{nc} \circ T_{nc,c} = T_{nc,c} \circ B_c : HC_n(\mathcal{H}(G)) \to HH_{n+1}(\mathcal{H}(G')).$$

Since $B_c = 0$ by (3.2.1), we have $B_{nc} \circ T_{nc,c} = 0$. Since $S_{nc} = 0$, again by (3.2.1), $B_{nc}$ is injective and consequently $T_{nc,c} = 0$ on cyclic homology.

(2) For cyclic homology, part (2) follows immediately from part (1). For Hochschild homology, let $x \in HH_n(\mathcal{H}(G))$. We have $I_{nc} \circ T_{nc,c}(x) = T_{nc,c} \circ I_c(x) = 0$ by part (1), and so $T_{nc,c}(x) = B_{nc}(y)$ for some $y \in HC_{n-1}(\mathcal{H}(G'))$. But now

$$T'_{c,nc} \circ T_{nc,c}(x) = T'_{c,nc} \circ B_{nc}(y) = B_c \circ T'_{c,nc}(y) = 0$$

because $B_c = 0$.

(3) By part (2), we have

$$T' \circ T = \begin{bmatrix} T'_c & T'_{c,nc} \\ T'_{nc,c} & T'_c \end{bmatrix} \circ \begin{bmatrix} T_c & T_{c,nc} \\ T_{nc,c} & T_c \end{bmatrix} = \begin{bmatrix} T'_c \circ T_c & * \\ * & * \end{bmatrix}.$$ 

\[ \square \]

**Corollary 3.2.8.** Let $G$ be as in Lemma 3.2.7, and let $E \in \mathcal{Z}(G)$ be an idempotent. The operator $E_c = 1_{G_c}E 1_{G_c}$ acts as an idempotent on $HH_s(\mathcal{H}(G))$ and on $HC_s(\mathcal{H}(G))$. If $E' \in \mathcal{Z}(G)$ is another idempotent with $EE' = E'E = 0$, then $E_cE'_c = E'_cE_c = 0$.

**Proof.** Apply Lemma 3.2.7 with $G' = G'' = G$, $T = E$, and $T' = E'$. 

\[ \square \]
Remark 3.2.9. (1) Corollary 3.2.8 would of course follow from the statement that the operator $E$ is diagonal with respect to the compact/noncompact decomposition of $HH_*(\mathcal{H}(G))$ and $HC_*(\mathcal{H}(G))$. Dat has proved this stronger statement in degree zero: see [Dat03, Proposition 2.8], or Proposition 4.1.13 below. For $G = \text{SL}_2(F)$, this commutativity property also holds in higher degrees: see Corollary 4.5.3.

(2) Note that part (1) of Lemma 3.2.7 implies that every endomorphism $T$ of the SBI sequence is upper-triangular on $HH_0(\mathcal{H}(G)) = HC_0(\mathcal{H}(G))$. But it is not true in general that the upper-right-hand matrix entry $T_{c,nc}$ vanishes; see Example 4.4.11.

3.2.4 Bernstein components in chamber homology

We continue in assuming the hypotheses (GAB1–3): $G$ is a totally disconnected locally compact group, acting on an oriented, locally finite, affine building $X$, in such a way that each isotropy subgroup $G_x$ is compact and open in $G$, and such that each fixed-point set $X^y$ is a subcomplex of $X$.

Let $E \in \mathcal{Z}(G)$ be an idempotent. In view of the isomorphism $H_*^G(X) \cong HH_*(\mathcal{H}(G))c$, we may consider $E_c$ as an idempotent operator on the chamber homology group $H_*^G(X)$.

Definition 3.2.10. For each idempotent $E \in \mathcal{Z}(G)$, we consider the direct-summand

$$H_*^G(X)_E := E_c(H_*^G(X))$$

of $H_*^G(X)$.

In their paper [BHP00], Baum, Higson and Plymen give an apparently different definition of the group $H_*^G(X)_E$; let us show that the two definitions are in fact equivalent.

Lemma 3.2.11. The composition

$$H_*^G(X) \xrightarrow{\sim} HH_*(\mathcal{H}(G))c \hookrightarrow HH_*(\mathcal{H}(G)) \xrightarrow{1} HC_*(\mathcal{H}(G))$$


induces an isomorphism

\[ H^G_n(X)_E \cong \ker \left( HC_n(\mathcal{H}(G)) \xrightarrow{S} HC_{n-2}(\mathcal{H}(G)) \right) \]
\[ \cap \text{image} \left( HC_{n+2}(\mathcal{H}(G)) \xrightarrow{S} HC_n(\mathcal{H}(G)) \right) \]
\[ \cap \text{image} \left( HC_n(\mathcal{H}(G)) \xrightarrow{E} HC_n(\mathcal{H}(G)) \right) \].

**Proof.** First recall that the isomorphism \( H_*(G, \mathcal{H}(G)_{Ad}) \xrightarrow{\cong} HH_*(\mathcal{H}(G)) \) commutes with the operators \( 1_{G_c} \) and \( E \), and so we immediately have

\[ H^G_n(X)_E \cong \text{image} \left( HH_n(\mathcal{H}(G)) \xrightarrow{E_c} HH_n(\mathcal{H}(G)) \right) \].

We must therefore show that the map

(3.2.12) \[ I : E_c(HH_n(\mathcal{H}(G))) \to HC_n(\mathcal{H}(G)) \]

is injective, with image equal to \( \ker(S) \cap \text{image}(S) \cap \text{image}(E) \).

For the injectivity, recall from (3.2.1) that \( B_c = 0 \). Thus the exactness of the SBI sequence ensures that \( I \) is injective on all of \( HH_n(\mathcal{H}(G))_c \).

The image of \( I : HH_n(\mathcal{H}(G)) \to HC_n(\mathcal{H}(G)) \) lies in the kernel of \( S \), by exactness. Once again using the fact that \( B_c = 0 \), we have

\[ B \circ I \circ E_c = B_c \circ I_c \circ E_c = 0. \]

So the image of the map (3.2.12) lies in \( \ker(B) = \text{image}(S) \). For each \( x \in HH_n(\mathcal{H}(G)) \), we have

\[ I \circ E_c(x) = E_c \circ I_c(x) = E \circ I_c(x) = E_{nc,c} \circ I_c(x), \]

and the term \( E_{nc,c} \circ I_c(x) \) is equal to zero by part (1) of Lemma 3.2.7. So the image of (3.2.12) lies in the image of \( E \).

It remains to show that (3.2.12) is surjective on \( \ker(S) \cap \text{image}(S) \cap \text{image}(E) \). Let \( y \in HC_n(\mathcal{H}(G)) \) be an element of this intersection. Since \( \text{image}(S) = \ker(B) = HC_n(\mathcal{H}(G))_c \),
we have $y \in HC_n(\mathcal{H}(G))_c$. Since $\ker(S) = \text{image}(I)$, we can find $x \in HH_n(\mathcal{H}(G))_c$ with $I_c(x) = y$. Then we have, as above,

$$I \circ E_c(x) = E \circ I_c(x) = E(y) = y,$$

and this proves surjectivity.

**Corollary 3.2.13.** For each idempotent $E \in \mathcal{Z}(G)$, the ideal $\mathcal{H}(G)_E := E(\mathcal{H}(G))$ of $\mathcal{H}(G)$ has periodic cyclic homology

$$HP_*(\mathcal{H}(G)_E) \cong \bigoplus_{n \in \mathbb{Z}} H^G_{*+2n}(X)_E.$$

**Proof.** Combine Lemma 3.2.11 and Theorem 2.4.13.\qed

The groups $H^G_*(X)_E$ may also be described in terms of the complex appearing in Corollary 2.4.14:

**Lemma 3.2.14.** The isomorphism $H^G_*(X) \cong HH_*(\mathcal{H}(G))_c$ identifies $H^G_*(X)_E$ with the cohomology of the cochain complex

$$\ldots \xrightarrow{BI} HH_{n-1}(\mathcal{H}(G)_E) \xrightarrow{BI} HH_n(\mathcal{H}(G)_E) \xrightarrow{BI} HH_{n+1}(\mathcal{H}(G)_E) \xrightarrow{BI} \ldots .$$

**Proof.** We have $HH_*(\mathcal{H}(G)_E) = E(HH_*(\mathcal{H}(G)))$, compatibly with the maps $B$ and $I$. So the given cochain complex is isomorphic to

$$\ldots \xrightarrow{BI} E(HH_n(\mathcal{H}(G))) \xrightarrow{BI} E(HH_{n+1}(\mathcal{H}(G))) \xrightarrow{BI} \ldots .$$

Define a map $\varphi$ from $E_c(HH_n(\mathcal{H}(G)))$ to the cohomology of (3.2.15), by

$$\varphi(E_c(x)) = [E1_{G_c}(x)],$$

the square brackets indicating a cohomology class. We claim that this map is a well-defined isomorphism.
To see that $E_{1G_c}(x)$ does indeed define a cohomology class, note that

$$BI(E_{1G_c}(x)) = EB_cI_c(x) = 0,$$

because $B_c = 0$.

To see that this cohomology class is well-defined, suppose that $E_{c}(x) = 0$. Then

$$E_{1G_c}(x) = 1_{G_{nc}}E_{1G_c}(x) = E_{nc,c}(x).$$

Part (1) of Lemma 3.2.7 implies that $I_{nc} \circ E_{nc,c}(x) = E_{nc,c} \circ I_c(x) = 0$, so $E_{1G_c}(x) \in \ker(I_{nc}) = \text{image}(B_{nc})$. Since $S_{nc} = 0$, $I_{nc}$ is surjective; thus $\text{image}(B_{nc}) = \text{image}(B_{nc} \circ I_{nc})$. This proves that $E_{1G_c}(x)$ is a coboundary, and so the map $\varphi$ is well-defined.

A similar argument shows that $\varphi$ is injective: if $E_{1G_c}(x)$ lies in $\text{image}(BI)$, then $E_{c}(x)$ lies in $\text{image}(B_cI_c)$; but $B_c = 0$, and so $E_{c}(x) = 0$. The surjectivity of $\varphi$ is obvious.

**Remark 3.2.16.** Suppose that the algebra $\mathcal{H}(G)_E$ is Morita equivalent to the algebra $\mathcal{O}(Y)$ of regular functions on a smooth, complex affine variety $Y$. Then, according to the Hochschild-Kostant-Rosenberg theorem (cf. [Lod92, Section 3.4]), each $HH_n(\mathcal{H}(G)_E)$ is isomorphic to the space $\Omega^n(Y)$ of (algebraic) differential forms on $Y$, and the cochain complex appearing in Lemma 3.2.14 is isomorphic to the (algebraic) de Rham complex for $Y$. Thus

$$H^G_*(X)_E \cong H^*_{\text{de Rham}}(Y).$$

**Examples 3.2.17.** (1) Let $\pi : G \to \text{GL}(V)$ be an irreducible compact representation of $G$, and let $E = E_{\pi}$ be the idempotent from Theorem 3.1.11. The algebra $\mathcal{H}(G)_{E_{\pi}}$ is isomorphic to the algebra $\text{End}_C(V)^0$ of finite-rank operators on $V$. This latter algebra is Morita equivalent to $\mathbb{C} \cong \mathcal{O}(\text{point})$. Thus Lemma 3.2.14 implies that

$$H^G_m(X)_{E_{\pi}} \cong H^m_{\text{de Rham}}(\text{point}) \cong \begin{cases} \mathbb{C} & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

(2) Let $\pi$ be a compact-mod-center representation of $G = \text{GL}_n(F)$, and let $E_{\pi} \in \mathcal{Z}(G)$ be
the idempotent from Theorem 3.1.20. Then \( \mathcal{H}(G)_{E_x} \) is Morita equivalent to the algebra of regular functions on a one-dimensional complex torus \( Y \) (cf. [BHP00, Theorem 7.16]), and so

\[
H^G_m(X)_{E_x} \cong H^m_{\text{de Rham}}(Y) \cong \begin{cases} 
\mathbb{C} & \text{if } m = 0 \text{ or } 1, \\
0 & \text{if } m > 1.
\end{cases}
\]

(3) Let \( G = \text{GL}_n(F) \). Bernstein’s Paley-Wiener theorem [BR92, III.5.2] implies that, if \( E \in \mathfrak{Z}(G) \) is a generic minimal idempotent, then \( \mathcal{H}(G)_E \) is Morita equivalent to the algebra of regular functions on a complex torus \( Y \) (see [ABP06, Theorem 1] for the terminology and a proof; note that this result holds for general reductive groups). Thus \( H^G_*(X)_E \cong H^*_\text{de Rham}(Y) \).

### 3.2.5 A complex to compute \( H^G_*(X)_E \)

We will give a general procedure for computing the image of an idempotent map on homology induced by a map of chain complexes, and then apply this result to the operator on chamber homology induced by an idempotent in the Bernstein center. Baum, Higson and Plymen have given another, conjectural construction of a complex computing the Bernstein components of chamber homology [BHP00, Conjecture 5.3], and we discuss this conjecture below.

Here is some labor-saving notation: if \( V \) is a module over some ring \( A \), and if \( f : V \to V \) is an endomorphism of \( V \), then we let

\[
\lim (V, f) := \lim \left( V \xrightarrow{f} V \xrightarrow{f} V \xrightarrow{f} \ldots \right).
\]

**Lemma 3.2.18.** Let \( C \) be a chain complex of \( A \)-modules, and let \( f : C \to C \) be a map of chain complexes whose induced map on \( H_*(C) \) is an idempotent. Then

\[
f(H_*(C)) \cong H_* \left( \lim (C, f) \right).
\]

**Proof.** If \( f \) is any idempotent operator on a module \( V \), then \( f(V) \cong \lim (V, f) \). Since
homology commutes with direct limits, this gives

\[ f(H_\ast(C)) \cong \lim\limits_p H_\ast(C), f) \cong H_\ast \left( \lim\limits_p (C, f) \right). \]

\[ \square \]

Applying this to the problem of computing \( H_\ast^G(X)_E \), we obtain:

**Theorem 3.2.19.** Let \( E \in \mathcal{Z}(G) \) be an idempotent, and let \( E_c : C_\ast(X, \mathcal{G}) \to C_\ast(X, \mathcal{G}) \) be a map of equivariant chain complexes making the diagram

\[
\begin{array}{ccc}
\mathcal{H}(G)_{Ad} & \xrightarrow{E_c} & \mathcal{H}(G)_{Ad} \\
\downarrow & & \downarrow \\
C_\ast(X, \mathcal{G}) & \xrightarrow{E_c} & C_\ast(X, \mathcal{G})
\end{array}
\]

commute. (Such a map exists and is unique up to homotopy.) Then

\[ H_\ast^G(X)_E \cong H_\ast \left( \lim\limits_p (C_\ast(X, \mathcal{G})_G, E_c) \right). \]

**Proof.** The existence and uniqueness of the lift \( E_c \) is standard homological algebra; see [Wei94, Theorem 2.2.6] for example.

Corollary 3.2.8 implies that \( E_c \) induces an idempotent on \( H_\ast^G(X) \). By definition, \( H_\ast^G(X)_E \) is the image of that idempotent. Now apply Lemma 3.2.18.

**Remark 3.2.20.** The utility of Theorem 3.2.19 depends, of course, on being able to find an explicit lifting of \( E_c \) to \( C_\ast(X, \mathcal{G}) \). In the following three sections, we apply the theorem to three examples; a further example is given in Section 3.4.

**Remark 3.2.21.** In [BHP00], Baum, Higson, and Plymen consider the following direct-summand \( \mathcal{G}_E \) of the equivariant coefficient system \( \mathcal{G} \):

\[ \mathcal{G}_E(\sigma) := \mathcal{H}(G) \cap E\mathcal{H}(G) \]

(the transition maps and \( G \)-action are inherited from \( \mathcal{G} \)). They conjecture that this coefficient system yields a finite-dimensional chain complex computing the Bernstein component.
$H^G_*(X)_E$ of chamber homology:

**Conjecture 3.2.22.** [BHP00, Conjectures 5.3 and 5.4] Let $G$ be a reductive $p$-adic group (e.g., $\text{GL}_n(F)$ or $\text{SL}_n(F)$) with Bruhat-Tits building $X$, and let $E \in \mathfrak{z}(G)$ be a minimal idempotent. Then:

1. $H^G_*(X,\mathcal{G}_E) \cong H^G_*(X)_E$.

2. The complex of coinvariants $C_*(X,\mathcal{G}_E)_G$ is finite-dimensional.

Baum, Higson, and Plymen also give conjectural descriptions of the complex $C_*(X,\mathcal{G}_E)_G$ in several cases [BHP00, Conjectures 6.2, 6.8, and 7.13].

For each polysimplex $\sigma \subset X$, the space $\mathcal{H}(G_\sigma)_{\text{Ad}} \cap E \mathcal{H}(G)$ is a $G_\sigma$-submodule—and therefore a direct summand—of $\mathcal{H}(G_\sigma)_{\text{Ad}}$. This implies that the chain complex $C_*(X,\mathcal{G}_E)$ is a direct-summand of $C_*(X,\mathcal{G})$, and that the same is true at the level of coinvariants. This inclusion $C_*(X,\mathcal{G}_E)_G \hookrightarrow C_*(X,\mathcal{G})_G$ induces a map

$$C_*(X,\mathcal{G}_E)_G \to \lim_{\text{Cone}} (C_*(X,\mathcal{G})_G, E_c)$$

for any lift of $E_c$ to $C_*(X,\mathcal{G})$. By Theorem 3.2.19, this map of complexes induces a map in chamber homology

(3.2.23) $$H^G_*(X,\mathcal{G}_E) \to H^G_*(X)_E,$$

which is independent of the choice of lift $E_c$. The following is a slight refinement of [BHP00, Conjecture 5.3]:

**Conjecture 3.2.24.** With the same hypotheses as Conjecture 3.2.22, the map (3.2.23) is an isomorphism.

In terms of smooth homology, this conjecture states the following: Consider the submodule $S_E \subset \mathcal{H}(G_\sigma)_{\text{Ad}}$ defined by

$$S_E = \sum_{K \subset G} \mathcal{H}(K)_{\text{Ad}} \cap E \mathcal{H}(G),$$
the sum being taken over the set of compact, open subgroups of $G$. Then the natural inclusion

$$S_E \hookrightarrow \varinjlim (\mathcal{H}(G_c)_{\text{Ad}}, E_c)$$

induces an isomorphism in smooth homology.

The following observation might be useful in approaching the finite-dimensionality assertion in Conjecture 3.2.22:

**Lemma 3.2.25.** (1) There is a lift of $E_c$ to $C_\s(X, \mathcal{G})$ which restricts to the identity on the direct-summand $C_\s(X, \mathcal{G}_E)$.

(2) For $E_c$ as in part (1), the map of chain complexes

$$C_\s(X, \mathcal{G}_E)_G \to \varinjlim (C_\s(X, \mathcal{G}), E_c)_G$$

is injective.

**Proof.** The map $E_c : \mathcal{H}(G_c)_{\text{Ad}} \to \mathcal{H}(G_c)_{\text{Ad}}$ restricts to the identity on the image of $C_0(X, \mathcal{G}_E)$ under the augmentation $\alpha : C_0(X, \mathcal{G}) \to \mathcal{H}(G_c)_{\text{Ad}}$. This immediately implies part (1). Part (2) is obvious. 

In each of the examples considered in the following three sections, we find a lift of $E_c$ to $C_\s(X, \mathcal{G})$ having the property that the complex $\varinjlim (C_\s(X, \mathcal{G})_G, E_c)$ is finite-dimensional. However, we have not been able to demonstrate that there is such a lifting that restricts to the identity on $C_\s(X, \mathcal{G}_E)$ (except in the trivial case of $G = \text{GL}_1(F)$), and so Conjecture 3.2.22 remains open.

**3.2.6 Example: GL$_1(F)$**

Let $G = \text{GL}_1(F) = F^\times \cong \mathcal{O}^\times \times \mathbb{Z}$ be the multiplicative group of a $p$-adic field. Then $G$ acts on the line $X = \mathbb{R}$, triangulated by placing a vertex at each integer point. The generator $\varpi$ of $\mathbb{Z}$ acts by $\varpi(x) = x + 1$, and $\mathcal{O}^\times$ acts trivially. In this short section, we compute the action of the central idempotents on chamber homology, which is a very straightforward matter.
Every representation of $G$ is compact-mod-center, and every irreducible representation of $G$ is one-dimensional. Each $\pi \in \hat{G}$ decomposes uniquely as a product $\psi \chi$, where $\psi \in \Psi(G) = \hat{\mathbb{Z}}$ is an unramified character, and $\chi \in \hat{O^\times} = \hat{\mathbb{O}^\times}$ is a character of $\mathbb{O}^\times$. The Hecke algebra $\mathcal{H}(G)$ decomposes as

$$\mathcal{H}(G) \cong \mathcal{H}(\mathbb{O}^\times) \otimes \mathcal{H}(\mathbb{Z}).$$

The idempotent $E_\chi \in \mathcal{H}(G)$ corresponding to an orbit $\Psi(G) \chi \subset \hat{G}$ acts by projecting $\mathcal{H}(\mathbb{O}^\times)$ onto its $\chi$-isotypical part. Every element of $\mathcal{H}(G)$ is a (possibly infinite) linear combination of these $E_\chi$.

Turning to chamber homology, we have $G_c = G^\sigma = \mathbb{O}^\times$. The restriction operator $1_{G_c}$ thus acts on $\mathcal{H}(G) = \mathcal{H}(\mathbb{O}^\times) \otimes \mathcal{H}(\mathbb{Z})$ by projecting $\mathcal{H}(\mathbb{Z})$ onto the space of multiples of the identity. Thus $1_{G_c}$ commutes with the Bernstein idempotents $E_\chi$ in this case.

Since $G$ is abelian, and all of the isotropy groups $G_\sigma$ are equal to $\mathbb{O}^\times$, the coefficient system $\mathcal{G}$ is constant:

$$\mathcal{G}(\sigma) = \mathcal{H}(\mathbb{O}^\times) \cong \bigoplus_{\chi \in \mathbb{O}^\times} \mathcal{C}_\chi,$$

with identity transition maps and trivial $G$-action.

Not only is there an obvious lift of each $(E_\chi)_c$ to an endomorphism of the chain complex $C_\bullet(X, \mathcal{G})$, but this lift comes from a map of coefficient systems:

$$(E_\chi)_c : \mathcal{H}(\mathbb{O}^\times) \to \mathcal{H}(\mathbb{O}^\times), \quad \sum_{\chi'} \lambda_{\chi'} \mapsto \lambda_\chi.$$

Since the map $(E_\chi)_c$ is already an idempotent at the level of chain complexes, there is no need to take the direct limit in Theorem 3.2.19: one has

$$H^G_\bullet(X)_{E_\chi} = H^G_\bullet(X, (E_\chi)_c \mathcal{G}) = H^G_\bullet(X, \mathcal{C}) \cong H_\bullet(X/G, \mathbb{C}) = H_\bullet(S^1, \mathbb{C}).$$

The case of $GL_1(F)$ is therefore very straightforward—too much so to give much insight into the general problem. In the next two sections, we consider some less trivial examples.
3.2.7 Example: Compact representations of $\text{SL}_n(F)$

Let $G = \text{SL}_n(F)$, $F$ a $p$-adic field, and let $X$ denote the corresponding Bruhat-Tits building (see Example 2.3.15). Let $\pi : G \to \text{GL}(V)$ be an irreducible, compact representation of $G$. Theorem 3.1.11 associates to $\pi$ an idempotent $E_\pi \in \mathcal{I}(G)$. We computed the chamber homology groups $H^G_\ast(X)_{E_\pi}$, in Example 3.2.17:

(3.2.26) \[ H^G_l(X)_{E_\pi} = \begin{cases} \mathbb{C} & \text{if } l = 0, \\ 0 & \text{if } l > 0. \end{cases} \]

This computation was based on identifying $HH_\ast(\mathcal{H}(G)_{E_\pi}) \cong HH_\ast(\mathbb{C})$. In this section, we present another proof of (3.2.26), using the building and Theorem 3.2.19. The computation is based on the following theorem of Bushnell and Kutzko:

**Theorem 3.2.27.** [BK93, Theorem 2.1] There is a compact open subgroup $K \subset G$ and an irreducible representation $\rho : K \to \text{GL}(W)$ such that $\pi \cong \text{ind}_K^G \rho$. \[ \square \]

**Remark 3.2.28.** The arguments below apply to arbitrary $G$ and any compact representation of the form $\text{ind}_K^G \pi$.

By the transitivity of induction, we might as well assume that $K$ is a maximal compact subgroup: i.e., the isotropy group $G_{v_0}$ of a vertex $v_0 \in X$. Let $e_\rho \in \mathcal{H}(K)$ be the idempotent associated to $\rho$:

$$e_\rho(k) = \frac{\dim(\rho)}{\text{vol}(K)} \text{Trace}(\rho(k^{-1})).$$

The idempotent $E = E_\pi : \mathcal{H}(G) \to \mathcal{H}(G)$ is defined as the composition

(3.2.29) \[ \mathcal{H}(G) \xrightarrow{\pi} \text{End}_\mathbb{C}(V)^0 \xrightarrow{d(\pi)m} \mathcal{H}(G), \]

where $m : \text{End}_\mathbb{C}(V)^0 \to \mathcal{H}(G)$ sends each finite-rank operator $T$ to the function

$$m(T)(g) = \text{Trace}(T\pi(g^{-1})), $$

and $d(\pi) = \frac{\dim(\rho)}{\text{vol}(K)}$ is the formal degree.
Lemma 3.2.30. The $G$-module $\text{End}_C(V)_0^{\text{Ad}}$ is projective.

Proof. The vector-space isomorphism $V \otimes_C \tilde{V} \to \text{End}_C(V)_0^{\text{Ad}}$, sending $v \otimes \tilde{v}$ to the corresponding rank-one operator, intertwines the adjoint action on $\text{End}_C(V)_0^{\text{Ad}}$ with the diagonal $G$-action on $V \otimes_C \tilde{V}$. Being a compact representation of $G$, $V$ is projective (Theorem 3.1.11). Lemma 2.2.6 then implies that $V \otimes_C \tilde{V}$, with the diagonal action of $G$, is also projective. □

Thus

$$0 \to \text{End}_C(V)_0^{\text{Ad}} \xrightarrow{\text{id}} \text{End}_C(V)_0^{\text{Ad}} \to 0$$

is a projective resolution.

Lemma 3.2.31. The trace $\text{End}_C(V)_0^{\text{Ad}} \to \mathbb{C}$ gives an isomorphism $(\text{End}_C(V)_0^{\text{Ad}})_G \cong \mathbb{C}$, and $\text{Trace}(\pi(e_\rho))$ is nonzero.

Proof. The representation $V$ is generated by the subspace $V_0$ of functions supported on $K$. It follows from this that $\text{End}_C(V)_0^{\text{Ad}}$ is generated, under the adjoint action, by its subspace $\text{Hom}_C(V, V_0)$. One has $gV_0 \cap V_0 = 0$ if $g \not\in K$, and so

$$(\text{End}_C(V)_0^{\text{Ad}})_G \cong (\text{Hom}_C(V_0, V)_\text{Ad})_K \cong (\text{Hom}_C(V, V_0)_\text{Ad})_K = \text{Hom}_K(V, V_0).$$

Frobenius reciprocity implies that

$$\text{Hom}_K(\rho, V|_K) \cong \text{Hom}_G(\text{ind}_K^G \rho, V) = \text{End}_G(V) = \mathbb{C},$$

and so the $\rho$-isotypical subspace $\pi(e_\rho)V$ of $V$ contains only one copy of $\rho$. Since $V_0 \cong \rho$, we have $V_0 = \pi(e_\rho)V$.

Now, since $K$ commutes with $e_\rho$, we have

$$(\text{End}_C(V)_0^{\text{Ad}})_G \cong \text{Hom}_K(V_0, V_0) \cong \text{Hom}_K(V_0, V_0) = \mathbb{C} \pi(e_\rho)$$

by Schur’s lemma. □

Remark 3.2.32. The fact that $(\text{End}_C(V)_0^{\text{Ad}})_G \cong \mathbb{C}$ also follows from Proposition 2.4.6,
Theorem 3.1.11, and the Morita equivalence $\text{End}_C(V)^0 \sim C$:

$$(\text{End}_C(V)^0_{\text{Ad}})_G = H_0(G, \text{End}_C(V)^0_{\text{Ad}}) \cong HH_0(\mathcal{H}(G), \text{End}_C(V)^0)$$

$$\cong HH_0(\text{End}_C(V)^0) \cong HH_0(C) = C.$$ 

We will lift $E_c$ to $C_*(X, \mathcal{G})$ by constructing a commuting diagram

\[
\begin{array}{cccccc}
\mathcal{H}(G_c) & \xrightarrow{\pi} & \text{End}_C(V)^0_{\text{Ad}} & \xrightarrow{d(\pi)_{1G_c m}} & \mathcal{H}(G_c) \\
\alpha \downarrow & & \text{id} \downarrow & & \alpha \downarrow \\
C_0(X, \mathcal{G}) & \xrightarrow{\pi_0} & \text{End}_C(V)^0_{\text{Ad}} & \xrightarrow{d(\pi)m_0} & C_0(X, \mathcal{G}) \\
\uparrow & & \uparrow & & \uparrow \\
C_1(X, \mathcal{G}) & \xrightarrow{\pi_1} & 0 & \xrightarrow{d(\pi)m_1} & C_1(X, \mathcal{G}) \\
\vdots & & \vdots & & \vdots \\
\end{array}
\]

of $G$-equivariant maps. Lemma 3.2.31 will then make it easy to compute the action of $E_c$ on coinvariants.

Clearly $\pi_0 = \pi\alpha$ is forced on us, as is $\pi_l = m_l = 0$ for $l \geq 1$. So we are left to construct $m_0$.

The space $\mathcal{H}(K)_{\text{Ad}}$ sits inside both $\mathcal{H}(G_c)_{\text{Ad}}$ and $C_0(X, \mathcal{G})$ as a $K$-invariant subspace. We choose and fix a $K$-equivariant map

$$s : \mathcal{H}(G_c)_{\text{Ad}} \to C_0(X, \mathcal{G})$$

with $\alpha s = \text{id}$ and $s|_{\mathcal{H}(K)} = \text{id}$.

(The existence of such a section is guaranteed by the compactness of $K$.)

**Lemma 3.2.34.** The composition

$$\text{Hom}_C(V, V_0) \xrightarrow{m} \mathcal{H}(G)_{\text{Ad}} \xrightarrow{1G_c} \mathcal{H}(G_c)_{\text{Ad}} \xrightarrow{s} C_0(X, \mathcal{G})$$

extends uniquely to a $G$-equivariant map $m_0 : \text{End}_C(V)^0_{\text{Ad}} \to C_0(X, \mathcal{G})$ making the diagram (3.2.33) commute.
Proof. Since $\text{End}_C(V)_0$ is generated over $G$ by $\text{Hom}_C(V,V_0)$, uniqueness is clear. The adjoint action of an element $g \in G$ stabilizes $\text{Hom}_C(V,V_0)$ if and only if $g \in K$, and $s, 1_{G_c}$ and $m$ are all $K$-equivariant; this proves the existence of $m_0$. The fact that $\alpha m_0 = 1_{G_c} m$ follows from $\alpha s = \text{id}$. \hfill \Box

To the representation $\rho$ we associate a chain $c_\rho \in C_0(X, \mathcal{G})$, supported at the vertex $v_0$ and having there

$$(c_\rho)_{v_0}(k) := e_\rho(k) = \frac{\dim(\rho)}{\text{vol}(K)} \text{Trace}(\rho(k^{-1})) = d(\pi) \text{Trace}(\rho(k^{-1})).$$

The same construction applies to each irreducible representation of each vertex-stabilizer $G_v$.

Let $E_s := d(\pi)m_s \circ \pi_*$. Then $E_s$ is an endomorphism of $C_s(X, \mathcal{G})$ lifting the map $E_c : \mathcal{H}(G_c)_{\text{Ad}} \to \mathcal{H}(G_c)_{\text{Ad}}$.

**Theorem 3.2.35.** (1) $E_0(c_\rho) = c_\rho$

(2) $\text{image}(E_s) = \mathbb{C}c_\rho$ on $G$-coinvariants.

(3) $\lim (C_s(X, \mathcal{G})_G, E_s)$ is isomorphic to the subcomplex

$$(3.2.36) \quad 0 \longrightarrow 0 \longrightarrow \mathbb{C}c_\rho \oplus 0 \longrightarrow 0$$

of $C_s(X, \mathcal{G})_G$.

**Remark 3.2.37.** If the representation $\rho$ of $K$ is not induced from the isotropy group of any higher-dimensional simplex in $X$, then the complex 3.2.36 coincides with Baum, Higson and Plymen’s conjectural description of the “local complex” $C_s(X, \mathcal{G}_G)$; see [BHP00, Conjecture 6.8] and Remark 3.2.21. Note that Theorem 3.2.35 offers evidence for, but not a proof of, Conjecture 3.2.22 in this case.

**Proof of Theorem 3.2.35.** (1) We have $\pi \alpha(c_\rho) = \pi(e_\rho) \in \text{Hom}_K(V, V_0)$ (see the proof of Lemma 3.2.31). Therefore, by the definition of $m_0$,

$$m_0 \circ \pi \circ \alpha(c_\rho) = s \circ 1_{G_c} \circ m \circ \pi(e_\rho).$$
We will show that \( d(\pi)m \circ \pi(e_\rho) = e_\rho \in \mathcal{H}(K) \); it will then follow from the definitions that

\[
E_0(c_\rho) = d(\pi)m_0 \circ s \circ \pi \circ \alpha(\rho) = c_\rho.
\]

For each \( g \in G \), we have

\[
d(\pi)m \circ \pi(e_\rho)(g) = \frac{\dim(\rho)}{\vol(K)} \Trace(\pi(e_\rho g^{-1})).
\]

If \( g \in K \), then \( \Trace(\pi(e_\rho g^{-1})) = \Trace(\rho(g^{-1})) \), because \( \pi \) contains \( \rho \) with multiplicity one; therefore \( d(\pi)m \circ \pi(e_\rho)(g)|_K = \rho|_K \).

Now suppose that \( g \notin G \). Since \( \pi(e_\rho) \) acts by restricting functions from \( G \) to \( K \), and \( Kg \cap K = \emptyset \), we have \( \pi(e_\rho g^{-1}e_\rho) = 0 \). Therefore \( \Trace(\pi(e_\rho g^{-1})) = 0 \), and so \( d(\pi)m \circ \pi(e_\rho)(g) = e_\rho(g) = 0 \).

(2) Lemma 3.2.31 implies that the image of \( E_a \) on coinvariants has dimension at most one. Part (1) showed that the image contains \( c_\rho \).

(3) Parts (1) and (2) together imply that \( E_a \) is an idempotent on coinvariants, and so the complex \( \lim_{\rightarrow} (C_\ast(X, \mathcal{G})^0, E_a) \) is isomorphic to the image of \( E_a \). Parts (1) and (2) imply that this image is isomorphic to (3.2.36).

\[ \square \]

**Remark 3.2.38.** It is easy to compute a precise formula for the action of \( E_a \) on \( G \)-coinvariants. Let \( \tau \) be an irreducible representation of \( K' \), where \( K' \) is the stabilizer of a vertex in \( X \). Let \( c_\tau \in C_0(X, \mathcal{G}) \) be the corresponding chain. By Lemma 3.2.31, the trace \( \End_C(V)^0 \to \mathbb{C} \) induces an isomorphism \( \left( \End_C(V)^0 \right)_G \cong \mathbb{C} \). Comparing with the proof of part (2) above, we must have

\[
E_0(c_\tau) = (a \Trace(\pi(e_\tau)))c_\rho = \left( a \dim(\tau) \dim(\Hom_{K'}(\tau, \pi)) \right) c_\rho,
\]

for some constant \( a \) independent of \( \tau \). Putting \( \tau = \rho \), we find that

\[
1 = a \dim(\rho) \dim(\Hom_{K'}(\rho, \pi)) = a \dim(\rho),
\]
and so

\[ E_0(c_\tau) = \left( \frac{\dim(\tau)}{\dim(\rho)} \dim(\Hom_K(\tau, \pi)) \right) c_\rho. \]

In particular, \( E_0(c_\tau) \) is nonzero if and only if \( \tau \) occurs in \( \pi \).

**Corollary 3.2.40.** Let \( G = \text{SL}_2(F) \), and consider the compact open subgroups

\[ K = \text{SL}_2(\mathcal{O}), \quad K' = \begin{bmatrix} \mathcal{O} & p^{-1} \\ p & \mathcal{O} \end{bmatrix}, \quad \text{and} \quad I = K_0 \cap K_1. \]

Suppose that \( \pi \) is a compact irreducible representation of \( G \), with \( \pi \cong \text{ind}_K^G \rho \cong \text{ind}_{K'}^{G'} \rho' \)

for representations \( \rho \) and \( \rho' \) of \( K \) and \( K' \), respectively. Then there is a virtual representation \( \rho'' \in R_C(I) \) such that

\[ \rho = \text{ind}_I^K \rho'' \quad \text{and} \quad \rho' = \text{ind}_I^{K'} \rho''. \]

**Proof.** First notice that \( \text{vol}(K) = \text{vol}(K') \), because these groups are conjugate in \( \text{GL}_2(F) \). The equality \( (3.1.14) \) thus implies that \( \dim(\rho) = \dim(\rho') = d(\pi) \).

Consider the chains \( c_\rho, c_{\rho'} \in C_0(X, \mathcal{G}) \), and denote by \([c_\rho]\) and \([c_{\rho'}]\) the corresponding classes in \( H_0^G(X) \). Theorem 3.2.35 and the formula \((3.2.39)\) imply that

\[ [c_\rho] = [E_0(c_\rho)] = \frac{\dim(\rho)}{\dim(\rho')} \dim(\Hom_K(\rho, \pi)) \frac{\dim(\rho)}{\dim(\rho')} [c_{\rho'}] = [c_{\rho'}]. \]

Using the the representation-theoretic picture of chamber homology (Lemma 2.3.26), we conclude that there is a virtual character \( \psi \in R_C(I) \) such that \( c_\rho = \text{ind}_I^K \psi \) and \( c_{\rho'} = \text{ind}_I^{K'} \psi \).

Since \( c_\rho \) is equal to \( d(\pi) \) times the character of the contragredient representation \( \hat{\rho} \), and similarly for \( c_{\rho'} \), it follows that the function \( \rho''(g) := d(\pi)^{-1} \psi(g^{-1}) \) on \( I \) satisfies \( \rho = \text{ind}_I^K \rho'' \) and \( \rho' = \text{ind}_I^{K'} \rho'' \).

**Remark 3.2.41.** Baum, Higson and Plymen note that a stronger version of Corollary 3.2.40
would follow from their conjecture [BHP00, Conjecture 6.8]: the truth of that conjecture would imply that $\rho''$ is an actual representation, rather than just a virtual representation.

### 3.2.8 Example: Compact-mod-center representations of $\text{GL}_2(F)$

Now let $G = \text{GL}_2(F)$, $F$ a $p$-adic field. Every irreducible, compact-mod-center representation $\pi$ of $G$ determines an idempotent $E_\pi$ in $\mathcal{Z}(G)$, as explained in Section 3.1.3. The associated direct-summands $H^G_\ast(X)_{E_\pi}$ of chamber homology were computed in Example 3.2.17, using the Morita equivalence $\mathcal{H}(G)_{E_\pi} \sim \mathcal{O}(\mathbb{C}^X)$. In this section, we show how Theorem 3.2.19 leads to an alternative computation of these homology groups.

The following notation will be used in this section:

\[
G^0 = \{ g \in G \mid \det(g) \in \mathcal{O}^\times \}, \quad \gamma = \begin{bmatrix} 0 & 1 \\ \wp & 0 \end{bmatrix}, \quad \Gamma = \{ \gamma^n \mid n \in \mathbb{Z} \},
\]

\[
K = \text{GL}_2(\mathcal{O}), \quad K^\gamma = \gamma^{-1}K\gamma, \quad I = K \cap K^\gamma, \quad Z = Z(G).
\]

Recall that $\wp$ denotes a generator of the ideal $\mathfrak{p} \subset \mathcal{O}$. Let us also recall that when $L$ is a subgroup of a group $H$, $\rho$ is a representation of $L$, and $h$ is an element of $H$, then $\rho^h$ is the representation $\rho(h\cdot h^{-1})$ of the group $L^h = h^{-1}Lh$.

The following identification of the compact-mod-center representations of $G$ is due to Kutzko.

**Theorem 3.2.42** ([Kut78]). Every irreducible, compact-mod-center representation of $G$ is compactly induced, either from a representation of the subgroup $ZK \subset G$, or from a representation of the subgroup $\Gamma I \subset G$. \qed

We will focus on the first case: representations induced from $ZK$ (the so-called unramified representations). The ramified case may be handled by a similar method, yielding essentially the same result: see Remark 3.2.51. The compact-mod-center representations of $\text{GL}_n(F)$ induced from $Z\text{GL}_n(\mathcal{O})$ can be dealt with by the same method. The analysis of the ramified cases is slightly more involved for $\text{GL}_n(F)$ than for $\text{GL}_2(F)$, but it is unlikely that this analysis would reveal any genuinely new difficulties.
We suppose, then, that $\rho$ is an irreducible representation of $\mathbb{Z}K$, such that the compactly induced representation $\pi = \text{ind}_{\mathbb{Z}K}^G \rho$ is irreducible.

**Lemma 3.2.43.** Restricting $\pi$ to the subgroup $G^o$ gives

$$\pi|_{G^o} \cong \text{ind}_K^{G^o} \rho \oplus \text{ind}_K^{G^o} \rho^\gamma,$$

and the two direct-summands are irreducible, inequivalent representations of $G^o$.

**Proof.** The double-coset space $\mathbb{Z}K\backslash G/G^o$ consists of two points, represented by the matrices $1$ and $\gamma$. The formula for $\pi|_{G^o}$ thus follows from Mackey’s induction-restriction formula (due, in this context, to Kutzko [Kut77]):

$$\text{res}_{G^o}^G \text{ind}_{\mathbb{Z}K}^G \rho \cong \bigoplus_{l \in \mathbb{Z}K\backslash G/G^o} \text{ind}_{G^o\cap(\mathbb{Z}K)^l}^G \rho^j = \text{ind}_{\mathbb{Z}K}^G \rho \oplus \text{ind}_{\mathbb{Z}K}^G \rho^\gamma.$$

We will show that $\text{ind}_{\mathbb{Z}K}^G \rho$ is an irreducible representation of $G^o$; this will imply that $\text{ind}_{\mathbb{Z}K}^G \rho^\gamma = \left(\text{ind}_{\mathbb{Z}K}^G \rho\right)^\gamma$ is irreducible as well.

Since $\text{ind}_{\mathbb{Z}K}^G \rho$ is a compact representation of $G^o$, it is isomorphic to a direct sum of irreducibles (Theorem 3.1.11), and so the converse of Schur’s lemma applies: $\text{ind}_{\mathbb{Z}K}^G \rho$ is irreducible if and only if $\text{End}_{G^o}(\text{ind}_{\mathbb{Z}K}^G \rho)$ is one-dimensional. Another application of the Mackey formula gives

(3.2.44) $$\text{End}_{G^o}(\text{ind}_{\mathbb{Z}K}^G \rho) \cong \bigoplus_{l \in \mathbb{Z}K\backslash G/K} \text{Hom}_{K\cap K^l}(\rho, \rho^j).$$

For each $l \in K\backslash G^o/K$, we have

(3.2.45) $$\text{Hom}_{K\cap K^l}(\rho, \rho^j) = \text{Hom}_{\mathbb{Z}K\cap \mathbb{Z}K^l}(\rho, \rho^j),$$

because $Z$ acts on both $\rho$ and $\rho^j$ by the same character. Since $\pi = \text{ind}_{\mathbb{Z}K}^G \rho$ is irreducible, Schur’s lemma and Mackey’s formula give

$$\text{End}_{G}(\text{ind}_{\mathbb{Z}K}^G \rho) \cong \bigoplus_{l \in \mathbb{Z}K\backslash G/\mathbb{Z}K} \text{Hom}_{\mathbb{Z}K\cap \mathbb{Z}K^l}(\rho, \rho^h) \cong \mathbb{C},$$
the only nonzero summand being the one associated to $h \in ZK$. Combined with (3.2.45), this implies that for each $l \in K \setminus G^0 / K$ we have

$$\text{Hom}_{K \cap K^l}(\rho, \rho^l) \cong \begin{cases} 
\mathbb{C} & \text{if } l \in ZK, \\
0 & \text{otherwise}.
\end{cases}$$

Since $ZK \cap G^0 = K$, we now conclude from (3.2.44) that $\text{End}_{G^0}(\text{ind}^G_{K^0} \rho)$ is one-dimensional. So $\text{ind}^G_{K^0} \rho$ is irreducible.

A similar argument shows that $\text{ind}^G_{K^0} \rho$ and $\text{ind}^G_{K^\gamma} \rho^\gamma$ are inequivalent: we have

$$\text{Hom}_{G^0}(\text{ind}^G_{K^0} \rho, \text{ind}^G_{K^\gamma} \rho^\gamma) = \bigoplus_{l \in K^\gamma \setminus G^0 / K} \text{Hom}_{K \cap K^\gamma l}(\rho, \rho^\gamma l)$$

$$= \bigoplus_{l \in K^\gamma \setminus ZK^\gamma / K} \text{Hom}_{ZK \cap ZK^\gamma l}(\rho, \rho^\gamma l)$$

$$\subseteq \bigoplus_{l \in ZK^\gamma \setminus G^0 / K} \text{Hom}_{ZK \cap ZK^\gamma h}(\rho, \rho^\gamma h).$$

The last line is isomorphic to the space of intertwiners of the irreducible representation $\text{ind}^G_{ZK^0} \rho$, and so the only nonzero summand is the one associated to $h = \gamma^{-1}$. The double coset $(ZK^\gamma)\gamma^{-1}(ZK)$ contains no element of $G^0$, and so for each $l \in K^\gamma \setminus G^0 / K$ we must have $\text{Hom}_{K \cap K^\gamma l}(\rho, \rho^l) = 0$. Therefore the irreducible representations $\text{ind}^G_{K^0} \rho$ and $\text{ind}^G_{K^\gamma} \rho^\gamma$ do not intertwine, and so they are inequivalent.

To simplify the notation, we write $\pi^0 = \text{ind}^G_{K^0} \rho$, and $\pi^\gamma = \text{ind}^G_{K^\gamma} \rho^\gamma$. According to Theorem 3.1.20, the idempotent $E_\pi \in \mathcal{H}(G)$ is given by

$$E_\pi = E_{\pi^0} + E_{\pi^\gamma}.$$

We are interested in the operator

$$(E_\pi)_c = (E_{\pi^0})_c + (E_{\pi^\gamma})_c = 1_{G_c} E_{\pi^0} 1_{G_c} + 1_{G_c} E_{\pi^\gamma} 1_{G_c} \in \text{End}_G(\mathcal{H}(G_c)_{\text{Ad}}).$$

Note that $G_c = G^0_c$. Note also that $E_{\pi^0} = \gamma^{-1} E_{\pi^\gamma}$, as operators on $\mathcal{H}(G^0)_{\text{Ad}}$. 


We now turn to chamber homology. Recall the definition of the Bruhat-Tits building of $G$ (Example 2.3.31): $X^0$ denotes the Bruhat-Tits building for $\text{SL}_2(F)$; the full group $G = \text{GL}_2(F)$ acts on $X^0$, and $Z$ acts trivially [Ser03, II.1.3]. We then let $X = X^0 \times L$, where $L \cong \mathbb{R}$ is a line, triangulated with a vertex at each integer point. The group $G$ acts on $X$, via
\[ g(x, l) = (gx, l + \text{val}(\det(g))). \]

There is an isomorphism of $G$-equivariant chain complexes
\[ C_\ast(X, G) := C_\ast(X^0, G^0) \otimes \mathbb{C} C_\ast(L, \mathbb{C}). \]

The augmentation $\alpha : C_0(X, G) \to \mathcal{H}(G_c)_{\text{Ad}}$ is then given by
\[ \alpha(c \otimes c') := \sum_{v \in (X^0)^{(0)}} (c_1)_v(c_2)_v. \]

Computing the complex of $G$-coinvariants $C_\ast(X, G)_G$, we find that $H_\ast^G(X)$ is the homology of the total complex of the bicomplex
\[ (3.2.46) \quad R_C(K) \oplus R_C(K^\gamma) \xleftarrow{\text{ind}_I^K - \text{ind}_I^{K\gamma}} R_C(I) \]
\[ \quad \text{id} - \gamma \downarrow \quad \text{id} - \gamma \downarrow \]
\[ R_C(K) \oplus R_C(K^\gamma) \xleftarrow{\text{ind}_I^K - \text{ind}_I^{K\gamma}} R_C(I) \]

Let $v_0 \in X^0$ be the vertex fixed by $K$; then $\gamma^{-1}v_0$ is the vertex fixed by $K^\gamma$. As in Section 3.1.2, choose a $K$-equivariant splitting
\[ s : \mathcal{H}(G_c)_{\text{Ad}} \to C_0(X^0, G) \]
of the augmentation $C_0(X^0, G) \to \mathcal{H}(G_c)$, such that $s|_{\mathcal{H}(K)} = \text{id}$. Use $s$ to define a $G^0$-equivariant lifting $E^0_\ast$ of $(E_{x^0})_c$ to $C_\ast(X^0, G)$, as in the previous section. (The fact that we are now working over $G^0 = \text{GL}_2(F)^0$, instead of over $\text{SL}_2(F)$, does not affect the construction in any way.)
Since $E_{\pi^*} = \gamma^{-1}E_{\pi^*}\gamma$, the map $E^{\gamma}_a := \gamma^{-1}E^{\gamma}_a\gamma$ similarly determines a $G^\circ$-equivariant lifting of $(E_{\pi^*})_c$.

**Lemma 3.2.47.** The endomorphism $E_a := (E^o_a + E^{\circ\gamma}_a) \otimes \text{id}$ of the chain complex $C_\bullet(X, \mathcal{G})$ is a $G$-equivariant lifting of the map $(E_\pi)_c : \mathcal{H}(G_c)_\text{Ad} \to \mathcal{H}(G_c)_\text{Ad}$.

**Proof.** The map $E_a$ is a lifting of $(E_\pi)_c$, because $E^o_a$ and $E^{\circ\gamma}_a$ are, respectively, liftings of $(E_\pi^o)_c$ and $(E_{\pi^*})_c$. Both $E^o_a$ and $E^{\circ\gamma}_a$ are $G^\circ$-equivariant, so $E_a$ is likewise. Since $G$ is generated by $G^\circ$ and $\gamma$, we are left to show that $\gamma^{-1}E_a\gamma = E_a$. This is true because $\gamma^2$ lies in the center of $G$, and therefore acts trivially on $C_\bullet(X, \mathcal{G})$.

As in the previous section, we consider the chain $c_\rho \in C_0(X, \mathcal{G})$ supported at $v_0$ and having there

$$(c_\rho)_{v_0}(k) = e_\rho(k) = \frac{\dim(\rho)}{\text{vol}(K)} \text{Trace}(\rho(k^{-1})).$$

We likewise consider $c_{\rho\gamma} = c_\rho(\gamma \star \gamma^{-1})$.

**Theorem 3.2.48.** (1) The map $E_a$ induces an idempotent operator on the complex of coinvariants $C_\bullet(X, \mathcal{G})_G$.

(2) The component $H^G_\bullet(X)_{E_a}$ of chamber homology is isomorphic to the homology of the total complex of the subcomplex

$$(3.2.49) \quad \begin{array}{ccc} Cc_\rho \oplus Cc_{\rho^\gamma} & \longrightarrow & 0 \\ \text{id} \downarrow & & \downarrow \\ Cc_\rho \oplus Cc_{\rho^\gamma} & \longrightarrow & 0 \end{array}$$

of $(3.2.46)$.

(3) One has

$$H^G_\bullet(X)_{E_a} \cong \begin{cases} \mathbb{C} & \text{if } n = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 3.2.50.** Baum, Higson and Plymen propose the complex 3.2.49 as one of two possibilities for the “local complex” $H^G_\bullet(X, \mathcal{G}_{E_a})$; see [BHP00, page 26].
Proof of Theorem 3.2.48. (1) Theorem 3.2.35 implies that the maps $E^0_\alpha$ and $E^{0\gamma}_\alpha$ are already idempotents on $G^\circ$-coinvariants. We will show that $E^{0\gamma}_\alpha E^0_\alpha = E^0_\alpha$ on $G^\circ$-coinvariants, from which it will follow that $E^0_\alpha$ is an idempotent on $G^\circ$-coinvariants and, a fortiori, on $G$-coinvariants.

The operator $\pi^{0\gamma} \alpha(c_\rho)$ is the projection of $\pi^{0\gamma}$ onto its $\rho$-isotypical component. Since

$$\text{Hom}_K(\rho, \pi^{0\gamma}) \cong \text{Hom}_G(\pi^0, \pi^{0\gamma}) = 0,$$

by Frobenius reciprocity and Lemma 3.2.43, we have $\pi^{0\gamma} \alpha(c_\rho) = 0$. By definition, the map $E^0_\alpha$ factors through $\pi^{0\gamma} \alpha$, and so $E^0_\alpha(c_\rho) = 0$. Theorem 3.2.35 implies that the image of $E^0_\alpha$ on $G^\circ$-coinvariants is spanned by $c_\rho$, and so $E^{0\gamma}_\alpha E^0_\alpha = 0$ on $G^\circ$-coinvariants. Conjugating by $\gamma$, we see that $E^0_\alpha E^{0\gamma}_\alpha$ is also zero on $G^\circ$-coinvariants, and this completes the proof of part (1).

(2) In view of part (1), Theorem 3.2.19 implies that $H^{G}_n(X)_{E^0}$ is the homology of the image of $E^0_\alpha$ on $G$-coinvariants. Theorem 3.2.35 implies that the image of $E^0_\alpha$ is the given complex.

Part (3) follows immediately from part (2). \qed

Remark 3.2.51. We conclude this section with a few remarks on how the above computation may be modified to deal with the case of a ramified representation. The end result is essentially the same as in the unramified case; in particular, one has

$$H^G_n(X)_{E^0} \cong \begin{cases} 
\mathbb{C} & \text{if } n = 0 \text{ or } 1, \\
0 & \text{otherwise}
\end{cases}$$

for every irreducible, compact-mod-center representation of $G = \text{GL}_2(F)$.

Let $\rho$ be a representation of the open, compact-mod-center subgroup $\Gamma I \subset G$, and suppose that the induced representation $\pi := \text{ind}^G_{\Gamma I}\rho$ is irreducible. Since the double-coset space $\Gamma\backslash G/G^\circ$ consists of a single point, Mackey’s formula gives $\pi|_{G^\circ} \cong \text{ind}^G_{I}\rho$. Another
computation with the Mackey formula shows that

\[ \text{End}_{G\gamma}(\pi_{G\gamma}) \cong \text{End}_{ZI}(\rho_{ZI}). \]

Since \( \Gamma I \) is the semidirect product for the action of the two-element group \( \Gamma/(Z \cap \Gamma) \) on the normal subgroup \( ZI \), Clifford theory (cf. [CR62, §49]) tells us that either:

1. \( \rho_{ZI} \) is irreducible, and \( \rho_{ZI} \cong (\rho_{ZI})\gamma \); or
2. \( \rho_{ZI} \cong \rho_1 \oplus \rho_1\gamma \), where \( \rho_1 \) is an irreducible representation of \( ZI \), and \( \rho_1 \not\cong \rho_1\gamma \).

In case (2), we have

\[ \pi_{G\gamma} \cong \text{ind}_{G\gamma}^{G\gamma} \rho_1 \oplus \text{ind}_{G\gamma}^{G\gamma} \rho_1\gamma \cong \text{ind}_{K\gamma}^{K\gamma} (\text{ind}_{I}^{K\gamma} \rho_1) \oplus \text{ind}_{K\gamma}^{K\gamma} (\text{ind}_{I}^{K\gamma} \rho_1)\gamma, \]

and the two summands do not intertwine. The construction from the unramified case now applies verbatim.

In case (1), we have \( \pi_{G\gamma} = \text{ind}_{I}^{G\gamma} \rho = \text{ind}_{I}^{G\gamma} (\text{ind}_{I}^{K\gamma} \rho) \), and so we may construct a lift \( E_\gamma \) of \( (E_{\pi|G\gamma})_c \) to \( C_s(X^\circ, G) \) as above. Since \( (\pi_{G\gamma})\gamma \cong \pi_{G\gamma} \), the map \( E_\gamma \) is also a lift of \( (E_{\pi|G\gamma})_c \). Therefore, the endomorphism \( E_\gamma := \frac{1}{2}(E_\gamma^0 + E_\gamma^{\gamma}) \otimes \text{id} \) of \( C_s(X, G) \) provides a \( G \)-equivariant lift of \( (E_{\pi|G\gamma})_c \).

As in the unramified case, \( E_\gamma^0 \) and \( E_\gamma^{\gamma} \) induce idempotents on \( G^0 \)-coinvariants. The formula for \( E_\gamma^0 \) and \( E_\gamma^{\gamma} \) given in Remark 3.2.38 imply that \( E_\gamma^0 E_\gamma^{\gamma} = E_\gamma^{\gamma} \) and that \( E_\gamma^{\gamma} E_\gamma^0 = E_\gamma^0 \) on \( G^0 \)-coinvariants, and it follows that \( E_\gamma \) is an idempotent on \( G^0 \)-coinvariants (and so also on \( G \)-coinvariants). Thus Theorem 3.2.19 identifies \( H^G_s(X)_{E_\gamma} \) with the image of \( E_\gamma \) on \( G \)-coinvariants. This image is equal to (the total complex of)

\[
\begin{array}{c}
\mathbb{C}c_{\text{ind}_I^K \rho} \oplus \mathbb{C}c_{\text{ind}_I^{K\gamma} \rho} \longrightarrow 0 \\
\text{id} \downarrow \\
\mathbb{C}c_{\text{ind}_I^K \rho} \oplus \mathbb{C}c_{\text{ind}_I^{K\gamma} \rho} \longrightarrow 0
\end{array}
\]

and so the homology computation is again the same as in the unramified case.
3.3 The Bernstein Decomposition

In Section 3.1.3 we recalled Bernstein’s construction of idempotents in $\mathfrak{z}(G)$ associated to compact-mod-center representations of $G = \text{GL}_n(F)$. In this section—which is likewise a review of results due to Bernstein and others—we build on this construction to give a description of the entire center $\mathfrak{z}(G)$. A crucial role is played by the Jacquet functors, which associate representations of $G$ to compact-mod-center representations of the block-diagonal subgroups of $G$ (and vice versa). Our purpose here is to recall the definition and main properties of these functors, and to underline their importance in studying the center $\mathfrak{z}(G)$. This will serve as motivation for Chapter 4, where we transport these functors into the context of chamber homology.

We shall continue focus on the group $G = \text{GL}_n(F)$, although everything discussed here extends to more general groups. For example, in later sections we shall consider $\text{SL}_n(F)$, and all of the results in this section apply to this group. Our exposition will closely follow that of Bernstein’s notes [BR92], where the case of a general reductive group is considered. Other sources for this material include [Ber84], [Ren10, VI], and [Roc09]. Some of the results already appear in the papers of Bernstein and Zelevinsky, [BZ76] and [BZ77].

3.3.1 The Structure of $\text{GL}_n(F)$

First, we introduce some terminology and notation related to the block-diagonal subgroups of $G = \text{GL}_n(F)$.

**Definition 3.3.1.** To each partition $n = n_1 + n_2 + \ldots + n_l$ of $n$, we associate a standard Levi subgroup

$$M_{(n_1,\ldots,n_l)} = \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_l}(F) \subset G$$

of block-diagonal matrices. We similarly define the standard parabolic subgroups $P_{(n_1,\ldots,n_l)}$ and $\overline{P}_{(n_1,\ldots,n_l)}$ of, respectively, block-upper-triangular and block-lower-triangular matrices. The subgroup

$$N_{(n_1,\ldots,n_l)} = \{p \in P_{(n_1,\ldots,n_l)} \mid \text{each diagonal block of } p \text{ is an identity matrix}\}$$
is normal in $P_{(n_1, \ldots, n_l)}$, and $P_{(n_1, \ldots, n_l)} = M_{(n_1, \ldots, n_l)} \ltimes N_{(n_1, \ldots, n_l)}$. There is an analogous decomposition $P = M \ltimes N$ of the block-lower-triangular subgroups. Let us emphasize that we consider $G$ to be a standard Levi subgroup of itself, with $P = G$ and $N = \overline{N} = \{1\}$.

Since each standard Levi subgroup $M \subseteq G$ is a product of general linear groups, all of our terminology for $G$ applies also to $M$. For example, a standard Levi subgroup of $M$ is, by definition, a product of standard Levi subgroups of the factors $\text{GL}_{n_i}(F)$. (Note that this is the same thing as a standard Levi subgroup of $G$ that is contained in $M$.) We likewise consider the subgroup $M^\circ = \text{GL}_{n_1}(F)^\circ \times \cdots \times \text{GL}_{n_l}(F)^\circ \subseteq M$, and the torus $\Psi(M) = \text{Hom}_\mathbb{Z}(M/M^\circ, \mathbb{Z})$ of unramified characters of $M$.

Each standard parabolic subgroup $P$ is cocompact in $G$. More precisely:

**Lemma 3.3.2** (The Iwasawa Decomposition). Let $K = \text{GL}_n(\mathcal{O})$. For each fixed parabolic subgroup $P$, every element $g \in G$ may be written (non-uniquely) as a product $g = kp$, where $k \in K$ and $p \in P$; briefly, $G = KP$. Moreover, writing $K_P = K \cap P$, and so on, one has a semi-direct product decomposition $K_P = K_M \ltimes K_N$.

**Proof.** Gaussian elimination; see [Ren10, V.5.1], for instance.

### 3.3.2 Jacquet Functors and Supercuspidal Representations

**Definition 3.3.3.** Let $M$ be a standard Levi subgroup of $G$, and let $P = MN$ and $P = M\overline{N}$ be the corresponding standard parabolic subgroups, with modular functions $\delta_P$ and $\delta_{P^{-1/2}}$ respectively. (See Warning 2.1.37 for a note on our convention regarding modular functions.)

We consider the following Jacquet functors:

1. The functors of parabolic induction:

$$i_M^G, \tau_M^G : \text{Mod}(M) \to \text{Mod}(G)$$

$$i_M^G \sigma = \text{ind}_P^M \left( \text{inf}_M^P \sigma \otimes \delta_P^{-1/2} \right), \quad \tau_M^G \sigma = \text{ind}_P^M \left( \text{inf}_M^P \sigma \otimes \delta_P^{-1/2} \right).$$
The functors of parabolic/Jacquet restriction:

\[ r^G_M, \ r^G_M : \text{Mod}(G) \rightarrow \text{Mod}(M) \]

\[ r^G_M \pi = \text{coinv}_M \text{res}^G_P(\pi) \otimes \delta_P^{1/2}, \quad r^G_M \pi = \text{coinv}_M \text{res}^G_P(\pi) \otimes \delta_P^{1/2}. \]

(The functors ind, res, infl, and coinv are defined as in Section 2.1.4.)

So, if \( \sigma : M \rightarrow \text{GL}(W) \) is a smooth representation of \( M \), we have

\[ i^G_M \sigma = \{ f : G \rightarrow W \mid f(mng) = \delta_P^{-1/2}(m)\sigma(m)f(g) \text{ for all } m \in M, \ n \in N \}^\infty, \]

with \( G \) acting by right-translation. (There is no distinction between induction and compact induction, since \( P \) is cocompact in \( G \).) If \( \pi : G \rightarrow \text{GL}(V) \) is a smooth representation of \( G \), then by definition

\[ r^G_M(V) = V_N = V/\text{span}\{v - \pi(n)v \mid v \in V, \ n \in N\}, \]

on which \( M \) acts by

\[ m(w + \text{span}\{v - \pi(n)v\}) = \delta_P^{1/2}(m)\pi(m)w - \text{span}\{v - \pi(n)v\}. \]

The next lemma, theorem and corollary record the most important facts about the Jacquet functors.

**Lemma 3.3.4.** (1) (Frobenius reciprocity) The functor \( r^G_M \) is left-adjoint to \( i^G_M \) (and \( r^G_M \) is left-adjoint to \( r^G_M \)):

\[ \text{Hom}_G(\pi, i^G_M \sigma) \cong \text{Hom}_M(r^G_M \pi, \sigma). \]

(2) All of the Jacquet functors are exact.

(3) If \( L \) is a standard Levi subgroup of \( M \), then \( i^G_L \equiv i^G_M i^L_M \) and \( r^G_L \equiv r^M_M r^G_M \). (And similarly for \( \bar{\tau} \) and \( \bar{\tau} \).

**Proof.** See [Ren10, Chapitre VI].
**Theorem 3.3.5** (The Second Adjoint theorem). The functor $r_M^G$ is right-adjoint to $i_M^G$ (and $i_M^G$ is right-adjoint to $r_M^G$):

$$\text{Hom}_G(i_M^G \sigma, \pi) \cong \text{Hom}_M(\sigma, r_M^G \pi).$$

**Proof.** Bernstein’s original proof is in [Ber87]; cf. [Bus01], [BK11] for alternative approaches.

**Corollary 3.3.6.** All of the Jacquet functors preserve the following classes of representations: Admissible; Finitely generated; Finite-length; Projective; Injective.

**Proof.** See [Ren10, Chapitre VI].

**Definition 3.3.7.** A smooth representation $\pi$ of $G$ is called supercuspidal if $r_M^G \pi = 0$ for every proper standard Levi subgroup $M$ of $G$. We denote by $\widehat{G}_{cusp}$ the set of equivalence classes of irreducible, supercuspidal representations of $G$.

The terms cuspidal, quasi-cuspidal, and absolutely cuspidal are also used in the literature, sometimes under an additional finiteness assumption (e.g., admissibility). The supercuspidal representations of $G$, and of its standard Levi subgroups, are fundamental to the representation theory of $G$, as can be seen in the following easy consequence of Frobenius reciprocity:

**Lemma 3.3.8.** Let $\pi$ be an irreducible representation of $G$. There is a standard Levi subgroup $M \subseteq G$ (allowing $M = G$) and an irreducible supercuspidal representation $\sigma \in \widehat{M}_{cusp}$ such that $\pi$ is a subquotient of $i_M^G \sigma$.

**Proof.** See [BR92, II.1 Lemma 17] or [Ren10, VI.2.1 Corollaire].

Here is the crucial theorem about supercuspidal representations:

**Theorem 3.3.9.** A smooth representation of $G$ is supercuspidal if and only if it is compact-mod-center.

**Proof.** See [BR92, II.1.3 Theorem 11] or [Ren10, VI.2.1].
Leaving aside for a moment the question of uniqueness, Lemma 3.3.8 and Theorem 3.3.9 together say that each irreducible representation of $G$ is associated to an irreducible, compact-mod-center representation of a standard Levi subgroup. Bernstein extends this partition of $\hat{G}$ to a decomposition of the category $\text{Mod}(G)$.

### 3.3.3 The Bernstein Decomposition

Every irreducible, supercuspidal representation of $G$ determines a decomposition of the category $\text{Mod}(G)$, as in Theorem 3.1.21. Moreover, two such representations $\pi_1$ and $\pi_2$ determine the same decomposition if and only if $\pi_1 \cong \psi \pi_2$, for some unramified character $\psi \in \Psi(G)$. Bernstein’s Uniform Admissibility Theorem (another consequence of the Second Adjoint theorem; see [Ber87], [BR92, I.1.4], or [Ren10, VI.2.3]) implies that the decompositions for all of the supercuspidal representations may be performed simultaneously:

**Theorem 3.3.10.** The category $\text{Mod}(G)$ decomposes as the product

$$\text{Mod}(G) \cong \text{Mod}(G)_{\text{cusp}} \times \text{Mod}(G)_{\text{ind}},$$

where

$$\text{Mod}(G)_{\text{cusp}} = \prod_{[\pi] \in (\hat{G}_{\text{cusp}})/\Psi(G)} \text{Mod}(G)_{[\pi]},$$

and $\text{Mod}(G)_{\text{ind}}$ comprises those representations having no supercuspidal subquotient.

**Proof.** See [BR92, II.3.2] or [Ren10, VI.3.5].
\( (M_1, \sigma_1) \) and \( (M_2, \sigma_2) \) are inertially equivalent if there exists \( \psi \in \Psi(M) \) such that \( (M_1, \sigma_1) \) and \( (M_2, \psi \sigma_2) \) are associate. We write \([M, \sigma]_G\) for the inertial equivalence class of \((M, \sigma)\), \(\mathcal{B}(G)\) for the set of all such classes, and \(s\) for a general element of \(\mathcal{B}(G)\).

**Example 3.3.12.** Every irreducible supercuspidal representation \(\pi\) of \(G\) determines a cuspidal datum \((G, \pi)\). Two such data \((G, \pi_1)\) and \((G, \pi_2)\) are associate if and only if \(\pi_1 \cong \pi_2\), and they are inertially equivalent if and only if \(\pi_1 \cong \psi \pi_2\) for some \(\psi \in \Psi(G)\). Let \(\mathcal{B}(G)_{\text{cusp}}\) denote the set of all inertial equivalence classes of this kind.

**Theorem 3.3.13.** (1) The cuspidal support map \(\text{cusp. supp} : \hat{G} \to \Omega(G),\)

\[
\text{cusp. supp}(\pi) = (M, \sigma)_G \iff \pi \text{ is a subquotient of } i_M^G \sigma
\]

is well-defined, surjective, and finite-to-one.

(2) The action of \(\Psi(M)\) endows each inertial equivalence class \([M, \sigma]_G\) with the structure of an irreducible complex algebraic variety, and so the set \(\Omega(G)\) is an infinite disjoint union of complex algebraic varieties.

**Proof.** See [BR92, II.2.1] or [Ren10, VI.7.1].

**Definition 3.3.14.** For each \(s \in \mathcal{B}(G)\), let \(\text{Mod}(G)_s\) denote the category of smooth representations of \(G\) whose irreducible subquotients all have cuspidal support contained in \(s\).

Here is the main theorem:

**Theorem 3.3.15 (The Bernstein Decomposition).** The category \(\text{Mod}(G)\) decomposes as the product

\[
\text{Mod}(G) \cong \prod_{s \in \mathcal{B}(G)} \text{Mod}(G)_s.
\]

**Proof.** We present the outline of Bernstein’s argument, to highlight the role played by the Jacquet functors. See [BR92, II.2.2] or [Ren10, VI.7.2] for the details.
Let π be a smooth representation, which we would like to decompose. For each standard Levi subgroup $M \subseteq G$, the representation $r^G_M(\pi)$ decomposes according to Theorem 3.3.10:

$$r^G_M(\pi) \cong \left( \prod_{t \in \mathcal{B}(M)_{\text{cusp}}} r^G_M(\pi)_t \right) \oplus r^G_M(\pi)_{\text{ind}}.$$ 

Frobenius reciprocity provides a map $\pi \to i^G_M r^G_M(\pi)$, which may be composed with the projection onto $r^G_M(\pi)_t$ to give a map $\pi \to i^G_M (r^G_M(\pi)_t)$ for each $t \in \mathcal{B}(M)_{\text{cusp}}$. These combine into a map

$$\pi \mapsto \bigoplus_{M \leq G \text{ standard Levi}} \prod_{t \in \mathcal{B}(M)_{\text{cusp}}} i^G_M (r^G_M(\pi)_t),$$

which is injective by Lemma 3.3.8. Collecting together the terms $t = [M, \sigma]_M$ associated to each inertial equivalence class $s = [M, \sigma]_G \in \mathcal{B}(G)$, we obtain the desired decomposition of $\pi$. □

Using the Second Adjoint Theorem, Bernstein obtains the following description of the center $\mathcal{Z}(G)$:

**Theorem 3.3.16.** The algebra $\mathcal{Z}(G)$ is isomorphic to the algebra $\mathcal{O}(\Omega(G)) = \prod_{s \in \mathcal{B}(G)} \mathcal{O}(s)$ of regular functions on $\Omega(G)$. Explicitly, the function $\hat{z}$ associated to an element $z \in \mathcal{Z}(G)$ is given by

$$\hat{z}(\text{cusp. supp}(\pi)) = z(\pi)$$

for each irreducible $\pi \in \hat{G}$.

For each $s \in \mathcal{B}(G)$, let $E_s \in \mathcal{Z}(G)$ be the idempotent given by Theorem 3.3.15. Then $\hat{E}_s$ is the characteristic function of the connected component $s \subset \Omega(G)$.

**Proof.** See [BR92, III.4.2 Theorem 24] or [Ren10, VI.10.3]. □

**Corollary 3.3.17.** Each category $\text{Mod}(G)_s$ is indecomposable. □

**Corollary 3.3.18.** The algebra $\mathcal{H}(G)$ decomposes as a direct sum of two-sided ideals,

$$\mathcal{H}(G) \cong \bigoplus_{s \in \mathcal{B}(G)} \mathcal{H}(G)_s.$$  

(3.3.19)
For each $s \in \mathcal{B}(G)$, $\text{Mod}(G)_s$ is equivalent to the category of nondegenerate modules over $\mathcal{H}(G)_s$. □

**Corollary 3.3.20.** $H^G_s(X) \cong \oplus_{s \in \mathcal{B}(G)} H^G_{s}(X)_{E_s}$.

**Proof.** Corollary 3.2.8 implies that the operators $(E_s)_c$ are mutually orthogonal idempotents summing to the identity on $H^G_s(X)$. □

**Definition 3.3.21.** For each Levi subgroup $M$, the map

$$\Omega(M) \to \Omega(G), \quad (M, \sigma)_M \mapsto (M, \sigma)_G$$

induces a map on the spaces of regular functions, called the Harish-Chandra homomorphism,

$$\Phi^G_M : \mathfrak{z}(G) \to \mathfrak{z}(M).$$

### 3.4 A Non-Cuspidal Component in Chamber Homology

Consider the group $G = \text{SL}_2(F)$. In Section 3.2.7, we computed the action on chamber homology of the idempotents in $\mathfrak{z}(G)$ associated to the supercuspidal (i.e., compact) representations of $G$. In this section, we obtain partial results for the idempotent associated to a generic principal-series component (see below for the definition). A complete understanding of the action of the Jacquet functors on chamber homology would allow a more satisfactory approach to the computation; see Chapter 4 for some initial steps in this direction.

The following notation will be used in this section:

$$G = \text{SL}_2(F), \quad M = \begin{bmatrix} F^\times & 0 \\ 0 & F^\times \end{bmatrix}, \quad N = \begin{bmatrix} 1 & F \\ 0 & 1 \end{bmatrix}, \quad P = MN$$

$$K = \begin{bmatrix} O & O \\ O & O \end{bmatrix}, \quad K' = \begin{bmatrix} O & p^{-1} \\ p & O \end{bmatrix}, \quad I = K \cap K' = \begin{bmatrix} O & O \\ p & O \end{bmatrix}.$$

(Recall that the notation indicates, for example, that $I$ consists of all those matrices in $G$ whose bottom-left entries lie in $p$ and whose other entries lie in $\mathcal{O}$.) The Haar measure on
$G$ is normalized so that $\text{vol}(K) = \text{vol}(K') = 1$. Then $\text{vol}(I) = \frac{1}{q+1}$, where $q = |\mathcal{O}/p|$ is the cardinality of the residue field of $F$. We let $X$ denote the Bruhat-Tits building of $G$, and

$$\Delta = \frac{v_0}{e_0} e_0 \frac{v_1}{v_0}$$

is the edge fixed by $I$, with $K = G_{v_0}$ and $K' = G_{v_1}$. We consider the coefficient system $G^+$ on $X$ from Definition 2.3.37:

$$G^+(\sigma) = \mathcal{H}(G^+_{\sigma}), \quad G^+_{\sigma} = \{ g \in G \mid \sigma \subset \min(g) \}.$$ 

### 3.4.1 Construction of central idempotents

Up to conjugation, each cuspidal datum for $G$ is either of the form $(G, \pi)$, where $\pi$ is an irreducible, supercuspidal representation of $G$; or of the form $(M, \chi)$, where $\chi : M \to \mathbb{C}^\times$ is a smooth character of $M \cong F^\times$. In this section we consider data of the latter sort. To save space, we write $\Psi$ instead of $\Psi(M)$.

Two characters $\chi_1$ and $\chi_2$ define the same inertial equivalence class in $\mathfrak{B}(G)$ if and only if $\chi_1|_{M^0} = \chi_2|_{M^0}$, where $M^0 = M \cap K \cong \mathcal{O}^\times$. We will consider the case of a regular character: i.e., one having $\chi|_{M^0} \neq \chi^{-1}|_{M^0}$. (Along with the supercuspidals, these account for all but finitely many of the components in $\mathfrak{B}(G)$.) Let $s = [M, \chi]_G$. The regularity hypothesis ensures that the map

$$\Psi \to \Omega(G), \quad \psi \mapsto (M, \psi \chi)_G$$

is one-to-one, and gives an isomorphism of algebraic varieties $\Psi \cong s$.

Let $\pi_\psi := i_M^G(\psi \chi)$, realized as right-translation operators on the space

$$V_\psi = \{ f : G \to \mathbb{C} \mid f(mng) = \delta_P^{-1/2}(m)\psi(m)\chi(m)f(g) \text{ for all } m \in M, \ n \in N \}^\infty.$$ 

The modular character $\delta_P$ is easily computed:

$$\delta_P \left( \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \right) = |a|^{-2}.$$
The spaces $V_\psi$ may all be identified, $K$-equivariantly, with the space $V_1$ associated to the trivial character $1 \in \Psi$: for each $f \in V_1$, define $f_\psi \in V_\psi$ by $f_\psi(mnk) = \psi(m)f(k)$. 

**Theorem 3.4.1.** For each $\psi \in \Psi$, the representation $\pi_\psi$ is irreducible. The representations $\pi_{\psi_1}$ and $\pi_{\psi_2}$ are equivalent if and only if $\psi_1 = \psi_2$.

**Proof.** See [GGPS69, Chapter 2 §3].

The cuspidal support map $\hat{G} \to \Omega(G)$ thus restricts to a bijection between the subset $\{\pi_\psi \mid \psi \in \Psi\} \subset \hat{G}$ and the inertial class $s \subset \Omega(G)$.

Each $\pi_\psi$ is an admissible representation of $G$, and therefore maps $\mathcal{H}(G)$ into the algebra $\text{End}_\mathbb{C}(V_\psi)^0$ of finite-rank operators. We consider, as in Section 3.1.2, the map

$$m : \text{End}_\mathbb{C}(V_\psi)^0 \to C^\infty(G), \quad m(T)(g) = \text{Trace}(T\pi_\psi(g^{-1})).$$

(Since $\pi_\psi$ is not supercuspidal, none of the $m(T)$ will lie in $\mathcal{H}(G)$.)

Consider the algebra $\mathcal{O}(\Psi) \otimes_\mathbb{C} \text{End}_\mathbb{C}(V_1)^0$ of regular functions on $\Psi$ with values in $\text{End}_\mathbb{C}(V_1)^0$. Identifying $V_1$ with $V_\psi$ as above, we may view the value $F(\psi)$ of a such a function $F$ at a point $\psi$ as lying in the algebra $\text{End}_\mathbb{C}(V_\psi)^0$. So we have produced an injective map

$$\mathcal{O}(\Psi) \otimes_\mathbb{C} \text{End}_\mathbb{C}(V_1)^0 \to \left\{ F : \Psi \to \bigsqcup_{\psi \in \Psi} \text{End}_\mathbb{C}(V_\psi)^0 \right\}.$$

We let $\mathcal{E}$ denote the image of this map. This space is an algebra, and a smooth representation of $G \times G$, under pointwise operations.

**Theorem 3.4.2** (Matrix Paley-Wiener Theorem). For each $f \in \mathcal{H}(G)$, the function $\pi(f)$ defined by

$$\pi(f)(\psi) = \pi_\psi(f)$$

lies in $\mathcal{E}$. The map $\pi : \mathcal{H}(G) \to \mathcal{E}$ is a $G \times G$-equivariant algebra homomorphism, and restricts to an isomorphism $\mathcal{H}(G)_s \xrightarrow{\cong} \mathcal{E}$.

**Proof.** [BR92, III.5.2]; cf. [BDK86].
The map $\psi \mapsto \psi(\pi)$ identifies $\Psi$ with $\mathbb{C}^\times$. Let $T \subset \Psi$ denote the preimage of the unit circle, and equip $T$ with the rotation-invariant measure having total volume 1. Let $c$ denote the conductor of the smooth character $\chi|_{M^z}$:

$$c = \min \{ n \geq 1 \mid \chi = 1 \text{ on the subgroup } 1 + p^n \subset O^\times \}.$$  

**Definition 3.4.3.** Let $\varphi : E \to C^\infty(G)$ be the linear map defined by

$$\varphi(F)(g) : = (q + 1)q^{l-1} \int_T m(F(\psi))(g) \, d\psi.$$

It turns out that the image of $\varphi$ lies in $\mathcal{H}(G)$. In fact:

**Theorem 3.4.4.** The idempotent $E = E_\pi \in \mathfrak{Z}(G)$ associated to the inertial class $\mathfrak{s}$ acts on $\mathcal{H}(G)$ through the composition

$$\mathcal{H}(G) \xrightarrow{\pi} E \xrightarrow{\varphi} \mathcal{H}(G).$$

**Proof.** Translated into a statement about distributions, the assertion is that $E$ acts by convolution with the distribution

$$f \mapsto \frac{q + 1}{q^{l-1}} \int_T \text{ch}_{\pi_s}(f) \, d\psi.$$

(Cf. Part (2) of Theorem 3.1.11.) This formula for $E$ follows from the Plancherel formula, as shown in [MT02, Section 3.4].

3.4.2 The Schneider-Stuhler resolution of $E$

We will construct a lifting of the map $E : \mathcal{H}(G) \to \mathcal{H}(G)$ to the complex $C_\pi(X, G^+)$. This will be done in two stages, using the factorization $E = \varphi \pi$, similarly to the supercuspidal case (Sections 3.2.7 and 3.2.8). To do this, we will need a projective resolution of the $G$-module $E_{\text{Ad}}$. Such a resolution may be constructed using a technique of Schneider and Stuhler [SS97], as we shall now explain.
Consider the congruence subgroups
\[ H = \begin{bmatrix} 1 + p^e & p^e \\ p^e & 1 + p^e \end{bmatrix} \quad \text{and} \quad H' = \begin{bmatrix} 1 + p^e & p^{e-1} \\ p^{e+1} & 1 + p^e \end{bmatrix}. \]

Each of these subgroups normalizes the other; let \( H'' = HH' = H'H \) be the subgroup of \( G \) generated by \( H \) and \( H' \).

**Definition 3.4.5.** For each simplex \( \sigma \subset X \), define an open normal subgroup \( H_{\sigma} \subset G_{\sigma} \) as follows:

\[ H_{g v_0} := g H g^{-1}; \quad H_{g v_1} := g H' g^{-1}; \quad H_{g e_0} := g H'' g^{-1}. \]

(Recall that \( v_0, v_1 \) and \( e_0 \) are the faces of the fundamental domain \( \Delta \subset X \).)

Note that \( \sigma_1 \subset \sigma_2 \) implies \( H_{\sigma_1} \subset H_{\sigma_2} \). The following construction is due to Schneider and Stuhler [SS97, II.2].

**Definition 3.4.6.** Let \( W \) be a smooth representation of \( G \). Define an equivariant coefficient system \( \mathcal{F}_W \) on \( X \) by \( \mathcal{F}_W(\sigma) := W^{H_{\sigma}} \). The transition map \( \mathcal{F}_W(\sigma_2) \to \mathcal{F}_W(\sigma_1) \), for \( \sigma_1 \subset \sigma_2 \), is the inclusion \( W^{H_{\sigma_2}} \hookrightarrow W^{H_{\sigma_1}} \). The \( G \)-action on \( \mathcal{F}_W \) is the one inherited from \( W \).

**Theorem 3.4.7** ([SS97, Theorem II.3.1]). The augmentation
\[ a : C_0(X, \mathcal{F}_W) \to W, \quad (w_v)_{v \in X(0)} \mapsto \sum_v w_v \]

turns \( C_* (X, \mathcal{F}_W) \) into a projective resolution of the \( G \)-submodule of \( W \) generated by \( W^H \cup W^{H'} \).

**Definition 3.4.8.** Consider the equivariant coefficient system on \( X \) defined by
\[ \mathcal{F}(\sigma) = \{ F \in \mathcal{E} \mid F(\psi) \in \text{Hom}_\mathbb{C}(V_\psi, V^{H_\psi}_\psi) \} \text{ for every } \psi \in \Psi \} \]

The transition maps are given by inclusion, and \( G \) acts by conjugation:
\[ \text{Ad}_g : \mathcal{F}(\sigma) \to \mathcal{F}(g \sigma), \quad \text{Ad}_g(F)(\psi) = g F(\psi) g^{-1}. \]
Consider $\mathcal{E}_{\text{left}}$, the representation of $G$ obtained by restricting the $G \times G$ action on $\mathcal{E}$ to the first copy of $G$. Then $\mathcal{F}$ is non-equivariantly isomorphic to the coefficient system $\mathcal{F}_{\mathcal{E}_{\text{left}}}$ of Definition 3.4.6. We consider the augmentation $a : C_0(X, \mathcal{F}) \to \mathcal{E}$ of Theorem 3.4.7.

**Lemma 3.4.9.** The complex $C_*(X, \mathcal{F})$ is a projective resolution of the $G$-module $\mathcal{E}_{\text{Ad}}$.

*Proof.* Modules of the form $C_*(X, \mathcal{F})$ are always projective (Lemma 2.3.22). The augmentation $a : C_0(X, \mathcal{F}) \to \mathcal{E}$ is obviously equivariant with respect to the adjoint action of $G$ on $\mathcal{E}$, and so we are left to prove exactness.

Since the $G$-action is not relevant to the question of exactness, it will suffice (thanks to Theorem 3.4.7) to prove that $\mathcal{E}_{\text{left}}$ is generated, as a $G$-module, by its $H$-fixed vectors.

For each $\psi \in \Psi$, let $f^H_\psi \in V_\psi$ be the unique function supported on $PH \subseteq G$ and having $f(1) = 1$. Such a function exists, because $\psi(m) = \chi(m) = \delta_p(m) = 1$ for $m \in M \cap H$. Thus each $V_\psi$ contains a nonzero $H$-fixed vector. Now, the left module $\mathcal{E}_{\text{left}} \cong \mathcal{H}(G)_s$ lies in $\text{Mod}(G)_s$, and so every irreducible subquotient of $\mathcal{E}_{\text{left}}$ is isomorphic to some $V_\psi$.

We are left to prove the following claim: Suppose that $W$ is a smooth representation of $G$, all of whose irreducible subquotients contain an $H$-fixed vector. Then $W$ is generated by $W^H$. To prove the claim, we may assume that $W$ is finitely generated. If the quotient $W/GW^H$ were nonzero, then it would admit an irreducible quotient, say $Q$. Every $H$-fixed vector in $Q$ would lift to an $H$-fixed vector in $W/GW^H$, because $H$ is compact; but $(W/GW^H)^H = 0$. This establishes the claim and the lemma.

We now compute the complex $C_*(X, \mathcal{F})_G$ of $G$-coinvariants. Lemma 2.3.23 implies that $C_*(X, \mathcal{F})_G$ is isomorphic to the complex $C_*(\Delta, \mathcal{F}^G)$ of chains on the fundamental domain $\Delta$, with coefficients $\mathcal{F}^G(\sigma) = \mathcal{F}(\sigma)^{G_\sigma}$. (We refer to Lemma 2.3.23 for the definition of the transition maps and the isomorphism.)

**Lemma 3.4.10.** For each $\psi \in \Psi$ and each simplex $\sigma \subset X$, one has

$$\text{Hom}_\mathcal{C}(V_\psi, V_{\psi}^{H_\sigma})^{\text{Ad}(G_\sigma)} \cong \text{End}_{G_\sigma}(V_{\psi}^{H_\sigma}).$$

*Proof.* We have $\text{Hom}_\mathcal{C}(V_\psi, V_{\psi}^{H_\sigma})^{\text{Ad}(G_\sigma)} = \text{Hom}_{G_\sigma}(V_\psi, V_{\psi}^{H_\sigma})$. Since $H_\sigma$ is a normal subgroup
of $G_\sigma$, every $G_\sigma$-equivariant map commutes with the projection $V \to V^{H_\sigma}$, and so the map

$$\text{Hom}_{G_\sigma}(V_\psi, V^{H_\sigma}) \to \text{End}_{G_\sigma}(V^{H_\sigma}), \quad T \mapsto T\big|_{V^{H_\sigma}}$$

is an isomorphism.

We shall now identify the spaces $V^{H_\sigma}_\psi$.

**Definition 3.4.11.** The character $\chi : F^\times \to \mathbb{C}^\times$ determines a character of the group $I_c \doteq \left[ \begin{smallmatrix} O & O \\ p & O \end{smallmatrix} \right]$, by $\chi : \left[ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right] \mapsto \chi(a)$. To simplify the notation below, we write $\chi_I = \text{ind}_{I_c}^K \chi$, $\chi_K = \text{ind}_{I_c}^I \chi$, and $\chi_{K'} = \text{ind}_{I_c}^{K'} \chi$. Similar notation will be used with respect to the character $\chi^{-1}$.

**Remark 3.4.12.** The representations $\chi_K$, $\chi_{K'}$, and $\chi_I$ appear in the paper [BHP93], where the chamber homology of $\text{SL}_2(F)$ is computed. The procedure $\chi \mapsto \chi_I, \chi_K, \chi_{K'}$ is an example of parahoric induction; see Chapter 5.

**Lemma 3.4.13.** Fix a character $\psi \in \Psi$.

1. There are isomorphisms

$$V^{H}_\psi \cong \chi_K, \quad V^{Hp}_\psi \cong \chi_{K'}, \quad \text{and} \quad V^{Hp'}_\psi \cong \chi_I \oplus \chi_I^{-1}.$$

These isomorphisms are equivariant for $K$, $K'$, and $I$, respectively.

2. $\chi_I$ and $\chi_I^{-1}$ are inequivalent, irreducible representations of $I$.

3. $\chi_K$ and $\chi_K^{-1}$ are equivalent, irreducible representations of $K$; similarly for $\chi_{K'}$ and $\chi_{K'}^{-1}$.

**Proof.** (1) By the Iwasawa decomposition, the restriction of functions from $G$ to $K$ determines a $K$-equivariant isomorphism $V_\psi \cong \text{ind}^K_{K \cap P} \chi$. Since $H$ is normal in $K$, we have

$$V^{H}_\psi \cong \{ f : G \to \mathbb{C} \mid f(phk) = \chi(m)f(k) \text{ for all } p \in K \cap P \text{ and } h \in H \}.$$

Now, $I_c = (K \cap P)H$, and $\chi(ph) = \chi(p)$, so the right-hand side is precisely $\text{ind}^K_{I_c} \chi$. The proof that $V^{Hp'}_\psi \cong \chi_{K'}$ is similar.
The double-coset space $P \backslash G / I$ contains two points, represented by the matrices $1$ and $[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$. The Mackey induction-restriction formula thus gives an $I$-equivariant isomorphism

$$V_{\psi} \cong \text{ind}_{I \cap P}^{I} \chi \oplus \text{ind}_{I \cap P}^{I} \chi^{-1}.$$ 

One above shows that $(\text{ind}_{I \cap P}^{I} \chi)^{H''} \cong \chi_{I}$, and that $(\text{ind}_{I \cap P}^{I} \chi^{-1})^{H''} \cong \chi_{I}^{-1}$, by a similar argument to the one above.

Parts (2) and (3) follow from the Mackey formula and the minimality of $\mathfrak{c}$; see [BHP93, Lemma 9.2 and Proposition 9.3], or Sections 5.3 and 5.4 below.

Let $e_{\chi_{K}} \in \mathcal{H}(K)$ denote the central idempotent

$$e_{\chi_{K}}(k) := \frac{\dim(\chi_{K})}{\text{vol}(K)} \text{Trace}(\chi_{K}(k^{-1})) = (q + 1)q^{c-1} \text{Trace}(\chi_{K}(k^{-1})).$$

We likewise consider $e_{\chi_{K'}} \in \mathcal{H}(K')$, and $e_{\chi_{I}}$, $e_{\chi_{I}^{-1}} \in \mathcal{H}(I)$.

**Lemma 3.4.14.** The functions $e_{\chi_{K}}$, $e_{\chi_{K'}}$, $e_{\chi_{I}}$ and $e_{\chi_{I}^{-1}}$ are all fixed by the operator $E : \mathcal{H}(G) \to \mathcal{H}(G)$.

**Proof.** This follows from the stronger assertion that the pair $(I_{c}, \chi)$ is a type for the component $\mathfrak{s}$; see [Kut04, 2 Proposition].

**Lemma 3.4.15.** For each $\psi \in \Psi$, we have

$$\text{End}_{K}(V_{\psi}^{H}) = \mathbb{C}\pi_{\psi}(e_{\chi_{K}}), \quad \text{End}_{K'}(V_{\psi}^{H'}) \cong \mathbb{C}\pi_{\psi}(e_{\chi_{K'}}),$$

and

$$\text{End}_{I}(V_{\psi}^{H''}) \cong \mathbb{C}\pi_{\psi}(e_{\chi_{I}}) \oplus \mathbb{C}\pi_{\psi}(e_{\chi_{I}^{-1}}).$$

**Proof.** The $K$-equivariant operator $\pi_{\psi}(e_{\chi_{K}})$ acts by projecting $V_{\psi}$ onto its $\chi_{K}$-isotypical component. Part (1) of Lemma 3.4.13 implies that this isotypical component is equal to $V_{\psi}^{H}$, so $\pi_{\psi}(e_{\chi_{K}})$ lies in $\text{End}_{K}(V_{\psi}^{H})$. Part (3) of Lemma 3.4.13 ensures that $\text{End}_{K}(V_{\psi}^{H})$ is one-dimensional. Similar arguments apply to $K'$ and $I$. 

\[\blacksquare\]
Proposition 3.4.16. The complex $C_*(X, F)_G$ is isomorphic to the complex

$$0 \to \mathcal{O}(\Psi) \oplus \mathcal{O}(\Psi) \xrightarrow{\partial} \mathcal{O}(\Psi) \oplus \mathcal{O}(\Psi) \to 0,$$

where the differential is $\partial(u_1, u_2) = (u_1 + u_2, -u_1 - u_2)$.

Proof. Lemma 2.3.23 identifies $C_*(X, F)_G$ with $C_*\left(\Delta, F^G\right)$. Lemma 3.4.15 implies that $F^G(v_0)$ is equal to the space

$$\{ F \in \mathcal{E} \mid F(\psi) \in \mathbb{C}_\pi\psi(e_{X^0}) \text{ for every } \psi \}.$$

Taking the fiberwise trace identifies this space with $\mathcal{O}(\Psi)$. We likewise have $F^G(v_1) \cong \mathcal{O}(\Psi)$, and $F^G(e_0) \cong \mathcal{O}(\Psi) \oplus \mathcal{O}(\Psi)$. The formula for the differential follows from Lemma 2.3.23. □

Corollary 3.4.17. Let $G = \text{SL}_2(F)$, and let $\chi$ be a regular character of the diagonal subgroup $M$. The ideal $\mathcal{H}(G)_s$ of $\mathcal{H}(G)$ corresponding to the cuspidal component $s = [M, \chi]_G$ has Hochschild homology

$$HH_n(\mathcal{H}(G)_s) \cong \begin{cases} \mathcal{O}(\Psi) & \text{for } n = 0, 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

Proof. Since the ideal $\mathcal{H}(G)_s$ is a direct-summand of $\mathcal{H}(G)$, we have

$$HH_*\left(\mathcal{H}(G)_s\right) \cong H_*\left(\mathcal{H}(G), \mathcal{H}(G)_s\right) \cong H_*\left(G, \mathcal{E}_{Ad}\right)$$

(the second isomorphism comes from Proposition 2.4.6 and Theorem 3.4.2). Proposition 3.4.16 allows us to compute $H_*\left(G, \mathcal{E}_{Ad}\right)$. □

3.4.3 Lifting $\pi$ and $\varphi$ to chamber homology

Recall (Definition 2.3.37) the coefficient system $\mathcal{G}^+$ on $X$:

$$\mathcal{G}^+(\sigma) = \mathcal{H}(G^+_\sigma), \quad \text{where } \ G^+_\sigma = \{ g \in G \mid \sigma \subset \min(g) \}.$$
The complex $C_\alpha(X, G^+)\text{ is a projective resolution of }\mathcal{H}(G)_{Ad}\text{ as a }G\text{-module, via the augmentation }\alpha : C_0(X, G^+) \to \mathcal{H}(G)_{Ad}.\text{ We write }K^+ = G_{v_0}^+, K^\ell = G_{v_1}^+,\text{ and }I^+ = G_{e_0}^+.

We shall construct maps of equivariant chain complexes

$$\pi_\alpha : C_\alpha(X, G^+) \to C_\alpha(X, F) \quad \text{and} \quad \varphi_\alpha : C_\alpha(X, F) \to C_\alpha(X, G^+)$$

lifting the maps $\pi : \mathcal{H}(G) \to E$ and $\varphi : E \to \mathcal{H}(G)$. The method is very similar to that of Section 3.2.7.

Consider the subspace $C_0(Gv_0, F) \subset C_0(X, F)$ consisting of chains supported on the orbit of the vertex $v_0$. In the proof of Lemma 3.4.9, we showed that $E = \bigcup_{g \in G} F(g v_0)$, and so the augmentation $a : C_0(Gv_0, F) \to E_{Ad}$ is surjective. The space $F(v_0) = E_{\text{left}}$ sits inside both $C_0(Gv_0, F)$ and $E$ as a $K$-invariant subspace. Choose a $K$-invariant linear map $t : E_{Ad} \to C_0(Gv_0, F)$ satisfying $a t = \text{id}_E$ and $t|_{E_{\text{left}}} = \text{id}$. The $K$-equivariant map

$$t \pi_\alpha : G^+(v_0) \to C_0(Gv_0, F)$$

extends uniquely to a $G$-equivariant map

$$T : C_0(Gv_0, G) \to C_0(Gv_0, F),$$

satisfying $a T = \pi_\alpha$.

Repeat the construction with $v_1$ in place of $v_0$, $K'$ in place of $K$, and $H'$ in place of $H$. That is, we choose a $K'$-equivariant splitting $t'$ of the surjective map $C_0(Gv_1, F) \to E$, such that $t'|_{E_{\text{left}}} = \text{id}$. We then let

$$T' : C_0(Gv_1, G^+) \to C_0(Gv_1, F)$$

be the unique $G$-equivariant extension of the map $t' \pi_\alpha : G^+(v_1) \to C_0(Gv_1, F)$. We have again $a T' = \pi_\alpha$.

Let $\pi_0 := T \oplus T' : C_0(X, G^+) \to C_0(X, F)$.

**Lemma 3.4.18.** (1) $a \pi_0 = \pi_\alpha$
(2) \( \pi_0\big|_{\mathcal{H}(H\setminus K^+)} = \pi\big|_{\mathcal{H}(H\setminus K^+)} : \mathcal{H}(H\setminus K^+) \to \mathcal{E}_{left}^H \)

(3) \( \pi_0\big|_{\mathcal{H}(H'\setminus K'^+)} = \pi\big|_{\mathcal{H}(H'\setminus K'^+)} : \mathcal{H}(H'\setminus K'^+) \to \mathcal{E}_{left}^{H'} \)

(4) There is a unique \( G\)-equivariant map \( \pi_1 : C_1(X, G^+) \to C_1(X, F) \) with the property that
\[ \partial \pi_1 = \pi_0 \partial. \]

(5) \( \pi_1\big|_{\mathcal{H}(H'\setminus I^+)} = \pi\big|_{\mathcal{H}(H''\setminus I^+)} : \mathcal{H}(H''\setminus I^+) \to \mathcal{E}_{left}^{H''} \)

Proof. Part (1) follows from the corresponding property of \( T \) and \( T' \). Parts (2) and (3) hold by virtue of the equalities \( t_{\varepsilon_{left}}^H = \text{id} \) and \( t_{\varepsilon_{left}}^{H'} = \text{id} \). Part (4) is true because the differential \( C_1 \to C_0 \) is injective in both of the complexes; we have \( \pi_1 = \partial F^{-1} \pi_0 \partial G^+ \). Part (5) follows from Parts (2) and (3), because both \( H \) and \( H' \) are contained in \( H'' \). \( \square \)

The maps \( \pi_0 \) and \( \pi_1 \) combine into a \( a \) a map \( \pi_a : C_0(X, G^+) \to C_0(X, F) \) of equivariant chain complexes, lifting the map \( \pi : \mathcal{H}(G) \to \mathcal{E} \).

We now turn to the construction of \( \varphi_a : C_0(X, F) \to C_0(X, G^+) \) lifting the map \( \varphi : \mathcal{E} \to \mathcal{H}(G) \). Choose a \( K \)-equivariant map \( s : \mathcal{H}(G) \to C_0(X, G^+) \) with \( s = \text{id} \) and \( s\big|_{\mathcal{H}(K^+)} = \text{id} \). The \( K \)-equivariant map
\[ s \varphi_a : F(v_0) \to C_0(X, G^+) \]
extends to a unique \( G \)-equivariant map \( S : C_0(Gv_0, F) \to C_0(X, G^+) \), satisfying \( \alpha S = \varphi a \).

Repeating the construction for \( v_1 \) and \( K' \), we choose a \( K' \)-equivariant map \( s' : \mathcal{H}(G) \to C_0(X, G^+) \) having \( \alpha s' = \text{id} \) and \( s'\big|_{\mathcal{H}(K'^+)} = \text{id} \), and use it to construct a \( G \)-equivariant map \( S' : C_0(Gv_1, F) \to C_0(X, G^+) \). Let \( \varphi_0 = S \oplus S' \). Once again, the choice of \( \varphi_1 = \partial^{-1} \varphi_0 \partial \) is forced on us. The maps \( \varphi_0 \) and \( \varphi_1 \) combine to give a \( G \)-equivariant map of chain complexes \( \varphi_a : C_0(X, F) \to C_0(X, G^+) \) lifting \( \varphi : \mathcal{E} \to \mathcal{H}(G) \).

Lemma 3.4.19. Consider the chain \( c_{\chi_K} \in C_0(X, G^+) \) supported on the vertex \( v_0 \), and having there \( c_{\chi_K}(v_0) = e_{\chi_K} \in \mathcal{H}(K) \). Then \( \varphi_0 \pi_0(c_{\chi_K}) = c_{\chi_K} \). Similar statements hold for the representations \( \chi_{K'}, \chi_I, \chi_I^{-1} \).

Proof. Combine Parts (2), (3), and (5) of Lemma 3.4.18 with Lemma 3.4.14. \( \square \)
3.4.4 The operator $E$ on chamber homology

The composition $\varphi \pi$ is an endomorphism of the chain complex $C_* (X, G^+)$, lifting the map $E : \mathcal{H}(G) \to \mathcal{H}(G)$. Passing to $G$-coinvariants gives

\[
\begin{align*}
\mathcal{H}(K^+) & \oplus \mathcal{H}(K'^+) \\
\oplus \mathcal{O}(\Psi) & \oplus \mathcal{O}(\Psi) \\
\overset{\pi_0}{\longleftarrow} & \overset{\varphi_0}{\longrightarrow} \\
\mathcal{H}(I^+) & \oplus \mathcal{H}(I'^+) \\
\end{align*}
\]

The complex $C_* (X, G^+)_{\sigma}$, appearing in the first and third columns of (3.4.20), is $\mathbb{N}$-graded, according to the function $g \mapsto d_g$:

\[
C_* (X, G^+)_{\sigma} = \bigoplus_{l \geq 0} C_* (X, G^+)_{\sigma} = \bigoplus_{l \geq 0} \bigoplus_{g \in G} \mathbb{C} \cap \mathbb{C} \cap \mathbb{C} \cap \mathbb{C}, \quad \mathbb{C}(c_1, c_2) = (c_1 + c_2, \sigma \cap \min(g) \text{ and } d_g = l}
\]

The zeroth component of this grading is the complex computing chamber homology. The nonzero components may be computed explicitly: one finds that the odd-degree components are all zero, while

\[
\begin{align*}
C_* (X, G^+)_{\sigma} & \equiv \bigoplus_{\ell \in \mathbb{Z}} \left( \mathbb{C} \oplus \mathbb{C} \overset{\delta}{\longrightarrow} \mathbb{C} \oplus \mathbb{C} \right), \quad \delta(c_1, c_2) = (c_1 + c_2, -c_1 - c_2)
\end{align*}
\]

for each $l \geq 1$. (See Proposition 4.5.5 for the detailed calculation.)

The complex $C_* (X, F)_{\sigma}$ appearing in the second column of (3.4.20) also carries a natural $\mathbb{N}$-grading, induced by the grading $\mathcal{O}(\Psi)_l := \text{span}\{\psi^l, \psi^{-l}\}$. Thus the $l$th graded component of $C_* (X, F)_{\sigma}$ is the complex

\[
(\mathcal{C}_*(X, F)_{\sigma})_l = \bigoplus_{\psi^l, \psi^{-l}} \left( \mathbb{C} \oplus \mathbb{C} \overset{\delta}{\longrightarrow} \mathbb{C} \oplus \mathbb{C} \right), \quad \delta(c_1, c_2) = (c_1 + c_2, -c_1 - c_2).
\]

Comparing (3.4.21) and (3.4.22), we are tempted to guess that the maps $\pi_0$ and $\varphi_0$ are given by very simple formulae; at present, this remains a guess, and it is not even clear that the maps $\pi_0$ and $\varphi_0$ preserve the gradings. Supposing such to be the case, we would obtain the following picture of the Bernstein component $H^G_*(X)_E$ of chamber homology.
Proposition 3.4.23. Let $\eta$ denote the projection of $C_*(X, \mathcal{F})_G$ onto its zero-graded component. Suppose either that $\pi_{*}1_{G_c} = \eta_{*}\pi_{*}$, or that $\eta_{*}\varphi_{*} = 1_{G_c}\varphi_{*}$. Then $E_{*} := 1_{G_c}\varphi_{*}\pi_{*}1_{G_c}$ is an idempotent endomorphism of $C_*(X, \mathcal{G})_G$, which identifies the component $H^G_*(X)_E$ with the homology of the subcomplex

$$C_{c_{X^I}} \oplus C_{c_{X^J}} \to C_{c_{X^K}} \oplus C_{c_{X^K'}}$$

of $C_*(X, \mathcal{G})_G$.

Proof. The map $\pi_{*}$ restricts to an isomorphism from (3.4.24) to the complex $\eta_{*}C_*(X, \mathcal{F})_G$; the map $\varphi_{*}\eta$ is the inverse of this isomorphism (Lemma 3.4.19). Thus $E_{*}$ is equal to the identity on the subcomplex (3.4.24).

Either of the stated hypotheses ensures that the image of $E_{*}$ is contained in the image of $\varphi_{*}\eta$, i.e., in the complex (3.4.24). Since $E_{*}$ is the identity on this subcomplex, it is an idempotent. Therefore Theorem 3.2.19 implies that $H^G_*(X)_E$ is the homology of (3.4.24).
Chapter 4

Jacquet Functors and Clozel’s Formula

In Chapter 3, we considered the action of the idempotents in the Bernstein center $\mathcal{Z}(G)$ on chamber homology (for, say, $G = \text{GL}_n(F)$ or $\text{SL}_n(F)$). This action is not easy to understand, owing to the fact that the Bernstein idempotents $E$ do not commute with the operator $1_{G_c}$ of “restriction to the compact part” as endomorphisms of the $G$-module $\mathcal{H}(G)_{\text{Ad}}$. We saw, nevertheless, that upon passing to homology, these operators do satisfy a weak kind of commutativity relation: namely, the operator $E_c = 1_{G_c}E1_{G_c}$ is an idempotent (Corollary 3.2.8).

In his paper [Dat03], Dat uses a formula of Clozel from [Clo89] to prove that the idempotents $1_{G_c}$ and $E$ in fact commute as operators on degree-zero homology (which, of course, implies that $E_c$ is an idempotent in degree zero). Dat’s result raises the question of whether this strengthening of Corollary 3.2.8 also holds in higher homology. The main result of this chapter, Theorem 4.5.2, answers this question in the affirmative for the group $G = \text{SL}_2(F)$. We do this by formulating (for $\text{GL}_n(F)$ and $\text{SL}_n(F)$) and proving (for $\text{SL}_2(F)$) an analog of Clozel’s formula in higher homology. The functors of parabolic induction and Jacquet restriction, and the maps they induce in homology, play an essential role.

We begin in Section 4.1 by recalling Clozel’s original formula, for $p$-adic groups, and presenting Dat’s proof that the Bernstein idempotents commute with $1_{G_c}$ in degree-zero
homology. By interpreting the Jacquet functors as maps between homology groups, we view
Clozel’s formula as a conjectural statement about higher homology (Conjecture 4.1.11).

In Section 4.3, we prove an analog of Clozel’s formula (in all degrees) for the affine
Coxeter group of type $\tilde{A}_n$ (Theorem 4.3.12). A similar result, with a similar proof, is valid
for any affine Coxeter group. Groups of this kind are intimately related to reductive $p$-adic
groups—for instance, $\tilde{A}_n$ is the automorphism group of a single apartment in the Bruhat-
Tits building of $\text{SL}_{n+1}(F)$—and so we consider the validity of the higher Clozel formula
for these groups as evidence for the $p$-adic case. The proof of Theorem 4.3.12 relies on an
explicit identification of the maps induced in homology by the functors $\text{ind}^G_M$ and $\text{res}^G_M$
of induction and restriction, for a finite-index subgroup $M$ of a discrete group $G$. These maps
are described in Section 4.2, in which we build on earlier work by Bentzen and Madsen
[BM90].

Motivated by the discrete case, in Section 4.4 we find explicit descriptions of the maps
induced by Jacquet restriction in Hochschild, smooth, and chamber homology; these results
overlap in places with earlier work of van Dijk [vD72] and Nistor [Nis01]. The maps induced
in homology by the parabolic induction functors are more difficult to understand, for reasons
explained below (Remark 4.4.14; see also Example 4.4.11). In Section 4.5, we partially
compute the induction map for the diagonal subgroup of $\text{SL}_2(F)$, and we thereby prove the
higher Clozel formula for this group (Theorem 4.5.2). Our investigation of the parabolic
induction map in homology continues in Chapter 5.

4.1 Clozel’s Formula for $\text{GL}_n(F)$

As in Chapter 3, we focus on the case of $G = \text{GL}_n(F)$, noting that all of the results of
this section hold for a general reductive $p$-adic group; for example, we will later consider
$G = \text{SL}_2(F)$. We write $M \leq G$ to indicate that $M$ is a standard Levi subgroup of $G$. (See
Section 3.3.1 for the terminology.)

**Definition 4.1.1.** For each $M \leq G$, denote by $M_{cz}$ the union of the compact-mod-center
subgroups of $M$. This is an open, conjugation-invariant subset of $M$, and we let $1_{M_{cz}} \in
\text{Cl}^G(M)$ denote its characteristic function.
Examples 4.1.2. (1) If $M \subset G$ is the diagonal subgroup, then $M_{cz} = M$ (because $M$ is abelian).

(2) Recall (Example 2.3.16) that $G = GL_n(F)$ acts on the Bruhat-Tits building $X$ of $SL_n(F)$, with the center $Z(G)$ acting trivially. The quotient $G/Z(G)$ acts properly, and the Bruhat-Tits fixed-point theorem (Theorem 2.3.18) implies that $G_{cz}$ is precisely the set of $g \in G$ fixing a point of $X$.

(3) A similar characterization applies to a general Levi subgroup $M = GL_{n_1}(F) \times \cdots \times GL_{n_d}(F)$: an element of $M$ lies in $M_{cz}$ if and only if it fixes a point in the product $X_1 \times \cdots \times X_d$, where $X_i$ is the Bruhat-Tits building of $SL_{n_i}(F)$.

Definition 4.1.3. For each standard Levi subgroup $M = GL_{n_1}(F) \times \cdots \times GL_{n_d}(F) \subset G$, we consider the open, normal subgroup

$$M^0 := GL_{n_1}(F)^0 \times \cdots \times GL_{n_d}(F)^0$$

of matrices whose diagonal blocks each have determinant lying in $O^\times$ (cf. Definition 3.1.17).

Examples 4.1.4. (1) For $M = GL_1(F)$, $M^0 = M_c = O^\times$ is the maximal compact subgroup.

(2) Recall (Example 2.3.16) that $GL_n(F)$ acts on a line $L$, with each matrix $g$ acting as translation by $\text{val}(\det(g))$. Thus $GL_n(F)^0$ consists of those $g$ which fix a point in $L$ (or equivalently, which act trivially on $L$).

(3) For general $M$ as in the definition, $M^0$ is the set of elements of $M$ fixing a point in (equivalently, acting trivially on) the product $L_1 \times \cdots \times L_d$.

Lemma 4.1.5. For each $M \leq G$ one has

$$M_c = M_{cz} \cap M^0.$$ 

Proof. It is enough to consider a single block $M = G = GL_n(F)$. This group acts properly on $X \times L$, where $X$ is the building for $SL_n(F)$ and $L$ is a line (Example 2.3.16). The
Bruhat-Tits fixed-point theorem (Theorem 2.3.18) identifies $G_c$ as the set of elements of $G$ fixing a point in $X \times L$. We observed above that $G_{c\circ}$ and $G^\circ$ consist of those $g \in G$ fixing a point of $X$ and of $L$, respectively.

**Definition 4.1.6.** Let $M = \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_d}(F) \subset G$ be a standard Levi subgroup. We consider the open, conjugation-invariant subset $M^+ \subset M$ consisting of those block-diagonal matrices $(X_1, X_2, \ldots, X_d) \in \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_d}(F)$ having

$$|\det(X_1)|_{F^{1/n}} < |\det(X_2)|_{F^{1/n}} < \cdots < |\det(X_d)|_{F^{1/n}}.$$

Let $\chi_M : M \to \{0, 1\}$ be the characteristic function of $M^+$.

Note that $\chi_M$ is a class function on $M$, which moreover factors through $M/M^\circ$.

**Examples 4.1.7.** (1) If $M = G$, then $G^+ = G$ and $\chi_G$ is the constant function 1.

(2) If $M$ is the diagonal subgroup in $\text{GL}_n(F)$, then $M^+$ is the set of diagonal matrices whose diagonal entries are strictly increasing in absolute value. Geometrically, the lattice $M/M^\circ$ acts by translations on an apartment $A$ in the Bruhat-Tits building of $\text{SL}_n(F)$, and $M^+$ consists of those elements which translate toward one particular open $(n-1)$-simplex in the boundary $\partial A$; cf. Section 4.3.

To each standard Levi subgroup $M$ is associated a quartet of Jacquet functors: $i_M^G$, $i_M^G$, $r_M^G$, and $r_M^G$ (see Section 3.3.2). These functors preserve the categories of finitely generated projective modules (Corollary 3.3.6), and so they induce a quartet of maps in Hochschild homology

$$HH_*(\mathcal{P}_G) \xrightarrow{r, \tau} HH_*(\mathcal{P}_M).$$

Using the results of Chapter 2, these functors also induce maps in smooth homology and chamber homology.

In degree zero, the maps induced by the Jacquet functors may be understood in terms of the duality between admissible and projective representations:
Lemma 4.1.8. Let $\pi$ and $\sigma$ be admissible representations of $G$ and $M$, respectively. For each $x \in HH_0(P_G)$ and $y \in HH_0(P_M)$ one has

$$\langle ch_{\pi}, x \rangle = \langle ch_{\sigma}, x \rangle \quad \text{and} \quad \langle ch_{\pi}, y \rangle = \langle ch_{\sigma}, y \rangle.$$ 

Proof. Let $x = (P, T)$ where $P$ is a finitely generated projective $G$-module, and $T \in \text{End}_G(P)$. Lemma 2.4.28 and Frobenius reciprocity give

$$\langle ch_{\sigma}, x \rangle = \text{Trace} \left( (rT) \big|_{\text{Hom}_M(rP, \sigma)} \right) = \text{Trace} \left( T \big|_{\text{Hom}_G(P, \sigma)} \right) = \langle ch_{\sigma}, x \rangle.$$ 

The proof of the second relation is similar, using the Second Adjoint theorem in place of Frobenius reciprocity.

The following result is due to Clozel; the formulation given here comes from Dat’s paper [Dat03].

Theorem 4.1.9 (Clozel’s formula [Clo89, Proposition 1]). One has

$$\sum_{M \in G} \Gamma_M^G \circ 1_{M_{cz}} \circ \chi_M \circ r_M^G = \text{id} \quad \text{as operators on } HH_0(P_G).$$ 

Note that both sides of Clozel’s formula are well-defined operators on higher Hochschild homology, $HH_* (P_G)$.

Conjecture 4.1.11. One has

$$\sum_{M \in G} \Gamma_M^G \circ 1_{M_{cz}} \circ \chi_M \circ r_M^G = \text{id} \quad \text{as operators on } HH_* (P_G).$$

Dat uses Clozel’s formula to prove the following:

Proposition 4.1.13. [Dat03, Proposition 2.8] Let $E \in \mathcal{Z}(G)$ be an idempotent in the Bernstein center of $G$. Then $E$ and $1_{G_{cz}}$ commute as operators on $HH_0(P_G)$. 
Remark 4.1.14. Proposition 4.1.13 may be restated as follows: the operator $E$ is diagonal with respect to the decomposition $HH_0(P_G) = HH_0(P_G)_c \oplus HH_0(P_G)_{nc}$. Cf. Remark 3.2.9.

We will now sketch Dat’s proof of Proposition 4.1.13.

Lemma 4.1.15. [Dat03, Lemme 2.9] Let $\psi$ be a complex-valued function on the abelian group $G$, viewed as a class function on $G$. Then $E$ commutes with $\psi$ as operators on $\mathcal{H}(G)_{Ad}$.

Proof. Dat first reduces to the case of character $\psi : G/G^0 \to \mathbb{C}^\times$, and then uses the fact that the Bernstein components are stable under unramified twists. \qed

Lemma 4.1.16. [Dat03, Proof of Proposition 2.8] $E$ commutes with $1_{G_{cz}}$ on $HH_0(P_G)$.

Proof. Dat’s proof is by induction on $n$. The base case $G = \text{GL}_1(F)$, in which $G = G_{cz}$, is easily disposed of.

For the inductive step, Dat combines the Clozel formula with the Harish-Chandra homomorphisms $\Phi^G_M : \mathfrak{z}(G) \to \mathfrak{z}(M)$ (see Definition 3.3.21): these are algebra homomorphisms with the property that

$$E \left( \tau^G_M(y) \right) = \tau^G_M \left( \Phi^G_M(E)(y) \right) \quad \text{and} \quad r^G_M \left( E(x) \right) = \Phi^G_M(E) \left( r^G_M(x) \right)$$

for all $E \in \mathfrak{z}(G)$, $x \in HH_0(P_G)$, and $y \in HH_0(P_M)$.

Assume the lemma holds when $G$ is replaced by each of its proper Levi subgroups $M$.

For $x \in HH_0(P_G)$, and each $M \leq G$, we have

$$\tau^G_M 1_{M_{cz}} \chi_M r^G_M(Ex) = \tau^G_M 1_{M_{cz}} \chi_M \Phi^G_M(E) r^G_M(x)$$

$$= \tau^G_M 1_{M_{cz}} \Phi^G_M(E) \chi_M r^G_M(x) \quad \text{(Lemma 4.1.15)}$$

$$= \tau^G_M \Phi^G_M(E) 1_{M_{cz}} \chi_M r^G_M(x) \quad \text{(Inductive hypothesis)}$$

$$= E \tau^G_M 1_{M_{cz}} \chi_M r^G_M(x).$$

Thus $E$ commutes with the operator $\tau^G_M 1_{M_{cz}} \chi_M r^G_M$ for every proper Levi subgroup $M \leq$
Clozel’s formula then implies that $E$ also commutes with the operator $\Gamma_G^1G_{ce}\chi_G\Gamma_G^1 = 1_{G_{ce}}$.

**Proof of Proposition 4.1.13.** Lemma 4.1.5 implies that $1_{G_c}$ is the pointwise product $1_{G_{ce}}1_{G^c}$. Lemmas 4.1.15 and 4.1.16 imply that $E$ commutes with both factors on $HH_0(\mathcal{P}_G)$. 

Note that, aside from the intervention of Clozel’s formula, every step of Dat’s proof is valid in higher-degree homology just as well as in degree zero. In other words:

**Corollary 4.1.17.** The validity of Conjecture 4.1.11 would imply that the idempotents $E \in \mathcal{H}(G)$ commute with $1_{G_c}$ as operators on $HH_a(\mathcal{P}_G)$.

Clozel’s proof of Theorem 4.1.9 relies upon explicit formulas, due to van Dijk and to Casselman, for the action of the Jacquet functors on the characters of admissible representations. In order to prove (or disprove) Conjecture 4.1.11, we would therefore like to better understand the operators induced on smooth homology by the functors $r_G^M$ and $i_G^M$. In the following two sections, we solve the analogous problem for affine Coxeter groups, leading to a version of Clozel’s formula for these groups. In Section 4.4, we give an explicit description of the map $r_G^M$ in homology, extending van Dijk’s formula for the character of an induced representation to higher degrees. In Section 4.5, we prove the higher Clozel formula for $G = SL_2(F)$, by partially computing the map $i_G^M$ for the diagonal subgroup $M$. The computation of $i_G^M$, in this special case, continues in Chapter 5.

### 4.2 Induction and Restriction in Homology: Discrete Groups

Let $G$ be a discrete group, with a finite-index subgroup $M \subseteq G$. Since these groups will be fixed throughout this section, we use the abbreviated notation $\text{ind} = \text{ind}_M^G$ and $\text{res} = \text{res}_M^G$ for the induction and restriction functors. We will occasionally leave out the “res”, using the same symbol to denote both a $G$-module, and its restriction to $M$. The example we have in mind is that of an affine Coxeter group $G$, and the isotropy group $M$ of a simplex in the boundary of the Coxeter complex; see Section 4.3.
The finite-index assumption implies that \( \text{ind} \) and \( \text{res} \) restrict to functors between the categories \( \mathcal{P}_G \) and \( \mathcal{P}_M \) of finitely generated projective modules (cf. Lemma 2.2.5). Therefore, by the results of Chapter 2, \( \text{ind} \) and \( \text{res} \) induce maps in Hochschild homology and in group homology—and, if \( G \) and \( M \) act on affine buildings, also in chamber homology. The object of this section is to describe these induced maps explicitly.

We choose representatives for the coset space \( M \backslash G \); nothing essential will depend on this choice.

**Lemma 4.2.1.** (1) For each \( q \geq 0 \), define a map \( R : \mathcal{H}(G^{q+1}) \to \mathcal{H}(M^{q+1}) \) by

\[
R(f)(m_0, \ldots, m_q) = \sum_{(k_0, \ldots, k_q) \in (M \backslash G)^{q+1}} f(k_0^{-1}m_0k_1, k_1^{-1}m_1k_2, \ldots, k_q^{-1}m_qk_0).
\]

These maps define a morphism of precyclic modules \( R : C(\mathcal{H}(G)) \to C(\mathcal{H}(M)) \), whose induced map in Hochschild homology is equal to \( \text{res} : HH_s(\mathcal{H}(G)) \to HH_s(\mathcal{H}(M)) \).

(2) The map \( \text{ind} : HH_s(\mathcal{H}(M)) \to HH_s(\mathcal{H}(G)) \) is the one induced by the natural inclusion of precyclic modules \( I : C(\mathcal{H}(M)) \to C(\mathcal{H}(G)) \).

(See Example 2.4.4 for the definition of \( C(\mathcal{H}(G)) \) and \( C(\mathcal{H}(M)) \).)

**Proof.** (1) The decomposition \( \mathcal{H}(G) = \bigoplus_{k \in M \backslash G} \mathcal{H}(M)k \) identifies \( \text{res}(\mathcal{H}(G)) \) as a free module over \( \mathcal{H}(M) \), giving a trace map \( C(\text{End}_M(\mathcal{H}(G)^{\text{opp}})) \to C(\mathcal{H}(M)) \) (as in (2.4.20)). A straightforward computation shows that the composition

\[
C(\mathcal{H}(G)) \to C(\text{End}_M(\mathcal{H}(G)^{\text{opp}})) \overset{\text{Trace}}{\longrightarrow} C(\mathcal{H}(M))
\]

is equal to \( R \).

(2) There is an isomorphism of \( \mathcal{H}(G) \)-\( \mathcal{H}(M) \)-bimodules,

\[
\text{ind}(\mathcal{H}(M)) \cong \mathcal{H}(G) \otimes_{\mathcal{H}(M)} \mathcal{H}(M) \cong \mathcal{H}(G).
\]

(See Proposition 2.1.39.) This realization of \( \text{ind}(\mathcal{H}(M)) \) as a free \( \mathcal{H}(G) \) module gives a trace
map $C(\text{End}_G(\text{ind}(\mathcal{H}(M))))^{\text{opp}} \rightarrow C(\mathcal{H}(G))$, such that the composition

$$C(\mathcal{H}(M)) \rightarrow C(\text{End}_G(\text{ind}(\mathcal{H}(M))))^{\text{opp}} \xrightarrow{\text{Trace}} C(\mathcal{H}(G))$$

is equal to the natural inclusion $C(\mathcal{H}(M)) \hookrightarrow C(\mathcal{H}(G))$. 

\textbf{Corollary 4.2.2.} One has $1_M \text{res} = \text{res}_G$ and $1_G \text{ind} = \text{ind}_M$ as maps on Hochschild homology.

\textit{Proof.} The relation for res follows from part (1) of Lemma 4.2.1, because

$$R(1_G f)(m_0, \ldots, m_q) = \sum (1_G f)(k_0^{-1}m_0k_1, \ldots, k_q^{-1}m_qk_0)$$
$$= 1_G (k_0^{-1}m_0m_1 \cdots m_qk_0) R(f)(m_0, \ldots, m_q)$$
$$= 1_G (m_0m_1 \cdots m_q) R(f)(m_0, \ldots, m_q)$$
$$= (1_M R(f))(m_0, \ldots, m_q).$$

The relation for ind follows immediately from part (2) of Lemma 4.2.1. 

\textbf{Remark 4.2.3.} Corollary 4.2.2 says that the operators ind and res are diagonal with respect to the compact/noncompact decomposition of Hochschild and cyclic homology (see Definition 3.2.2):

$$\text{ind} = \begin{bmatrix} \text{ind}_c & 0 \\ 0 & \text{ind}_{nc} \end{bmatrix} \quad \text{and} \quad \text{res} = \begin{bmatrix} \text{res}_c & 0 \\ 0 & \text{res}_{nc} \end{bmatrix}.$$

We now move on to group homology.

\textbf{Definition 4.2.4.} Let $V$ be a $G$-module, and consider the vector-space map $V_G \rightarrow V_M$ between coinvariant spaces induced by the map

$$V \rightarrow V, \quad v \mapsto \sum_{k \in M/G} kv. \quad (4.2.5)$$

(The map on $V$ depends on the choice of coset representatives, but the induced map on
coinvariants does not.) This map $V_G \to V_M$ is natural in $V$, and it therefore induces a natural transformation between the respective derived functors,

$$H_a(G, V) \to H_a(M, \text{res } V).$$

(See [Wei94, Theorem 2.4.7], for example.)

Explicitly, if $V_\bullet \to V$ is a projective resolution of $V$ as a $G$-module, then applying the map (4.2.5) in each degree gives a map of chain complexes

$$p_{\bullet} V_G \to p_{\bullet} \text{res } V_M,$$

and the map (4.2.6) is the one induced in homology by (4.2.7).

On the other hand, the identity $V \to V$ induces a map $V_M \to V_G$, which in turn induces a map in homology, as above:

$$H_a(M, \text{res } V) \to H_a(G, V).$$

**Remark 4.2.9.** The maps (4.2.6) and (4.2.8) may alternatively be defined as follows. The two adjunction relations between ind and res (Theorem 2.1.35) give $G$-equivariant maps

$$V \xrightarrow{\text{ind res }} \text{ind res } V,$$

which in turn induce maps in homology. One also has the *Shapiro isomorphism*

$$H_a(G, \text{ind } W) \cong H_a(M, W),$$

and the maps (4.2.6) and (4.2.8) are, respectively, the compositions

$$H_a(G, V) \xrightarrow{\text{adjunction}} H_a(G, \text{ind res } V) \xrightarrow{\text{Shapiro}} H_a(M, \text{res } V)$$
and

\[ H_\bullet(M, \text{res } V) \xrightarrow{\text{Shapiro adjunction}} H_\bullet(G, \text{res } V) \xrightarrow{\text{adjunction}} H_\bullet(G, V). \]


**Definition 4.2.10.** Let \( W \) be an \( M \)-module, and let \( V \) be a \( G \)-module. If \( f : V \to W \) is an \( M \)-equivariant map, define \( f_\bullet : H_\bullet(G, V) \to H_\bullet(M, W) \) as the composition

\[ H_\bullet(G, V) \xrightarrow{(4.2.6)} H_\bullet(M, \text{res } V) \xrightarrow{f} H_\bullet(M, W). \]

If \( h : W \to V \) is an \( M \)-equivariant map, define \( h_\bullet : H_\bullet(M, W) \to H_\bullet(G, V) \) as the composition

\[ H_\bullet(M, W) \xrightarrow{h} H_\bullet(M, \text{res } V) \xrightarrow{(4.2.8)} H_\bullet(G, V). \]

Now consider the modules \( \mathcal{H}(G)_{\text{Ad}} \) and \( \mathcal{H}(M)_{\text{Ad}} \), and the \( M \)-equivariant maps

\[ \mathcal{H}(G)_{\text{Ad}} \xrightarrow{r} \mathcal{H}(M)_{\text{Ad}} \]

given by restriction and extension of functions:

\[ r(f)(m) = f(m) \quad \text{and} \quad i(f)(g) = \begin{cases} f(g) & \text{if } g \in M, \\ 0 & \text{otherwise}. \end{cases} \]

Part (1) of the following proposition is due to Bentzen and Madsen [BM90, Proposition 1.4]; see also [BB92, Section 8].

**Proposition 4.2.11.** Let \( M \) be a finite-index subgroup of a discrete group \( G \).

1. \( r_\bullet = \text{res as maps } H_\bullet(G, \mathcal{H}(G)_{\text{Ad}}) \to H_\bullet(M, \mathcal{H}(M)_{\text{Ad}}). \)

2. \( i_\bullet = \text{ind as maps } H_\bullet(M, \mathcal{H}(M)_{\text{Ad}}) \to H_\bullet(G, \mathcal{H}(G)_{\text{Ad}}). \)

**Proof.** Computing using the standard resolutions, Bentzen and Madsen prove that the
Part (2) of the proposition follows from a similar (and simpler) computation: the map $i_*$ is the one induced by the natural inclusion $B_*(M, \mathcal{H}(M)_{\text{Ad}}) \hookrightarrow B_*(G, \mathcal{H}(G)_{\text{Ad}})$ of the standard resolutions (see Definition 2.2.10). Consulting the formula given in [BB92, Proposition 2.3] for the isomorphism between Hochschild and group homology, one sees that the diagram
\[ HH_*(\mathcal{H}(G)) \xrightarrow{R} HH_*(\mathcal{H}(M)) \]
\[ \cong \]
\[ H_*(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{r_*} H_*(M, \mathcal{H}(M)_{\text{Ad}}) \]
commutes. The equality $i_* = \text{ind}$ follows, by part (2) of Lemma 4.2.1.

**Example 4.2.12.** When $G$ is finite, Proposition 4.2.11 reduces to the usual formulas for the characters of induced/restricted representations; see Example 2.1.43.

Now suppose that $G$ acts by automorphisms of a locally finite affine building $X$, such that the conditions (GAB1–3) of Section 2.3.2 are satisfied: $X$ is oriented, and $G$ preserves the orientation; the isotropy group $G_\sigma$ of every polysimplex $\sigma \subset X$ is finite; and for each $g \in G$, the fixed-point set $X^g$ is a subcomplex of $X$. The action of the subgroup $M$ on $X$ then also satisfies the hypotheses (GAB1–3).

The relations $\text{res}_{G_e} = 1_{M_e}$, res and $\text{ind}_{M_e} = 1_{G_e}$, ind of Corollary 4.2.2 imply that $\text{res}$ and $\text{ind}$ restrict to maps between the chamber homology groups $H_*^G(X)$ and $H_*^M(X)$. (See Corollary 2.3.34 for the relationship between chamber and group homology.) We will show that these maps are given by the corresponding induction and restriction functors for the finite subgroups of $G$ and $M$.

Consider the coefficient systems $\mathcal{R}_G(\sigma) = R_C(G_\sigma)$ and $\mathcal{R}_M(\sigma) = R_C(M_\sigma)$ of Lemma 2.3.26.
Definition 4.2.13. For each polysimplex $\sigma \subset X$, we define linear maps

$$r' : R_C(G_\sigma) \to R_C(M_\sigma), \quad \pi \mapsto \frac{|M_\sigma|}{|G_\sigma|} \res_{M_\sigma}^{G_\sigma} \pi$$

and

$$i' : R_C(M_\sigma) \to R_C(G_\sigma), \quad \rho \mapsto \ind_{M_\sigma}^{G_\sigma} \rho.$$

The maps $r'$ and $i'$ induce $M$-equivariant maps

$$C_n(X, \mathcal{R}_G) \xrightarrow{r'_n} C_n(X, \mathcal{R}_M)$$

for each $n$. The $i'$ commute with the boundary operators, by the transitivity of induction, and so define a map of chain complexes. The maps $r'$ do not commute with the boundaries, because $\ind \circ \res \neq \res \circ \ind$.

The $M$-equivariant maps $r'_n$ and $i'_n$ induce maps on coinvariants

(4.2.14) $$C_n(X, \mathcal{R}_G) \xrightarrow{r'_n} C_n(X, \mathcal{R}_M)_M$$

by composition with (4.2.6) and (4.2.8), as above.

Proposition 4.2.15. (1) The maps (4.2.14) are maps of chain complexes, and so they induce maps in chamber homology

$$H^G_*(X) \xrightarrow{i'_*} H^M_*(X).$$

(2) $r'_* = \res$ as maps $H^G_*(X) \to H^M_*(X)$.

(3) $i'_* = \ind$ as maps $H^M_*(X) \to H^G_*(X)$.

Proof. Consider the coefficient systems

$$\mathcal{G}(\sigma) = \mathcal{H}(G_\sigma) \quad \text{and} \quad \mathcal{M}(\sigma) = \mathcal{H}(M_\sigma)$$
of Definition 2.3.24. The complex $C_*(X, G)$ is a projective resolution of $\mathcal{H}(G_c)_{Ad}$, and similarly for $C_*(X, M)$ and $\mathcal{H}(M_c)_{Ad}$. Restriction and extension of functions over each polysimplex induce $M$-equivariant maps of chain complexes

$$C_*(X, G) \xrightarrow{r} C_*(X, M),$$

lifting the corresponding maps between $\mathcal{H}(G_c)_{Ad}$ and $\mathcal{H}(M_c)_{Ad}$. Proposition 4.2.11 implies that the induced maps in homology satisfy $r_* = \text{res}$ and $i_* = \text{ind}$.

Recall (Lemmas 2.3.23 and 2.3.26) that we have isomorphisms

$$\varphi_G : C_*(X, R_G) \xrightarrow{\cong} C_*(X, G) \quad \text{and} \quad \varphi_M : C_*(X, R_M) \xrightarrow{\cong} C_*(X, M),$$

defined over each polysimplex $\sigma$ by

$$\varphi_G(\pi) = \frac{1}{|G_\sigma|} \chi_\pi \quad \text{and} \quad \varphi_M(\rho) = \frac{1}{|M_\sigma|} \chi_\rho$$

for $\pi \in R_G(G_\sigma)$ and $\rho \in R_G(M_\sigma)$.

We claim that the diagrams

$$
\begin{array}{ccc}
C_n(X, R_G) & \xrightarrow{r_n} & C_n(X, R_M) \\
\varphi_G \downarrow & & \varphi_M \downarrow \\
C_n(X, G) & \xrightarrow{r_n} & C_n(X, M)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C_n(X, R_M) & \xrightarrow{i_n} & C_n(X, R_G) \\
\varphi_M \downarrow & & \varphi_G \downarrow \\
C_n(X, M) & \xrightarrow{i_n} & C_n(X, G)
\end{array}
$$

commute. Proposition 4.2.15 follows from this claim, because $\varphi_G$ and $\varphi_M$ are isomorphisms of chain complexes, and $r_*$ and $i_*$ are maps of complexes whose respective induced maps on homology are equal to res and ind.

Considering the diagram for $r$, we fix a polysimplex $\sigma$ in $X$, and let $\pi$ be a representation of $G_\sigma$. We then have

$$r_n \circ \varphi_G(\pi) = \frac{1}{|G_\sigma|} \sum_{k \in M \setminus G} r(k \cdot \chi_\pi) = \frac{1}{|G_\sigma|} \sum_{k \in M \setminus G} (\chi_{\sigma k^{-1}}) \big|_{M_\sigma},$$
while
\[
\varphi_M \circ r_n'(\pi) = \varphi_M \left( \sum_{k \in M \setminus G} r'(\pi^{k-1}) \right) = \sum_{k \in M \setminus G} \frac{|M_\sigma|}{|M_{k\sigma}|} \frac{|M_{k\sigma}|}{|G_\sigma|} \text{ch}(\text{res}_{M_{k\sigma}} \pi^{k-1}).
\]

Since \(|M_{k\sigma}| = |M_\sigma|\), and \(\text{ch}(\text{res}_{M_{k\sigma}} \pi^{k-1}) = (\text{ch}_{\pi^{k-1}})_{M_{k\sigma}}\), we find that \(r_n \circ \varphi_G = \varphi_M \circ r_n'\).

Now for the \(i\) diagram: let \(\rho\) be a representation of \(M_\sigma\). We have
\[
i_n \circ \varphi_M(\rho) = \frac{1}{|M_\sigma|} i(\text{ch}_\rho), \quad \text{while} \quad \varphi_G \circ i_n'(\rho) = \frac{1}{|G_\sigma|} \text{ch}(\text{ind}_{M_\sigma} \rho).
\]

Frobenius’s formula for induced characters (cf. [Ser77, Theorem 12]) implies that
\[
\text{ch}(\text{ind}_{M_\sigma} \rho) = \frac{|G_\sigma|}{|M_\sigma|} i(\text{ch}_\rho)
\]
as elements of \((\mathcal{H}(G_\sigma)_{Ad} G_\sigma)\), and so \(i_n \circ \varphi_M = \varphi_G \circ i_n'\) as claimed.

4.3 Application: Clozel’s Formula for \(\widetilde{A}_n\)

We have seen that the induction and restriction functors for discrete groups correspond, in homology, to extension and restriction of functions. In this section we will use this correspondence to translate into representation-theoretic terms a natural geometric partition of the conjugacy classes in the affine Coxeter group \(\widetilde{A}_n\). The final result, Theorem 4.3.12, is motivated by and closely related to Clozel’s formula ([Clo89, Proposition 1], or Theorem 4.1.9 above) in the representation theory of \(p\)-adic groups. We have chosen to concentrate on \(\widetilde{A}_n\) to simplify the exposition, but it will be clear that the methods discussed here apply in greater generality.

**Definition 4.3.1.** Let
\[
X = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \middle| \sum x_i = 0 \right\} \quad \text{and} \quad \Lambda = X \cap \mathbb{Z}^{n+1}.
\]

For each point \(x \in X\), we let \(t_x : X \to X\) denote the corresponding translation operator. Let
Figure 4.3.1: The affine Coxeter complex of type $\widetilde{A}_1$ and its boundary

$T = \{t_\lambda \mid \lambda \in \Lambda\}$ be the group of translations by elements of the lattice $\Lambda$. The symmetric group $S = S_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permuting the standard basis, and this action restricts to $X$ and to $\Lambda$. We consider the semidirect product group $G := T \rtimes S$, acting on the space $X$.

**Remark 4.3.2.** This group $G$ is an example of an *affine Coxeter group*. See [AB08, 10.1–2] for a general discussion of such groups. Most of this section extends without difficulty to the more general context of *crystallographic groups*, for which see [Wol11, 3.2] for example.

The Euclidean space $X$ is a complete CAT(0) space, and we accordingly revive the terminology and notation of Section 2.3.1. In particular, recall that a *hyperbolic* isometry of $X$ is one that acts as a nontrivial translation along some geodesic line in $X$, and that an *elliptic* element is one that fixes a point of $X$. The elliptic elements of $G$ are precisely those of finite order (Corollary 2.3.19).

The space $X$ carries a canonical $G$-invariant simplicial structure, isomorphic to an apartment in the Bruhat-Tits building of $\text{SL}_{n+1}(F)$ (Example 2.3.15). The boundary $\partial X$ also carries a natural $G$-invariant simplicial structure, isomorphic to the barycentric subdivision of the boundary of an $n$-simplex (cf. [AB08, Example 1.10]). The simplex

$$\Delta = \left\{ \gamma(\infty) \in \partial(X) \mid \gamma(l) = l(x_1, \ldots, x_{n+1}), \sum x_i = 0, \quad x_1 \leq x_2 \leq \cdots \leq x_n \right\}$$

is a fundamental domain for the action of $G$ on $\partial X$. As a matter of convenience, we consider the empty set $\emptyset \subset \partial X$ to be a minus-one-dimensional face of $\Delta$. The $G$-orbit of a face $\sigma \subset \Delta$ will be denoted by $[\sigma]$.

The space $X$, along with its boundary, is pictured for $n = 1$ and $n = 2$ in Figures 4.3.1 and 4.3.2, respectively. The elements of $\Lambda$ are indicated by black circles. The squares represent vertices in $\partial X$; their colors in Figure 4.3.2 indicate two distinct $G$-orbits.

**Definition 4.3.3.** Define a map $\partial : \{\text{isometries of } X\} \to \{\text{simplices in } \partial X\}$ as follows.
Figure 4.3.2: The affine Coxeter complex of type $\tilde{A}_2$ and its boundary

For each elliptic isometry $f$, $\partial(f) = \emptyset$. For each hyperbolic $f$, choose $x \in \text{min}(f)$ (cf. Definition 2.3.5), and let $\gamma_{f,x}$ be the unique geodesic ray which has $\gamma_{f,x}(0) = x$, and which is translated (in the positive direction) by $f$. Then $\partial(f)$ is defined to be the unique open simplex containing the point $\gamma_{f,x}(\infty)$. It does not depend on $x$ (Theorem 2.3.7).

**Example 4.3.4.** For each point $y \in X$, the translation $t_y$ is hyperbolic, with $\text{min}(t_y) = X$. In particular, the origin 0 lies in $\text{min}(t_y)$, and the geodesic ray $\gamma_{t_y,0}$ is given by

$$\gamma_{t_y,0}(l) = ty \quad (l \geq 0).$$

Thus, on the subgroup group of translations, $\partial$ acts as the radial projection of $X$ onto the sphere at infinity $\partial X$. Figure 4.3.3 illustrates the case $n = 2$.

**Lemma 4.3.5.** For each translation $t \in T$ one has

$$G_{\partial(t)} = Z_G(t),$$

where $G_{\partial(t)}$ is the pointwise isotropy group of the simplex $\partial(t) \subset \partial X$, and $Z_G(t)$ is the
centralizer of $t$ in $G$.

Proof. Let $t = t_\lambda$ for some lattice-point $\lambda \in \Lambda$. Since $T$ acts trivially on $\partial X$, we have $G_{\partial(t)} = T \times S_{\partial(t)}$. Since $T$ is abelian, $Z_G(t) = T \times Z_S(t)$. For each $s \in S$ we have:

$s$ fixes $\partial(t) \iff s \cdot \gamma_{t,0}$ is parallel to $\gamma_{t,0} \iff s$ fixes $\gamma_{t,0}$ pointwise (since $s$ fixes 0)

$\iff s$ fixes $\lambda \iff s$ commutes with $t_\lambda$ (since $st_\lambda s^{-1} = t_{s \cdot \lambda}$).

Thus $S_{\partial(t)} = Z_S(t)$, and so $G_{\partial(t)} = Z_G(t)$. \qed

Lemma 4.3.6. For every $g, h \in G$ and $m > 0$ one has

(1) $\partial(hgh^{-1}) = h\partial(g)$.

(2) $\partial(g^m) = \partial(g)$.

Proof. Both statements are obviously true if $g$ is elliptic, so assume that $g$ is hyperbolic, and choose $x \in \min(g)$. Part (1) holds because $hx \in \min(hgh^{-1})$ and $\gamma_{hgh^{-1},hx} = h\gamma_{g,x}$.

Part (2) holds because $x \in \min(g^m)$ and $\gamma_{g^m,x} = \gamma_{g,x}$. \qed

Part (1) of the above lemma implies that the partition

$$G = \bigsqcup_{\sigma \in \Delta_{\text{face}}} \partial^{-1}[\sigma]$$
is conjugation-invariant, giving a direct-sum decomposition of the $G$-module $\mathcal{H}(G)_{\text{Ad}}$ and of the homology group $H_\ast(G, \mathcal{H}(G)_{\text{Ad}})$:

$$H_\ast(G, \mathcal{H}(G)_{\text{Ad}}) \cong \bigoplus_{\sigma \in \Delta} H_\ast(G, \mathcal{H}(\partial^{-1}[\sigma])_{\text{Ad}}).$$

Note that this is a refinement of the decomposition into compact/noncompact parts, because $G_c = \partial^{-1}[\varnothing]$. We will derive a formula for the idempotent operators on $H_\ast(G, \mathcal{H}(G)_{\text{Ad}})$ effecting this decomposition (Theorem 4.3.12).

First some notation:

**Definition 4.3.7.** For each discrete group $H$, define:

1. $H_{cz} = \{ h \in H \mid h \text{ has finite order modulo the center of } H \}$, and
2. $H^0 = \bigcap_{\psi \in \text{Hom}(H, \mathbb{Z})} \ker \psi$.

The subset $H_{cz} \subset H$ is conjugation-invariant, so its characteristic function $1_{H_{cz}}$ is a class function on $H$. The group $H/H^0$ is abelian, and so every function on $H/H^0$ lifts to a class function on $H$.

**Lemma 4.3.8.** For each $g \in G$ one has $g \in (G_{\partial(g)})_{cz}$, where $G_{\partial(g)}$ denotes the pointwise isotropy group of the simplex $\partial(g) \subset \partial X$.

**Proof.** The quotient $S = G/T$ is finite, and so the positive power $g^{[S]}$ of $g$ lies in $T$. Now Lemmas 4.3.6 and 4.3.5 imply that $G_{\partial(g)} = G_{\partial(g^{[S]})} = Z_G(g^{[S]})$, so $g^{[S]}$ lies in the center of $G_{\partial(g)}$. \qed

**Definition 4.3.9.** For each face $\sigma \subset \Delta$, define:

- A map $\alpha_\sigma : G_\sigma \to \Lambda$ by
  $$\alpha_\sigma(g) = \sum_{s \in S_\sigma} s \cdot g(0),$$
- A set $G^+_\sigma := \{ g \in G_\sigma \mid \partial(\alpha_\sigma(g)) = \sigma \}$, and
- A function $\chi_\sigma : G_\sigma \to \{0, 1\}$, the characteristic function of $G^+_\sigma$. 

Examples 4.3.10. (1) If $\sigma = \emptyset$, then $\alpha_\emptyset : G \to \Lambda$ is the zero map, $G^+_{\sigma} = G$, and $\chi_\emptyset : G \to \{0, 1\}$ is the constant function 1.

(2) At the other extreme, if $\sigma$ is the top-dimensional face of $\Delta$, then $\alpha_\sigma : T \to \Lambda$ is the map $t_\lambda \mapsto \lambda$, and

$$G^+_{\sigma} = \{t_\lambda \mid \lambda_1 < \lambda_2 < \ldots < \lambda_{n+1}\}.$$ 

The case $n = 2$ is illustrated in Figure 4.3.4, where $G^+_{\sigma}$ is the set of $t_\lambda$ with $\lambda$ lying in the shaded region.

(3) The case of $n = 2$, $\sigma$ a vertex, is pictured in Figure 4.3.5. The group $S_\sigma$ consists of two elements: the identity, and the reflection across the horizontal black line. The map $\alpha_\sigma$ sends $g \in G_\sigma$ to twice the orthogonal projection of the point $g(0)$ onto this line. The set $G^+_{\sigma}$ consists of all those $g$ for which $g(0)$ lies in the shaded region.

Lemma 4.3.11. Let $\sigma$ be a face of $\Delta$.

(1) $\alpha_\sigma$ is a homomorphism from $G_\sigma$ to the free-abelian group $\Lambda$.

(2) $(G_\sigma)^{\circ} \subseteq \ker(\alpha_\sigma)$.
Figure 4.3.5: The set $G_\sigma^+$ for $\tilde{A}_2$, $\sigma$ a vertex

\[
G_\sigma^+
\]

(3) $\partial \circ \alpha_\sigma = \partial$ on $(G_\sigma)_{cz}$.

Proof. Part (1) is a straightforward computation, using the decomposition $G_\sigma = T \rtimes S_\sigma$. Part (2) follows from part (1), by the definition of $(G_\sigma)^\circ$.

For part (3): If $g \in G_\sigma$ is elliptic, and therefore of finite order, then part (2) implies that $\alpha_\sigma(g) = 0$. So $\alpha_\sigma(g)$ is also elliptic, and $\partial(\alpha_\sigma(g)) = \partial(g) = \emptyset$.

Now suppose that $g \in (G_\sigma)_{cz}$ is hyperbolic. Since

\[
\partial(g^m) = \partial(g) \quad \text{and} \quad \partial(\alpha_\sigma(g^m)) = \partial(\alpha_\sigma(g)^m) = \partial(\alpha_\sigma(g)),
\]

by Lemma 4.3.6 and part (1) of the present lemma, we can assume that $g$ itself lies in the center of $G_\sigma$. Note that this center is a subgroup of $T$, because no nontrivial element of $S$ centralizes $T$.

We have

\[
\alpha_\sigma(g) = \sum_{s \in S_\sigma} s \cdot g(0) = \sum_{s \in S_\sigma} g \cdot s(0) = |S_\sigma| \cdot g(0),
\]

showing that $\alpha_\sigma(g)$ is a multiple of $g$. Lemma 4.3.6 now gives $\partial(\alpha_\sigma(g)) = \partial(g)$. \qed
By Lemma 4.3.11, the function $\chi_\sigma$ factors through the quotient map $G_\sigma \to G_\sigma/(G_\sigma)^s$, and is therefore a class function on $G_\sigma$. Thus the functions $1_{(G_\sigma)_{c_s}}$ and $\chi_\sigma$ both define endomorphisms of the $G_\sigma$-module $\mathcal{H}(G_\sigma)_{\text{Ad}}$. Since $[G : G_\sigma] = [S : S_\sigma] < \infty$ for every $\sigma$, the functors $\text{ind}_{G_\sigma}^G$ and $\text{res}_{G_\sigma}^G$ induce maps in homology, as in Section 4.2. Define an endomorphism $P_\sigma$ of $H_* (G, \mathcal{H}(G)_{\text{Ad}})$ by

$$P_\sigma := \text{ind}_{G_\sigma}^G \circ 1_{(G_\sigma)_{c_s}} \circ \chi_\sigma \circ \text{res}_{G_\sigma}^G.$$ 

**Theorem 4.3.12.** The operator $P_\sigma$ is the projection of $H_* (G, \mathcal{H}(G)_{\text{Ad}})$ onto its direct-summand $H_* (G, \mathcal{H}(\partial^{-1}[\sigma])_{\text{Ad}})$. Consequently,

$$\sum_{\sigma \in \Delta} P_\sigma = \text{id} : H_* (G, \mathcal{H}(G)_{\text{Ad}}) \to H_* (G, \mathcal{H}(G)_{\text{Ad}}).$$

**Remark 4.3.13.** The operator $P_\sigma$ is just $1_{G_{c_s}}$ (which is equal to $1_{G_c}$, because the center of $G$ is trivial). Thus the theorem expresses the operator $1_{G_{c_s}}$ in terms of the corresponding operators for the proper subgroups $G_\sigma \subsetneq G$.

**Proof of Theorem 4.3.12.** We compute using the standard complex $B_* (G, \mathcal{H}(G)_{\text{Ad}})$ of Definition 2.2.10. Our notation will ignore the distinction between group elements and their associated characteristic functions in $\mathcal{H}(G)$.

Proposition 4.2.11 implies that $P_\sigma$ is the map induced in homology by the following map of complexes:

$$\Phi : B_* (G, \mathcal{H}(G)_{\text{Ad}})_G \to B_* (G, \mathcal{H}(G)_{\text{Ad}})_G,$$

$$\Phi (g_0 \otimes \cdots \otimes g_l \otimes g) = \left( \sum_{k \in G_\sigma \backslash G} 1_{(G_\sigma)_{c_s}} (kgk^{-1}) \chi_\sigma (kgk^{-1}) \right) (g_0 \otimes \cdots \otimes g_l \otimes g).$$

So we wish to prove that

$$\sum_{k \in G_\sigma \backslash G} 1_{(G_\sigma)_{c_s}} (kgk^{-1}) \chi_\sigma (kgk^{-1}) = \begin{cases} 1 & \text{if } \partial(g) \in [\sigma], \\ 0 & \text{otherwise.} \end{cases}$$

(4.3.14)
Part (3) of Lemma 4.3.11 implies that for each \( k \in G_\sigma \setminus G \) with \( kgk^{-1} \in (G_\sigma)_{cz} \), we have

\[
\chi_\sigma(kgk^{-1}) = \begin{cases} 
1 & \text{if } k\partial(g) = \sigma, \\
0 & \text{otherwise}.
\end{cases}
\]  

(4.3.15)

If \( \partial(g) \) does not lie in the orbit \([\sigma]\), then (4.3.15) implies that both sides of (4.3.14) are zero.

On the other hand, suppose that \( \partial(g) \in [\sigma] \). Replacing \( g \) by one of its conjugates, we might as well assume that \( \partial(g) = \sigma \). Since \( g \in (G_{\partial(g)})_{cz} \), by Lemma 4.3.8, the term corresponding to the identity coset on the left-hand side of (4.3.14) is equal to 1. The terms corresponding to \( k \notin G_\sigma \) vanish, by (4.3.15), and so both sides of (4.3.14) equal 1 in this case. This completes the proof of part (1), from which part (2) follows immediately.

4.4 Jacquet Restriction in Homology

Returning from discrete groups to \( p \)-adic groups, we now compute the maps induced in Hochschild, smooth, and chamber homology by the Jacquet restriction functors \( r^G_M \). The results of this section will be used in the next section to prove the higher Clozel formula for \( SL_2(F) \).

Let \( G = GL_n(F) \) (or \( SL_n(F) \)), \( F \) a \( p \)-adic field, and let \( P \subset G \) be a standard parabolic subgroup with Levi decomposition \( P = MN \). Let \( P = M\overline{N} \) be the opposite parabolic. Let \( K = GL_n(\mathcal{O}) \) (or \( SL_n(\mathcal{O}) \)). We write \( K_P = K \cap P \), \( K_M = K \cap M \), and so on.

We may (and do) normalize the Haar measure \( d_G \) on \( G \) so that its restriction \( d_K \) to \( K \) has total volume 1. We likewise fix Haar measures \( d_M \) and \( d_N \), on \( M \) and \( N \) respectively, so that \( \text{vol}(K_M) = \text{vol}(K_N) = 1 \). Then the product measure \( d_P(mn) = d_M(m)d_N(n) \) is a (left) Haar measure on \( P \), and with these normalizations one has

\[
\int_G f(g) \, d_G(g) = \int_K \int_M \int_N f(nmk) \, \delta_P(m) \, d_N(n) \, d_M(m) \, d_K(k)
\]

(4.4.1)

for every \( f \in \mathcal{H}(G) \). (Here \( \delta_P = \delta \) is the modular function on \( P \); see Warning 2.1.37 for a
note on our convention regarding modular functions.) We will henceforth write $dn$ instead of $d_N(n)$, and likewise for $G$, $M$, and $K$, when no confusion is likely to arise.

We consider the (normalized) Jacquet functors $i^G_M$, $\tau^G_M$, $r^G_M$, and $\tilde{r}^G_M$, as defined in Section 3.3. Since the groups involved will be fixed throughout this section, we will denote these functors simply by $i$, $\tau$, $r$, and $\tilde{r}$.

As we observed in Section 4.1, the Jacquet functors induce maps between the Hochschild, smooth, and chamber homology groups associated to $G$ and $M$. We will compute the map $r$ in each of these three settings. The methods and results parallel those of Section 4.2 quite closely.

### 4.4.1 The map $r$ in Hochschild homology

We begin by computing the map $r : HH_u(\mathcal{H}(G)) \to HH_u(\mathcal{H}(M))$, which will turn out to be equal to one appearing in Nistor’s paper [Nis01]; see Remark 4.4.3. The result is similar to the corresponding result for discrete groups (Lemma 4.2.1 part (1)). In degree zero, we recover a formula of van Dijk [vD72, Theorem 2]; see Remark 4.4.6. A corollary of our result is the fact that $r$ is diagonal with respect to the compact/noncompact decomposition of homology; in degree zero, this is due to Dat [Dat03, Lemme 2.6].

**Theorem 4.4.2.** For each $q \geq 0$, define a map $R : \mathcal{H}(G^{q+1}) \to \mathcal{H}(M^{q+1})$ by

$$R(f)(m_0, \ldots, m_q) = \int_{K^{q+1}} \int_{N^{q+1}} f(k_0^{-1} n_0 m_0 k_1, k_1^{-1} n_1 m_1 k_2, \ldots, k_q^{-1} n_q m_q k_0) \delta_{\mathcal{H}}^1(m_0 m_1 \cdots m_q) \ dn \ dk$$

where $n = (n_0, \ldots, n_q)$ and $k = (k_0, \ldots, k_q)$. These maps define a morphism of precyclic modules $R : C(\mathcal{H}(G)) \to C(\mathcal{H}(M))$, whose induced map in Hochschild homology is equal to $r : HH_u(\mathcal{H}(G)) \to HH_u(\mathcal{H}(M))$.

**Remark 4.4.3.** The map $R$ appears in Nistor’s paper [Nis01], where it is denoted $\text{inf}_M^P \text{ind}_G^H$. Nistor proposes that this map be considered an analog, in Hochschild homology, of the Jacquet functor $r$. Theorem 4.4.2 makes the analogy precise.

The following lemma will be used to prove Theorem 4.4.2.
Lemma 4.4.4. (1) The map

\[ \Phi : r(\mathcal{H}(G)) \to \text{ind}_{K_M}^{K} \mathcal{H}(K_N \backslash K), \quad \Phi(f)(m)(k) = \delta_{P}^{-1/2}(m) \int_{N} f(nm^{-1}k) \, dn \]

(where \( f \in \mathcal{H}(G), m \in M \) and \( k \in K \)) is an isomorphism, equivariant both for the left action of \( M \) and for the right action of \( K \).

(2) For each \( f_1, f_2 \in \mathcal{H}(G), m_0 \in M \) and \( k \in K \) one has

\[ \Phi(f_1 * f_2)(m_0)(k_0) = \int_{K} \left( \int_{M} \Phi(f_1)(m)(k) \Phi(kf_2)(m_0m^{-1})(k_0) \, dm \right) dk. \]

Proof. (1) Routine computations show that \( \Phi \) is well-defined and equivariant. Integration over \( N \) gives a vector-space isomorphism \( r(\mathcal{H}(G)) \cong \mathcal{H}(N \backslash G) \), and the Iwasawa decomposition \( G = NMK \) implies that

\[ \mathcal{H}(N \backslash G) \cong \mathcal{H}(M \times_{K_M} (K_N \backslash K)) \cong \text{ind}_{K_M}^{K} \mathcal{H}(K_N \backslash K). \]

This shows that \( \Phi \) is a vector-space isomorphism; the equivariance properties are easily verified.

(2) Using the integration formula (4.4.1), we compute

\[ \Phi(f_1 * f_2)(m_0)(k_0) = \int_{N} (f_1 * f_2)(um_0^{-1}k_0) \delta_{P}^{-1/2}(m_0) \, du \]

\[ = \int_{N \times K \times M \times N} f_1(nm^{-1}k) f_2(k^{-1}mn^{-1}um_0^{-1}k_0) \delta_{P}^{-1/2}(m_0) \delta_{P}^{-1}(m) \, dn \, dm \, dk \, du \]

\[ = \int_{N \times K \times M \times N} f_1(nm^{-1}k) f_2(k^{-1}nm^{-1}um_0^{-1}k_0) \delta_{P}^{-1/2}(m_0) \, dn \, dk \, du \quad (\nu = mn^{-1}um^{-1}) \]

\[ = \int_{K \times M} \left( \int_{N} f_1(nm^{-1}k) \delta_{P}^{-1/2}(m) \, dn \right) \left( \int_{N} (kf_2)(\nu mm^{-1}k_0) \delta_{P}^{-1/2}(m_0m^{-1}) \, dv \right) \, dk \, du \]

\[ = \int_{K \times M} \left( \int_{N} f_1(nm^{-1}k) \delta_{P}^{-1/2}(m) \, dn \right) \Phi(f_2)(m_0m^{-1})(k_0) \, dk \, du. \]

Proof of Theorem 4.4.2. Given \( f = f_0 \otimes \cdots \otimes f_q \in \mathcal{H}(G^{q+1}) \), choose a compact, open, normal subgroup \( H \subset K \) such that each \( f_i \) is \( H \)-bi-invariant. Let \( e_H \in \mathcal{H}(G) \) be the normalized characteristic function of this subgroup.
Let $\rho$ denote the representation of $K_M$ on $H(K_N\backslash K/H)$ by left translation. Choose representatives $\kappa_1, \ldots, \kappa_d \in K$ for $K_N\backslash K/H$, and let $w_1, \ldots, w_d \in H(K_N\backslash K/H)$ denote the characteristic functions of the corresponding double cosets.

The map $\Phi$ of Lemma 4.4.4 restricts to an isomorphism

$$r(H(G)e_H) \cong \text{ind}_{K_M}^K \rho.$$ 

Letting $e_\rho \in H(M)$ be the idempotent associated to the representation $\rho$, we choose coordinates

$$\text{ind}_M^K \rho \overset{\alpha_i}{\longrightarrow} H(M)e_\rho$$

for $\text{ind}_M^K \rho$ as in Example 2.4.27:

$$(\alpha_i f)(m) = f(m^{-1})(\kappa_i)$$

and

$$(\beta_i f)(m) = \int_{K_M} f(m^{-1}k)kw_i \, dk.$$ 

Let $\text{Trace} : C(\text{End}_M(\text{ind}_M^K \rho)) \to C(e_\rho \mathcal{H}(M)e_\rho)$ be the morphism of precyclic modules induced by this choice of coordinates, as in (2.4.20):

$$\text{Trace}(T_0 \otimes \cdots \otimes T_q) = \sum_{(i_0, \ldots, i_q)} \alpha_{i_0}T_0\beta_{i_0}(e_\rho) \otimes \alpha_{i_1}T_1\beta_{i_1}(e_\rho) \otimes \cdots \otimes \alpha_{i_q}T_q\beta_{i_q}(e_\rho).$$

Each of the functions $f_i \in e_H \mathcal{H}(G)e_H$ defines an $M$-equivariant endomorphism of $r(H(G)e_H)$, by right multiplication. The isomorphism $\Phi$ therefore turns $f_i$ into an element of $\text{End}_M(\text{ind}_M^K \rho)$. We will show that

$$\text{Trace}(f_0 \otimes \cdots \otimes f_q) = R(f_0 \otimes \cdots \otimes f_q).$$

By the definition of the map $r$, this will establish Theorem 4.4.2.

We first compute $\beta_i(e_\rho)$:

$$\beta_i(e_\rho)(m) = \int_{K_M} e_\rho(m^{-1}k)kw_i \, dk = \begin{cases} mw_i & \text{if } m \in K_M, \\ 0 & \text{if } m \notin K_M, \end{cases}$$

(4.4.5)
because \( e_\rho \) is supported on \( K_M \) and acts as the identity on \( \rho \). Now we use part (2) of Lemma 4.4.4 to compute \( \alpha_j f \beta_i(e_\rho) \), for \( f \in e_H \mathcal{H}(G)e_H \):

\[
\alpha_j(f \beta_i(e_\rho))(m_0) = (f \beta_i(e_\rho))(m_0^{-1})(\kappa_j)
\]

\[
= \int_K \int_M (\beta_i(e_\rho))(m)(k) \Phi(kf)(m_0^{-1}m^{-1})(\kappa_j) \ dm \ dk \quad \text{(Lemma 4.4.4 (2))}
\]

\[
= \int_K \int_{K_M} w_k(m^{-1}k) \Phi(kf)(m_0^{-1}m^{-1})(\kappa_j) \ dm \ dk \quad \text{(4.4.5)}
\]

\[
= \int_K \int_{K_M} w_k(l) \Phi(mlf)(m_0^{-1}m^{-1})(\kappa_j) \ dm \ dl \quad \text{(l = m^{-1}k)}
\]

\[
= \int_{K_N \kappa_i H} \Phi(lf)(m_0^{-1})(\kappa_j) \ dl \quad \text{(\( \Phi \) is \( M \)-equivariant)}
\]

\[
= \text{vol}_K(K_N \kappa_i H) \Phi(\kappa_i f)(m_0^{-1})(\kappa_j),
\]

the final equality holding because the map \( \Phi : \mathcal{H}(G) \rightarrow \text{ind}_{K_M}^K \rho \) is left-\( N \)-invariant, and because \( f \) was assumed \( H \)-invariant.

Write \( v_i = \text{vol}(K_N \kappa_i H) \). For \( f_0 \otimes \cdots \otimes f_q \in (e_H \mathcal{H}(G)e_H)^{\otimes(q+1)} \) we have

\[
R(f_0 \otimes \cdots \otimes f_q)(m_0, \ldots, m_q)
\]

\[
= \int_{K^{q+1}} \Phi(k_0 f_0)(m_0^{-1})(k_1) \cdots \Phi(k_q f_q)(m_q^{-1})(k_0) \ dk
\]

\[
= \sum_{(i_0, \ldots, i_q)} (v_{i_0} \cdots v_{i_q}) \Phi(\kappa_{i_0} f_0)(m_0^{-1})(\kappa_{i_1}) \cdots \Phi(\kappa_{i_q} f_q)(m_q^{-1})(\kappa_{i_0})
\]

\[
= \sum_{(i_0, \ldots, i_q)} (\alpha_{i_0} f_0 \beta_{i_0}(e_\rho))(m_0) \cdots (\alpha_{i_q} f_q \beta_{i_q}(e_\rho))(m_q)
\]

\[
= \text{Trace}(f_0 \otimes \cdots \otimes f_q)(m_0, \ldots, m_q).
\]

This completes the proof of Theorem 4.4.2.

\[\square\]

**Remark 4.4.6.** Let \( \sigma \) be an admissible representation of \( M \). Combining Theorem 4.4.2 with Lemma 4.1.8 gives

\[
\langle \text{ch}_i \sigma, f \rangle = \langle \text{ch}_\sigma, Rf \rangle
\]

\[
(4.4.7)
\]
for each $f \in HH_0(\mathcal{H}(G))$. In degree zero, we have

$$Rf(m) = \int_K \int_N f(kmnk^{-1}) \delta_p^{1/2}(m) \, dn \, dk.$$  

This is van Dijk’s formula for $\text{ch}_{1_\sigma}$; see [vD72, Theorem 2].

Since the characters of admissible representations separate the points of $HH_0(\mathcal{H}(M))$ [Kaz86, Theorem 0], the formula (4.4.7) and Lemma 4.1.8 together imply that $R = r$ in degree-zero homology. Theorem 4.4.2 may therefore be viewed as an extension of van Dijk’s formula to higher-degree homology. The relation of the map $R$ to van Dijk’s formula was noted by Nistor [Nis01].

Recall from Example 2.4.4 that the algebra $\text{Cl}^c(G)$ acts on $HH_0(\mathcal{H}(G))$. In particular, the characteristic function $1_{G_c}$ defines an idempotent linear operator on $HH_0(\mathcal{H}(G))$. The same considerations apply to $HH_0(\mathcal{H}(M))$.

**Theorem 4.4.8.** The following diagram is commutative:

$$
\begin{array}{ccc}
HH_0(\mathcal{H}(G)) & \xrightarrow{1_{G_c}} & HH_0(\mathcal{H}(G)) \\
\downarrow r & & \downarrow r \\
HH_0(\mathcal{H}(M)) & \xrightarrow{1_{M_c}} & HH_0(\mathcal{H}(M)).
\end{array}
$$

In degree zero, this was proved by Dat [Dat03, Lemme 2.6].

**Proof.** Theorem 4.4.2 allows us to replace $r$ by $R$. For $f \in \mathcal{H}(G^{q+1})$, we have

$$\delta_p^{-1/2}(m_0 \cdots m_q) R(1_{G_c} f)(m_0, \ldots, m_q)$$

$$= \int_{K^{q+1}} \int_{N^{q+1}} (1_{G_c} f)(k_0m_0k_1^{-1}, \ldots, k_qm_qk_0^{-1}) \, dn \, dk$$

$$= \int_{K^{q+1}} \int_{N^{q+1}} 1_{G_c}(k_0m_0 \cdots n_qm_qk_0^{-1}) f(k_0m_0k_1^{-1}, \ldots, k_qm_qk_0^{-1}) \, dn \, dk$$

$$= \int_{K^{q+1}} \int_{N^{q+1}} 1_{G_c}(\nu m_0 m_1 \cdots m_q) f(k_0m_0k_1^{-1}, \ldots, k_qm_qk_0^{-1}) \, dn \, dk,$$

where $\nu \in N$ is a function of $n_0, \ldots, n_q, m_0, \ldots, m_q$. 

We claim that for \( n \in N \) and \( m \in M \), one has \( nm \in G_c \) if and only if \( m \in M_c \). Indeed, the quotient map \( P \to M \) sends the subgroup \( \langle nm \rangle \) generated by \( nm \) onto the subgroup \( \langle m \rangle \) generated by \( m \). If the latter is not precompact, then neither is the former; thus \( m \notin M_c \) implies \( nm \notin G_c \). Conversely, suppose \( m \in M_c \). Then \( L := \{ m^d nm^{-d} \mid d \in \mathbb{Z} \} \) is a precompact subset of \( N \). Since \( N \) is an increasing union of compact open subgroups, the subgroup \( \langle L \rangle \subset N \) generated by \( L \) is precompact. Then \( \langle nm \rangle \subset \langle L \rangle \) is precompact, and so \( nm \in M_c \). This proves the claim.

It follows that \( 1_{G_c}(vm_0m_1 \cdots m_q) = 1_{M_c}(m_0m_1 \cdots m_q) \), and so returning to the computation we find

\[
R(1_{G_c}f)(m_0, \ldots, m_q) = 1_{M_c}(m_0 \cdots m_q)R(f)(m_0, \ldots, m_q) = (1_{M_c}R(f))(m_0, \ldots, m_q). \quad \square
\]

**Remark 4.4.9.** Theorem 4.4.8 implies that the operator \( r \) is diagonal with respect to the decomposition of \( HH_a(\mathcal{H}(G)) \) and \( HH_a(\mathcal{H}(M)) \) into their compact and noncompact parts:

\[
(4.4.10) \quad r : \begin{bmatrix} HH_a(\mathcal{H}(G))_c \\ HH_a(\mathcal{H}(G))_{nc} \end{bmatrix} \xrightarrow{\begin{bmatrix} r_c & 0 \\ 0 & r_{nc} \end{bmatrix}} \begin{bmatrix} HH_a(\mathcal{H}(M))_c \\ HH_a(\mathcal{H}(M))_{nc} \end{bmatrix}.
\]

(Notation of Definition 3.2.2.)

The following example, taken from a paper of Dat, shows that the same is not true of the operator \( i \).

**Example 4.4.11.** [Dat03, Remarque, p.77] Let \( G = \text{SL}_2(F) \), \( M \) the diagonal subgroup, \( P \) the upper-triangular subgroup. Let \( f \in HH_0(\mathcal{H}(M)) \) be the class of the characteristic function of the compact open subset

\[
U := \begin{bmatrix} \varpi \mathcal{O}^\times & 0 \\ 0 & \varpi^{-1} \mathcal{O}^\times \end{bmatrix} \subset M_{nc}.
\]

Clozel’s formula (Theorem 4.1.9) implies that

\[
1_{G_c} i_{M}^G(f) = i_{M}^G(f) - \tau_M^G \circ \chi_M \circ i_{M}^G \circ i_{M}^G(f).
\]
Bernstein and Zelevinsky’s Geometrical Lemma [BZ77, 2.12] gives

\[ i^G_M \circ i^G_M (f) = f + f^w, \]

where \( w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) acts on \( \mathcal{H}(M) \) by conjugation; explicitly, \( f^w(m) = f(m^{-1}) \).

Now, \( U \subset M^+ \), and \( U^{-1} \cap M^+ = \emptyset \), so \( \chi_M(f + f^w) = f \). Thus

\[ 1_{G_c} i^G_M(f) = i^G_M(f) - r^G_M(f). \]

Pairing with the character \( \text{ch}_{\text{triv}_G} \) of the trivial representation of \( G \), we find

\[
\langle \text{ch}_{\text{triv}_G}, 1_{G_c} i^G_M(f) \rangle = \langle \text{ch}_{\text{triv}_G}, i^G_M(f) \rangle - \langle \text{ch}_{\text{triv}_G}, r^G_M(f) \rangle \\
= \langle \text{ch}_{\text{triv}_G}, f \rangle - \langle \text{ch}_{\text{triv}_G}, f \rangle \quad \text{(Lemma 4.1.8)} \\
= \langle \text{ch}_{\delta^{1/2}_p}, f \rangle - \langle \text{ch}_{\delta^{-1/2}_p}, f \rangle \\
= \int_U (\delta^{1/2}_p - \delta^{-1/2}_p)(u) \, d_M(u) \neq 0.
\]

This shows that the upper-right matrix entry \( i_{c,nc} \) is nonzero. (We will see in Corollary 4.5.11 that \( i^G_M \) is diagonal in degree one.)

### 4.4.2 The map \( r \) in smooth homology

We now turn to the map \( r : H_* (G, \mathcal{H}(G)_{Ad}) \to H_* (M, \mathcal{H}(M)_{Ad}) \) in smooth homology. The identification of this map follows from Theorem 4.4.2 and from computations of Nistor [Nis01, Theorems 6.2 and 6.5].

Recall the following notation from Definition 2.1.38: for each smooth representation \( W \) of \( M \), \( W_{\delta^{1/2}_p} \) denotes twisting of \( W \) by the character \( \delta^{1/2}_p \) of \( M \to \mathbb{C}^\times \):

\[ W_{\delta^{1/2}_p} := W \otimes_{\mathbb{C}} \delta^{1/2}_p. \]
Lemma 4.4.12. For each smooth $G$-module $V$, the vector-space map

$$V \to V, \quad v \mapsto \int_K k v \, dk$$

induces a natural map $\kappa : V_G \to (r(V)_{\delta^{1/2}})_M$.

Proof. Since $\delta$ is trivial on $N$, we have $(r(V)_{\delta^{1/2}})_M \cong (V_\delta)_P$ (recall that $r$ already incorporates a twist by $\delta^{1/2}$). For every $V$ we have a natural isomorphism $V \cong \mathcal{H}(G) \otimes_{\mathcal{H}(G)} V$, and the map

$$\kappa : \mathcal{H}(G) \to \mathcal{H}(G), \quad \kappa(f)(g) = \int_K f(k^{-1}g) \, dk$$

is equivariant for the right-translation action of $G$. Thus it will suffice to prove that $\kappa$ induces a well-defined map $\mathcal{H}(G)_G \to (\mathcal{H}(G)_\delta)_P$.

The maps

$$\varphi : \mathcal{H}(G)_G \to \mathbb{C}, \quad f \mapsto \int_G f(g) \, dg$$

and

$$\psi : (\mathcal{H}(G)_\delta)_P \to \mathcal{H}(P \backslash G), \quad f \mapsto \int_P f(pg) \delta^{-1}(p) \, dp$$

are isomorphisms [Cas81, A.6], and the integration formula (4.4.1) implies that $\varphi = \psi \circ \kappa$. Thus, if $f = 0$ in $\mathcal{H}(G)_G$ then $\kappa(f) = 0$ in $(\mathcal{H}(G)_\delta)_P$. This proves that $\kappa$ is well-defined.

Applying $\kappa$ to each term in a projective resolution of $V$ gives a map in smooth homology,

$$\kappa : H_\bullet(G, V) \to H_\bullet(M, r(V)_{\delta^{1/2}})$$

(cf. Definition 4.2.10).

Definition 4.4.13. Let $f : r(V)_{\delta^{1/2}} \to W$ be an $M$-equivariant map, where $V$ is a $G$-module and $W$ is an $M$-module. Let $f_\bullet : H_\bullet(G, V) \to H_\bullet(M, W)$ be the composition

$$H_\bullet(G, V) \xrightarrow{\Delta} H_\bullet(M, r(V)_{\delta^{1/2}}) \xrightarrow{f} H_\bullet(M, W).$$

As in the discrete case (Remark 4.2.9), the map $\kappa$ may be described more naturally, as
follows. Frobenius reciprocity gives a natural map $V \to i \circ r(V)$, and there is a Shapiro-type isomorphism (cf. [Cas81, A.8])

$$H_\ast(G, iW) \cong H_\ast(M, W_{\delta/2}).$$

Then $\kappa$ is the composition

$$H_\ast(G, V) \xrightarrow{\text{Frobenius}} H_\ast(G, i \circ r(V)) \xrightarrow{\text{Shapiro}} H_\ast(M, r(V)_{\delta/2}).$$

**Remark 4.4.14.** Bernstein’s Second Adjoint theorem provides a second natural transformation, $i \circ r(V) \to V$, and an associated map

$$H_\ast(M, r(V)_{\delta/2}) \xrightarrow{\text{Shapiro}} H_\ast(G, i \circ r(V)) \xrightarrow{\text{Bernstein}} H_\ast(G, V).$$

The appearance of the Second Adjoint theorem, in place of the much easier Frobenius reciprocity, is what makes the maps $i$ rather more difficult to compute than the maps $r$.

**Theorem 4.4.15.** (cf. [Nis01, Theorems 6.2 and 6.5]) The map $r : \mathcal{H}(G) \to \mathcal{H}(M)$ defined by

$$r(f)(m) = \int_N f(nm) \delta^{1/2}(m) \, dn$$

descends to an $M$-equivariant map $r : r(\mathcal{H}(G))_{\delta/2} \to \mathcal{H}(M)_{\text{Ad}}$. The induced map

$$r_* : H(G, \mathcal{H}(G))_{\text{Ad}} \to H(M, \mathcal{H}(M))_{\text{Ad}}$$

is equal to $r$, the map induced by Jacquet restriction.

**Proof.** Computations of Nistor in [Nis01] show that the diagram

$$
\begin{array}{c}
HH_\ast(\mathcal{H}(G)) \xrightarrow{R} HH_\ast(\mathcal{H}(M)) \\
\cong \downarrow \quad \cong \\
H_\ast(G, \mathcal{H}(G)_{\text{Ad}}) \xrightarrow{r_*} H_\ast(M, \mathcal{H}(M)_{\text{Ad}})
\end{array}
$$

commutes, where $R$ is the map appearing in Theorem 4.4.2, and the vertical arrows are the
canonical isomorphisms. (We note that our maps $R$ and $r_*$ differ from their counterparts in [Nis01] by a modular factor). In view of Theorem 4.4.2, this implies that $r_*=r$. \hfill \Box

### 4.4.3 The map $r$ in chamber homology

Now suppose that $G$ and $M$ act on buildings $X_G$ and $X_M$, satisfying the hypotheses (GAB1–3) of Section 2.3.2. Theorem 4.4.8 says that the map $r$ is diagonal with respect to the compact/noncompact decomposition of Hochschild homology: $r = r_c + r_{nc}$, where $r_c : HH_*(\mathcal{H}(G))_c \to HH_*(\mathcal{H}(M))_c$. Identifying the compact part of Hochschild homology with chamber homology, we may view $r_c$ as a map

$$r_c : H^G_*(X_G) \to H^M_*(X_M).$$

We shall compute this map explicitly for $G = \text{SL}_n(F)$, $X_G$ the Bruhat-Tits building, $M$ the diagonal subgroup, $X_M$ the apartment in $X_G$ stabilized by $M$, and $N$ the unipotent upper-triangular subgroup. We will find, as in the discrete case, that this map comes from applying the obvious analog of the Jacquet functor $r$ to the compact subgroups of $G$.

The first step is to define $r$ as a functor on coefficient systems.

**Definition 4.4.17.** Let $\mathcal{F}$ be a $G$-equivariant coefficient system on $X_G$. Define an $M$-equivariant coefficient system $r(\mathcal{F})_{\delta^{1/2}}$ on $X_M$ as follows.

- For each simplex $\sigma \subset X_M$,
  $$r(\mathcal{F})_{\delta^{1/2}}(\sigma) = \mathcal{F}(\sigma)_{N_\sigma},$$
  the space of $N_\sigma$-coinvariants for the representation $\mathcal{F}(\sigma)$ of the group $G_\sigma$.

- The transition maps in $r(\mathcal{F})_{\delta^{1/2}}$ are the ones induced by the transition maps in $\mathcal{F}$: i.e., for each simplex $\sigma$ and each face $\tau \subset \sigma$, the diagram

$$
\begin{array}{ccc}
\mathcal{F}(\sigma) & \xrightarrow{\mathcal{F}(\tau,\sigma)} & \mathcal{F}(\tau) \\
\downarrow \text{quotient} & & \downarrow \text{quotient} \\
\mathcal{F}(\sigma)_{N_\sigma} & \xrightarrow{r(\mathcal{F})_{\delta^{1/2}}(\tau,\sigma)} & \mathcal{F}(\tau)_{N_\tau}
\end{array}
$$
commutes. (Note that this is a valid definition, because $N_\sigma \subseteq N_\tau$.)

- The $M$-action is defined by twisting the original $M$-action by $\delta$:

$$m : F(\sigma)_N \to F(m\sigma)_N, \quad m \cdot v := \delta(m)\left(m \cdot v\right).$$

**Lemma 4.4.18.** There is a natural isomorphism of $M$-equivariant chain complexes,

$$r\left(C_*(X_G, F)\right)_{1/2} \cong C_*(X_M, r(F)_{1/2}).$$

**Proof.** The inclusion of $X_M$ into $X_G$ induces an isomorphism $X_M \cong X_G/N$: indeed, the Iwasawa decomposition $G = \bigsqcup_{w \in N_G(M)/M} NMwI$ ($I$ the standard Iwahori subgroup) implies that $X_G = NX_M$; the fact that for each vertex $v \in X_M$ one has $G_v \cap MN = (G_v \cap M) \ltimes (G_v \cap N)$ ensures that no two distinct vertices in $X_M$ lie in the same $N$-orbit (cf. [Ren10, V.5.1]). Thus the space of $N$-coinvariants $C_*(X, F)_N$ identifies with a complex of simplicial chains on $X_M$, the coefficients over a simplex $\sigma \subset X_M$ being the space of coinvariants $F(\sigma)_N \sigma$. It remains to observe that the modular factors coincide; recall that the functor $r$ already incorporates a twist by $\delta^{1/2}$.

Lemmas 4.4.12 and 4.4.18 give a map of chain complexes

$$\kappa : C_*(X_G, F)_G \to C_*(X_M, r(F)_{1/2})_M,$$

and every map $f : r(F)_{1/2} \to E$ of $M$-equivariant coefficient systems induces a map $f_* : H_*^G(X_G, F) \to H_*^M(X_M, E)$, by composition with $\kappa$.

Now consider the coefficient system $R_G(\sigma) = R_C(G_\sigma)$, in which the transition maps are given by induction of representations, and $G$ acts by

$$g : R_C(G_\sigma) \to R_C(G_{g\sigma}), \quad \pi \mapsto \pi^{g^{-1}},$$

(see Lemma 2.3.26). Let $R_M$ be the corresponding coefficient system on $X_M$. For each
simplex $\sigma$ in $X_M$, the group $G_\sigma$ acts trivially on $R_C(G_\sigma)$, and so we have
$$r(R_G)_{G/\Sigma}(\sigma) = R_C(G_\sigma).$$

For each representation $\pi$ of $G_\sigma$, the coinvariants space $\pi_{N_\sigma}$ is a representation of $M_\sigma$.

**Proposition 4.4.19.** For each simplex $\sigma \in X_M$, define a linear map
$$r : r(R_G)_{G/\Sigma}(\sigma) \to R_M(\sigma), \quad r(\pi) = \frac{\text{vol}_F(P_\sigma)}{\text{vol}_G(G_\sigma)} \pi_{N_\sigma}.$$

Then $r$ is a map of $M$-equivariant coefficient systems, and the induced map $r_\ast : H^G_\ast(X_G) \to H^M_\ast(X_M)$ is equal to $r_c$, the map induced by Jacquet restriction.

**Proof.** The equalities
$$(\pi^{m^{-1}})_{N_\ast^{m^{-1}}} \cong (\pi_{N_\ast})^{m^{-1}}$$
and
$$\delta(m) \frac{\text{vol}_P(P_\sigma^{m^{-1}})}{\text{vol}_G(G_\sigma^{m^{-1}})} = \frac{\text{vol}_F(P_\sigma)}{\text{vol}_G(G_\sigma)}$$
imply that $r$ is $M$-equivariant.

Consider the coefficient systems $G(\sigma) = \mathcal{H}(G_\sigma)$ and $M(\sigma) = \mathcal{H}(M_\sigma)$ (Definition 2.3.24). For each $\sigma \in X_M$, define a map
$$r' : \mathcal{H}(G_\sigma) \to \mathcal{H}(M_\sigma) \quad r'(f)(m) := \int_{N_\sigma} f(nm) \, dn.$$

The diagram
\begin{align*}
(4.4.20) & \quad \frac{\text{ch}_\pi}{\text{vol}_G(G_\sigma)} \quad \downarrow \pi \quad \frac{\text{ch}_\rho}{\text{vol}_M(M_\sigma)} \\
& \quad \mathcal{H}(G_\sigma) \quad \frac{r'}{r} \quad \mathcal{H}(M_\sigma) \quad \mathcal{H}(G_\sigma) \quad \mathcal{H}(M_\sigma)
\end{align*}
commutes:
$$r'(\text{ch}_\pi)(m) = \int_{N_\sigma} \text{ch}_\pi(nm) \, dn = \text{vol}_N(N_\sigma) \text{ch}_{\pi_{N_\sigma}}(m) = \frac{\text{vol}_F(P_\sigma)}{\text{vol}_M(M_\sigma)} \frac{\text{ch}_{\pi_{N_\sigma}}}{\text{vol}_M(M_\sigma)}(m).$$
The vertical arrows in (4.4.20) are injective maps of coefficient systems, which become isomorphisms in homology (Lemma 2.3.26). The top horizontal arrow \( r' \) induces a map of \( M \)-equivariant coefficient systems \( r(G)_{\delta^{1/2}} \to \mathcal{M} \), and the induced map in homology \( r'_* : H^G_M(X_G) \to H^M_M(X_M) \) is equal to \( r_c \) (Theorem 4.4.15; note that the modular factor vanishes on \( M_c \)). So the commutativity of (4.4.20) implies that \( r \) is a map of coefficient systems, and that \( r_* = r_c \) on chamber homology.

**Example 4.4.21.** Let \( G = \text{SL}_2(F) \), and deploy the following notation:

\[
M = \begin{bmatrix} F^x & 0 \\ 0 & F^x \end{bmatrix}, \quad L = \begin{bmatrix} \mathcal{O}^x & 0 \\ 0 & \mathcal{O}^x \end{bmatrix}, \quad N = \begin{bmatrix} 1 & F \\ 0 & 1 \end{bmatrix}, \quad \overline{N} = \begin{bmatrix} 1 & 0 \\ F & 1 \end{bmatrix},
\]

\[
K = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}, \quad K' = \begin{bmatrix} \mathcal{O} & p^{-1} \\ p & \mathcal{O} \end{bmatrix}, \quad I = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} \end{bmatrix}, \quad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]

\[
N_K = N \cap K, \quad N_{K'} = N \cap K', \quad N_I = N \cap I, \quad \overline{N}_I = \overline{N} \cap I.
\]

The complex \( C_a(X_G, R_G)_G \) is isomorphic to

\[
R_C(I) \xrightarrow{\delta_G} R_C(K) \oplus R_C(K'), \quad \delta_G(\pi) = \text{ind}_I^K \pi \oplus -\text{ind}_{I'}^{K'} \pi,
\]

while \( C_a(X_M, R_M)_M \) is isomorphic to

\[
R_C(L) \oplus R_C(L) \xrightarrow{\delta_M} R_C(L) \oplus R_C(L), \quad \delta_M(\rho_0, \rho_1) = (\rho_0 + \rho_1, -\rho_0 - \rho_1).
\]

The map \( r_* : H^G_M(X_G) \to H^M_M(X_M) \) is given by the map of complexes

\[
\pi \quad \xrightarrow{} \quad R_C(I) \xrightarrow{\delta_G} R_C(K) \oplus R_C(K') \xrightarrow{} \quad (\pi, \pi')
\]

\[
\left( \pi_{N_I}, (\pi_{\overline{N} I})^w \right) \quad R_C(L) \oplus R_C(L) \xrightarrow{\delta_M} R_C(L) \oplus R_C(L) \xrightarrow{} \quad (\pi_{N_K}, \pi'_{N_{K'}})
\]

as can be verified by a routine computation.
Corollary 4.4.22. Let \( M \) be the diagonal subgroup of \( G = \text{SL}_2(F) \). The Jacquet restriction map \( r_c : H^G_1(X_G) \to H^M_1(X_M) \) is injective.

**Proof.** The group \( H^G_1(X_G) \) was computed by Baum, Higson and Plymen in [BHP93], and their results imply that the map

\[
    \mathcal{R}_C(I) \to \mathcal{R}_C(M_c), \quad \pi \mapsto \pi^{N_I}
\]

is injective on \( H^G_1(X_G) \) (see also Sections 5.3 and 5.5, below). By Proposition 4.4.19, this ensures that \( r_c : H^G_1(X_G) \to H^M_1(X_M) \) is injective. \( \square \)

## 4.5 The Higher Clozel Formula for \( \text{SL}_2(F) \)

Clozel’s formula (Theorem 4.1.9) for \( G = \text{SL}_2(F) \) says the following: Let \( M \subset G \) be the diagonal subgroup, and let \( \chi : M \to \{0,1\} \) be the characteristic function of the subset \( M^+ := \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid 0 < |a| < 1 \} \).

Writing \( \mathfrak{T} = \mathfrak{T}^G_M \) and \( r = r^G_M \), the following is an equality of operators on \( HH^0(\mathcal{H}(G)) \):

\[
    (4.5.1) \quad \text{id} = 1_{G_c} + \mathfrak{T} \circ \chi \circ r.
\]

(Note that this is equivalent to the assertion that \( \mathfrak{T} \circ \chi \circ r = 1_{G_{nc}} \).)

In this section, we prove:

**Theorem 4.5.2.** The formula (4.5.1) also holds on \( HH_1(\mathcal{H}(G)) \).

Since \( HH_n(\mathcal{H}(G)) = 0 \) for all \( n \geq 2 \), this means that (4.5.1) holds in all degrees.

**Corollary 4.5.3.** Let \( G = \text{SL}_2(F) \), and let \( E \in \mathfrak{Z}(G) \) be an idempotent in the Bernstein center. Then \( E \) and \( 1_{G_c} \) commute as operators on \( HH_1(\mathcal{H}(G)) \).

**Proof.** This follows from the Clozel formula, via the argument of Dat; see Corollary 4.1.17. \( \square \)
The proof of Theorem 4.5.2 occupies the rest of this chapter. Note that the function $\chi$ is supported on the noncompact part $M_{nc}$ of $M$, and so in the matrix notation of Definition 3.2.2 we have

$$\mathbb{T}_c \circ \chi \circ r = \begin{bmatrix}
\mathbb{T}_c & \mathbb{T}_{c,nc} \\
\mathbb{T}_{nc,c} & \mathbb{T}_{nc}
\end{bmatrix} \circ \begin{bmatrix}
0 & 0 \\
0 & \chi
\end{bmatrix} \circ \begin{bmatrix}
r_c & 0 \\
0 & r_{nc}
\end{bmatrix} = \begin{bmatrix}
0 & \mathbb{T}_{c,nc} \circ \chi \circ r_{nc} \\
0 & \mathbb{T}_{nc} \circ \chi \circ r_{nc}
\end{bmatrix}. $$

(Recall that $r$ is diagonal, by Theorem 4.4.8.)

We therefore wish to prove that

$$\mathbb{T}_{c,nc} \circ \chi \circ r_{nc} = 0 \quad \text{and} \quad \mathbb{T}_{nc} \circ \chi \circ r_{nc} = 1_{G_{nc}}$$

on $HH_1(\mathcal{H}(G))$. We begin by computing $r_{nc}$.

### 4.5.1 The map $r_{nc}$

Conjugation by the matrix $w = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ defines a linear automorphism of $\mathcal{H}(M)$; explicitly, $(wf)(m) = f(m^{-1})$. The Weyl group $W = \{1, w\}/\pm 1$ thus acts on $\mathcal{H}(M)$, and this action restricts to $\mathcal{H}(M_{nc})$. As usual, we let $\mathcal{H}(M_{nc})^W$ denote the space of $W$-invariant functions.

**Proposition 4.5.5.** The noncompact part $HH_*(\mathcal{H}(G))_{nc}$ of the Hochschild homology of $G$ is the homology of the complex

$$\mathcal{H}(M_{nc}) \xrightarrow{\partial_G} \mathcal{H}(M_{nc})^W \oplus \mathcal{H}(M_{nc})^W, \quad \partial_G(f) = (f + w f, -f - w f).$$

**Proof.** Consider the action of $G$ on its Bruhat-Tits tree $X_G$ (Example 2.3.15). This action satisfies the hypotheses (GAB1–3+), and so Proposition 2.3.39 implies that

$$HH_*(\mathcal{H}(G))_{nc} \cong H_*^G(X_G, \mathcal{G}^{++}),$$

where $\mathcal{G}^{++}$ is the coefficient system

$$\mathcal{G}^{++}(\sigma) = \mathcal{H}(G^{++}_\sigma), \quad G^{++}_\sigma = \{g \in G_{nc} \mid \sigma \subset \text{min}(g)\}$$
(Definition 2.3.35).

As in Example 2.3.29, we have $H^G_*(X_G, G^{++}) \cong H^G_*(\Delta, (G^{++})^G)$, where $\Delta$ is the standard chamber

$$v \rightarrow e \rightarrow v'$$

$G_v = K = \text{SL}_2(\mathcal{O}), \quad G_{v'} = K' = \left[ \begin{array}{cc} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{array} \right], \quad G_\epsilon = I = \left[ \begin{array}{cc} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{array} \right],$

and $(G^{++})^G$ is the coefficient system

$$(G^{++})^G(\sigma) = \mathcal{H}(G^{++})^G_{\text{Ad}}.$$  

The isotropy group $K = G_v$ acts transitively on the set of oriented lines through $v$: this is equivalent (cf. [Ser03, p. 72]) to the assertion that $\text{SL}_2(\mathcal{O})$ acts doubly-transitively on the projective line $\mathbb{P}^1(F)$, which assertion is easily verified. Thus every oriented line through $v$ is $K$-conjugate to the apartment $X_M \subset X_G$ stabilized by the diagonal subgroup $M$, and restriction of functions gives an isomorphism

$$\mathcal{H}(K^{++})^{K}_{\text{Ad}} \cong \mathcal{H}(M_{nc})^W.$$  

The same argument applies to the stabilizer $K'$ of $v'$, giving $\mathcal{H}(K'^{++})^{K'}_{\text{Ad}} \cong \mathcal{H}(M_{nc})^W$.

The Iwahori subgroup $I = G_\epsilon$ acts transitively on the set of lines containing $\epsilon$, but preserves the orientation of each line. Restriction of functions thus gives an isomorphism

$$\mathcal{H}(I^{++})^{I}_{\text{Ad}} \cong \mathcal{H}(M_{nc}).$$  

The formula for the differential $\partial_G$ follows from Lemma 2.3.23.  

\begin{remark}
Computing the homology of \eqref{4.5.6}, we find that

$$HH_n(\mathcal{H}(G))_{\text{nc}} \cong \begin{cases} \{ f \in \mathcal{H}(M_{nc}) \mid wf = f \} & \text{if } n = 0, \\
\{ f \in \mathcal{H}(M_{nc}) \mid wf = -f \} & \text{if } n = 1, \\
0 & \text{if } n \geq 2. \end{cases}$$
\end{remark}
This result also follows from [BB92, Proposition 6.2], which was proved by a different method.

The noncompact part of Hochschild homology for $M$ may likewise be computed using the building $X_M$ and Proposition 2.3.39: one finds that $HH_*(\mathcal{H}(M))_{nc}$ is the homology of the complex

$$
\mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \xrightarrow{\partial_M} \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}), \quad \partial_M(f_1, f_2) = (f_1 + f_2, -f_1 - f_2).
$$

(Cf. Example 2.3.28.)

**Lemma 4.5.8.** The map $r_{nc} : HH_*(\mathcal{H}(G))_{nc} \to HH_*(\mathcal{H}(M))_{nc}$ is the one induced by the map of complexes

$$
\begin{aligned}
&\xymatrix{ f & \mathcal{H}(M_{nc}) \ar[r]^-{\partial_G} & \mathcal{H}(M_{nc})^W \oplus \mathcal{H}(M_{nc})^W \ar[d] & (f, f') \\
(\xi \cdot f, \xi \cdot w f) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \ar[r]^-{\partial_M} & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \ar[d] & (\xi \cdot f, \xi \cdot f')
}
\end{aligned}
$$

where $\xi$ is the function

$$
\xi ([\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}]) = \max\{ |a|, |a|^{-1} \}
$$

and the products $\xi \cdot f$ etc. are pointwise products of functions on $M$.

**Proof.** Theorem 4.4.15 implies that $r_{nc}$ is the map induced in homology by the composition

$$
C_*(X_G, G^{++})_G \xrightarrow{\kappa} C_*(X_M, t(G^{++})_{\delta^{1/2}})_M \xrightarrow{r'} C_*(X_M, M^{++})_M,
$$

where $\kappa$ is the integration-over-$K$ map (Lemma 4.4.12), and $r'$ is the $M$-equivariant map of coefficient systems

$$
r' : \mathcal{H}(G^{++})_G \to \mathcal{H}(M^{++}), \quad r'(f)(m) = \int_N f(nm) \delta^{1/2}(m) \, dn.
$$

We will show that $r'\kappa(f) = \xi \cdot f$ for $f \in \mathcal{H}(K^{++})$; the computations for $K'$ and $I$ are similar.
Fix \( f \in \mathcal{H}(K^{++}) \); we might as well assume from the beginning that \( f \) is invariant under conjugation by \( K \), so that \( \kappa(f) = f \). We also fix \( m = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in M_{nc} \), so that \( \delta(m) = |a|^{-2} \). Let \( n = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \) be an element of \( N \). The matrix \( nm \) lies in \( K^{++} \) if and only if it is conjugate, via an element of \( K \), to \( m \). This is the case if and only if \( b(1 - a^2)^{-1} \in \mathcal{O} \), and so we have

\[
\rho'(f)(m) = \int_N f(nm) |a|^{-1} \, dn = \int_{(1-a^2) \mathcal{O}} f(m) |a|^{-1} \, dn.
\]

If \( |a| > 1 \), then \( (1 - a^2) \mathcal{O} = \mathcal{O} \) has volume 1. If, on the other hand, \( |a| < 1 \), then \( (1 - a^2) \mathcal{O} = a^2 \mathcal{O} \) has volume \( |a|^2 \). Thus

\[
\rho'(f)(m) = \begin{cases} f(m) |a|^{-1} & \text{if } |a| < 1, \\ f(m) |a| & \text{if } |a| > 1. \end{cases}
\]

So \( \rho'(f) = \xi \cdot f \) as claimed. \( \square \)

**Corollary 4.5.9.** The map \( r_{nc} \) is injective.

**Proof.** This follows from Lemma 4.5.8 by an easy computation. \( \square \)

### 4.5.2 The action of the Weyl group

We noted above that the Weyl group \( W \) acts on \( \mathcal{H}(M) \) by conjugation. This is an action by algebra automorphisms, so \( W \) acts on \( HH_\bullet(\mathcal{H}(M)) \). The decomposition \( M = M_c \cup M_{nc} \) is \( W \)-invariant, and so the action of \( W \) is diagonal with respect to the compact/noncompact decomposition of Hochschild homology.

The following is a version of Bernstein and Zelevinsky’s Geometrical Lemma (see [BZ77, 2.12], [BR92, III.1.2], or [Ren10, VI.5.1]):

**Theorem 4.5.10.** \( r \circ \iota = r \circ \iota = \text{id} + w \) as operators on \( HH_\bullet(\mathcal{H}(M)) \).

**Proof.** The cited result of Bernstein and Zelevinsky implies that \( r \circ \iota = r \circ \iota = \text{id} \oplus w \) as functors on \( \mathcal{P}_M \) (note that the filtration in [BZ77, 2.12] becomes a direct sum when one restricts to projective modules). The theorem therefore follows from the isomorphism \( HH_\bullet(\mathcal{P}_M) \cong HH_\bullet(\mathcal{H}(M)) \) of Proposition 2.4.23. \( \square \)
This theorem, along with the injectivity properties of \( r \) established above, yields the following information about the maps \( i \) and \( \overline{r} \).

**Corollary 4.5.11.** (1) \( \tau_{nc} = i_{nc} \) as maps \( HH_0(\mathcal{H}(M))_{nc} \to HH_0(\mathcal{H}(G))_{nc} \).

(2) \( \overline{r} = i \) as maps \( HH_1(\mathcal{H}(M)) \to HH_1(\mathcal{H}(G)) \).

(3) \( \tau : HH_0(\mathcal{H}(M)) \to HH_0(\mathcal{H}(G)) \) is upper-triangular: \( \tau_{nc,c} = 0 \) in degree zero.

(4) \( \tau : HH_1(\mathcal{H}(M)) \to HH_1(\mathcal{H}(G)) \) is diagonal: \( \tau_{c,nc} = \tau_{nc,c} = 0 \) in degree one.

**Proof.** Parts (1) and (2): Theorem 4.5.10 states that \( r \circ \tau = r \circ i \). Since \( r_{nc} \) is injective in both degree zero and in degree one (Corollary 4.5.9), we have \( \tau_{nc} = i_{nc} \) in both degrees. In degree one, \( r_c \) is also injective (Corollary 4.4.22), giving \( \tau = i \) in degree one.

Parts (3) and (4): We have

\[
\tau \circ \overline{r} = \begin{bmatrix} r_c & 0 \\ 0 & r_{nc} \end{bmatrix} \circ \begin{bmatrix} \overline{\tau}_c & \overline{\tau}_{c,nc} \\ \overline{\tau}_{nc,c} & \overline{\tau}_{nc} \end{bmatrix} = \begin{bmatrix} r_c \circ \overline{\tau}_c & r_c \circ \overline{\tau}_{c,nc} \\ r_{nc} \circ \overline{\tau}_{nc,c} & r_{nc} \circ \overline{\tau}_{nc} \end{bmatrix}.
\]

Theorem 4.5.10 implies that \( r \circ \overline{r} \) is diagonal, and so the compositions \( r_{nc} \circ \tau_{nc,c} \) and \( r_c \circ \tau_{c,nc} \) are both zero. Parts (3) and (4) now follow from the injectivity properties of \( r \), as in parts (1) and (2).

In terms of chamber homology, the action of \( W \) on \( HH_*(\mathcal{H}(M))_{nc} \) is given by the natural action of \( W \) on the building \( X_M \) of \( M \):

**Lemma 4.5.12.** The automorphism \( w : HH_*(\mathcal{H}(M))_{nc} \to HH_*(\mathcal{H}(M))_{nc} \) is the one induced by the following map of chain complexes:

\[
\begin{array}{ccc}
(f_0, f_1) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \\
(f_0, f_1) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \\
(wf_1, wf_0) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \\
(wf_0, wf_1) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc})
\end{array}
\]

\[\xymatrix{ \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \ar[r]^{\partial_M} & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) \ar[r]^{\partial_M} & \mathcal{H}(M_{nc}) \oplus \mathcal{H}(M_{nc}) }\]
Proof. According to Lemma 2.4.12, \( w \) may be computed as follows. Consider the \( M \)-equivariant chain complex \( C_*(X_M, \mathcal{M}^{++}) \) which is obtained from \( C_*(X_M, \mathcal{M}^{++}) \) by twisting the \( M \)-action by \( w \). Choose an \( M \)-equivariant map of complexes

\[
\varphi : C_* (X_M, \mathcal{M}^{++}) \to C_* (X_M, \mathcal{M}^{++})_w
\]

lifting \( w : \mathcal{H}(\mathcal{M})_{Ad} \to \mathcal{H}(\mathcal{M})_{Ad} \). Then the map \( w : H_*^M (X_M, \mathcal{M}^{++}) \to H_*^M (X_M, \mathcal{M}^{++}) \) is the one induced by \( \varphi \), upon passing to \( M \)-coinvariants and taking homology.

There is a natural choice for \( \varphi \) : identifying \( C_* (X_M, \mathcal{M}^{++}) \) with \( C_* (X_M, \mathcal{C}) \otimes \mathcal{H}(\mathcal{M}) \), we let \( \varphi \) act on \( C_* (X_M, \mathcal{C}) \) by reflecting through the origin, and on \( \mathcal{H}(\mathcal{M})_{Ad} \) via the map \( w \). Then (4.5.13) is map induced by \( \varphi \) on the complex \( C_* (X, \mathcal{M}^{++})_M \) of coinvariants.

4.5.3 The map \( \iota_{nc} \)

Definition 4.5.14. Define \( \iota_{nc} : HH_* (\mathcal{H}(\mathcal{M}))_{nc} \to HH_* (\mathcal{H}(\mathcal{G}))_{nc} \) via the map of complexes

\[
\begin{array}{ccc}
\mathcal{H}(\mathcal{M})_{nc} & \xrightarrow{\delta_G} & \mathcal{H}(\mathcal{M})_{nc}^{W} \oplus \mathcal{H}(\mathcal{M})_{nc}^{W} \\
(\mathcal{H}(\mathcal{M})_{nc}^{W} \oplus \mathcal{H}(\mathcal{M})_{nc}^{W}) & \xrightarrow{\delta_M} & (\mathcal{H}(\mathcal{M})_{nc}^{W} \oplus \mathcal{H}(\mathcal{M})_{nc}^{W})
\end{array}
\]

Proposition 4.5.15. One has \( \iota_{nc} = \tau_{nc} = i_{nc} \) as endomorphisms of \( HH_* (\mathcal{H}(\mathcal{M}))_{nc} \).

Proof. A straightforward computation, using the descriptions of \( r_{nc} \) and \( w \) given in Lemmas 4.5.8 and 4.5.12, shows that

\[
r_{nc} \circ \iota_{nc} = \text{id} + w.
\]

The equality \( r_{nc} \circ \iota_{nc} = r_{nc} \circ i_{nc} = \text{id} + w \) (Theorem 4.5.10), together with the injectivity of \( r_{nc} \) (Corollary 4.5.9), implies that \( \iota_{nc} = \tau_{nc} = i_{nc} \).
4.5.4 Proof of Theorem 4.5.2

Proof. We observed in equation (4.5.4) that Theorem 4.5.2 reduces to the assertions

\[ \iota_{c,nc} \circ \chi \circ r_{nc} = 0 \quad \text{and} \quad \iota_{nc} \circ \chi \circ r_{nc} = 1_{G_{nc}} \]

as operators on $HH_1(\mathcal{H}(G))$.

The first equality follows from part (4) of Corollary 4.5.11 (which showed that $\iota_{c,nc} = 0$). Proposition 4.5.15 reduces the second equality to $\iota_{nc} \circ \chi \circ r_{nc} = 1_{G_{nc}}$, which is easily verified at the level of complexes:

\[ \iota \chi r(f) = \iota (\xi \cdot (wf), \xi \cdot (wf)) = \iota (\chi \xi \cdot f, \chi \xi \cdot (wf)) = \chi \cdot f + \chi \cdot (wf) = f \]

for $f \in \mathcal{H}(M_{nc})$. This completes the proof of Theorem 4.5.2.

Remark 4.5.16. In the course of proving Theorem 4.5.2, we obtained the following partial description of the map $\iota : HH_*(\mathcal{H}(M)) \to HH_*(\mathcal{H}(G))$:

\[ \iota = \begin{bmatrix} \iota_e & \iota_{e,nc} \\ 0 & \iota_{nc} \end{bmatrix}, \]

where $\iota_{nc}$ is given by a certain explicit map of chain complexes, and $\iota_{c,nc} = 0$ in degree one. The map $\iota_e : HH_*(\mathcal{H}(M))_e \to HH_*(\mathcal{H}(G))_e$ will be computed in the next chapter.
Chapter 5

Parahoric Induction

Let $G = \text{SL}_2(F)$ (with $F$ a $p$-adic field), and let $M \subset G$ be the diagonal subgroup. In Chapter 4, we observed that the Jacquet functors $i = i^G_M$ and $r = r^G_M$ induce maps between the various homology groups associated to $G$ and to $M$. In Section 4.4, we explicitly computed the maps induced by the functor $r : \pi \mapsto \pi_N$. This restriction functor has an obvious analog for representations of compact open subgroups of $G$: if $\rho$ is a representation of such a subgroup $H$, then the space of coinvariants $\rho_{H \cap N}$ is a representation of $H \cap M$.

The map induced by $r$ in chamber homology is given by applying this “compact” analog of $r$ to the isotropy groups of points in the Bruhat-Tits building of $G$ (Proposition 4.4.19).

In this chapter, we obtain a counterpart of Proposition 4.4.19 for the parabolic induction functor $i$ (in the special case of $G = \text{SL}_2(F)$). As we noted in the previous chapter, the maps induced by $i$ are more difficult to understand than those induced by $r$. In chamber homology, the difficulty is manifested in the fact that the appropriate analog of parabolic induction for compact subgroups is not the naïve one. Indeed, let $\rho$ be a finite-dimensional representation of the diagonal subgroup

$$L = \begin{bmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{bmatrix} \subset \text{SL}_2(\mathcal{O}) = K,$$

Mimicking the definition of parabolic induction, one would first inflate $\rho$ to a representation of the upper-triangular subgroup of $K$, and then induce up to $K$. But the induced repre-
sentation would not be finite-dimensional, and so this procedure would not define a map from $R_C(L)$ to $R_C(K)$.

Dat has defined in [Dat09] an analog of parabolic induction—called parahoric induction—for the kind of compact groups relevant to chamber homology. In the main result of this chapter, Theorem 5.5.2, we use parahoric induction to define a map in chamber homology

$$\iota_c : H^M_s(X_M) \rightarrow H^G_s(X_G),$$

and prove that this map is equal to $i_c$, the upper-left corner of the matrix of $i$, relative to the compact/noncompact decomposition of Hochschild homology.

A similar map $\iota_c : H^M_s(X_M) \rightarrow H^G_s(X_G)$ may be defined for $M$ the diagonal subgroup in $G = \text{SL}_n(F)$, and we conjecture that $\iota_c = i_c$ in this case as well; we prove this in degree zero. Difficulties arise in the definition of $\iota_c$ for non-minimal Levi subgroups, mirroring problems noted by Dat (eg., [Dat09, Question 2.14]). It is possible that new methods will be required to deal with such cases.

The contents of this chapter are as follows. In Section 5.1 we review Dat’s basic construction. We have taken a slightly different approach to that in [Dat09], and accordingly we give full proofs, even though most of the results appear already in [Dat09, Section 2]. (Let us emphasize that, unlike [Dat09], here we work exclusively over $\mathbb{C}$.) Section 5.2 gives an alternative characterization of Dat’s construction, in terms of characters. In Sections 5.3 and 5.4, we focus on the Iwahori subgroup in $\text{SL}_2(F)$, where one finds that the parahoric induction functor is one that was already encountered in Section 3.4. Section 5.5 contains the statement and proof of the main result, Theorem 5.5.2.

5.1 Inflation for Groups with an Iwahori Decomposition

This section reviews a construction due to Dat [Dat09, Section 2], from a different point of view.

Definition 5.1.1. Let $J$ be a compact, totally disconnected group. An Iwahori decomposition of $J$ is a triple $(U, L, \overline{U})$ of closed subgroups of $J$, such that
(1) \( L \) normalizes \( U \) and \( \overline{U} \), and

(2) The product map \( U \times L \times \overline{U} \rightarrow J \) is a homeomorphism.

(Note that if (2) holds, then thanks to (1) the same is true for any ordering of the factors \( U, L, \) and \( \overline{U} \).)

We will say “let \( J = U L \overline{U} \) be an Iwahori decomposition” for short.

**Example 5.1.2.** Let \( F \) be a \( p \)-adic field, and let \( J = \left[ \begin{array}{cc} \mathcal{O}_p & \mathcal{O}_p \\ \mathcal{O}_p & \mathcal{O}_p \end{array} \right] \) be the standard Iwahori subgroup in \( SL_2(F) \). Let \( U = \left[ \begin{array}{cc} 1 & \mathcal{O}_p \\ 0 & 1 \end{array} \right], \overline{U} = \left[ \begin{array}{cc} 1 & 0 \\ \mathcal{O}_p & 1 \end{array} \right], \) and \( L = \left[ \begin{array}{cc} \mathcal{O}_p^\times & 0 \\ 0 & \mathcal{O}_p^\times \end{array} \right] \). Then \( J = U L \overline{U} \) is an Iwahori decomposition.

**Example 5.1.3.** Let \( J \) be a congruence subgroup of \( GL_n(F) \): i.e.,

\[
J = \{ g \in GL_n(\mathcal{O}) \mid g \equiv 1 \mod p^r \}
\]

for some \( r \geq 1 \). Let \( M \subset G \) be a standard Levi subgroup, and let \( N \) and \( \overline{N} \) be the corresponding subgroups of unipotent upper- and lower-triangular matrices. Then

\[
J = J_N J_M J_{\overline{N}}
\]

is an Iwahori decomposition of \( J \). (Recall that \( J_N \) means \( J \cap N \), and so on.)

Thus \( GL_n(F) \) (and likewise \( SL_n(F) \)) contains arbitrarily small compact, open subgroups, each admitting an Iwahori decomposition with respect to all the standard Levi subgroups of \( G \). (See [Ren10, V.5.2] for a more general result.) Such subgroups play a prominent role in the representation theory of \( p \)-adic groups: for example, they are central to some proofs of the Second Adjoint Theorem and its precursors ([BR92, III.3], [Ren10, VI.9]), and to Bushnell and Kutzko’s theory of types and covers [BK98].

**Lemma 5.1.4.** Let \( J = U L \overline{U} \) be a group with an Iwahori decomposition, and let \( du, dl \) and \( d\overline{u} \) be Haar measures on \( U, L \) and \( \overline{U} \) respectively. The product measure \( du \, dl \, d\overline{u} \) is a Haar measure on \( J \).
Proof. Since \(L\) is compact, its conjugation action on \(\overline{U}\) preserves \(d\overline{u}\), and this ensures that the product \(dl\,d\overline{u}\) is a Haar measure on the group \(L\overline{U}\). The standard theory of invariant measures on homogeneous spaces (as in [Wei40, II §9], for example) implies that there is a unique (up to a scalar) \(J\)-invariant measure on the homogeneous space \(J/LU \simeq U\). Such a measure is in particular a Haar measure on \(U\), so it is a scalar multiple of \(du\). Thus the product measure \(du\,dl\,d\overline{u}\) is a Haar measure on \(J\) (again by the standard theory of invariant measures).

Consider \(J = UL\overline{U}\), a group with a fixed Iwahori decomposition. From now on we will assume that the Haar measures on \(J, U, L,\) and \(\overline{U}\) are all normalized to have total volume 1.

Since \(L\) is a quotient of \(LU\), and also of \(L\overline{U}\), one can inflate representations from \(L\) to \(LU\) and \(L\overline{U}\) in the usual way (see Section 2.1.4).

**Definition 5.1.5.** For each smooth, finite-dimensional representation \(\rho\) of \(L\), we define \(\iota_J^L\rho\) to be the image of the \(J\)-equivariant map

\[
I_U : \text{ind}_{LU}^J \rho \to \text{ind}_{L\overline{U}}^J \rho,
\]

\((I_U f) (j) := \int_{\overline{U}} f(\overline{u}, j) d\overline{u}\).

(We are using the standard realization of the induced representations as spaces functions on \(J\), as in Definition 2.1.34.)

**Example 5.1.6.** Suppose that \(\overline{U}\) is trivial, so that \(J = LU\). Then \(\text{ind}_{LU}^J \rho = \rho\), and \(I_U\) is the obvious embedding. So in this case, \(\iota_J^L\) is the usual inflation of representations from \(L\) to \(LU\).

**Lemma 5.1.7.** (1) The map \(I_U\) is nonzero.

(2) \(\iota_J^L\) is an exact functor from \(\text{Mod}(L)\) to \(\text{Mod}(J)\), commuting with direct sums.

(3) If \(\rho\) is irreducible, then \(\iota_J^L\rho\) is the unique irreducible representation of \(J\) common to both \(\text{ind}_{LU}^J \rho\) and \(\text{ind}_{L\overline{U}}^J \rho\).

**Proof.** (1) Fix a nonzero vector \(v\) in the representation \(\rho\), and let \(f \in \text{ind}_{LU}^J \rho\) be the unique function having \(f(\overline{u}) = v\) for all \(\overline{u} \in \overline{U}\). Then \((I_U f)(1) = v \neq 0\).
(2) Let $T \in \text{Hom}_L(\rho_1, \rho_2)$ be an intertwiner between two representations of $L$. The diagram

$$
\begin{array}{ccc}
\text{ind}^J_{LU} \rho_1 & \xrightarrow{\tau} & \text{ind}^J_{LU} \rho_1 \\
\text{ind}^J_{LU} (T) \downarrow & & \downarrow \text{ind}^J_{LU} (T) \\
\text{ind}^J_{LU} \rho_2 & \xrightarrow{\tau} & \text{ind}^J_{LU} \rho_2
\end{array}
$$

is commutative, and this implies that $\text{ind}^J_{LU}(T)$ restricts to a $J$-equivariant map $\iota^J_L: \nu^J_L \rho_1 \to \nu^J_L \rho_2$. The fact that $\iota^J_L$ is an exact functor commuting with direct sums follows from the corresponding properties of the functors $\text{ind}^J_{LU}$ and $\text{ind}^J_{LU}$.

(3) By Frobenius reciprocity, we have

$$
\text{Hom}_J(\text{ind}^J_{LU} \rho_1, \text{ind}^J_{LU} \rho_2) \cong \text{Hom}_{LU}(\text{ind}^J_{LU} \rho_1, \rho_2) \\
\cong \text{Hom}_L \left( (\text{ind}^J_{LU} \rho_1)^\mathcal{U}, \rho_2 \right).
$$

The decomposition $J = LU \mathcal{U}$ implies that $(\text{ind}^J_{LU} \rho_1)^\mathcal{U} \cong \rho_1$, as representations of $L$, and so

$$
\text{Hom}_J(\text{ind}^J_{LU} \rho_1, \text{ind}^J_{LU} \rho_2) \cong \text{Hom}_L(\rho_1, \rho_2).
$$

If $\rho$ is irreducible, it follows from Schur’s lemma that $\text{Hom}_J(\text{ind}^J_{LU} \rho, \text{ind}^J_{LU} \rho)$ is one-dimensional, and so is spanned by the nonzero intertwiner $\iota^J_L$. This implies that the two induced representations share a unique irreducible component, and that component is $\nu^J_L \rho$.

**Example 5.1.8.** The trivial representation triv$_J$ of $J$ sits inside both $\text{ind}^J_{LU}(\text{triv}_L)$ and $\text{ind}^J_{LU}(\text{triv}_L)$, as the space of constant functions in each case. So $\nu^J_L(\text{triv}_L) = \text{triv}_J$.

**Remark 5.1.9.** The symmetry between $U$ and $\mathcal{U}$ in part (3) of Lemma 5.1.7 implies that we can interchange the roles of $U$ and $\mathcal{U}$ in the definition of $\iota^J_L \rho$ and obtain an isomorphic functor.

**Lemma 5.1.10.** For each finite-dimensional representation $\rho$ of $L$, one has

$$
\nu^J_L \rho \cong \nu^J_{\tilde{L}} \rho.
$$
Proof. Assume, as we may, that \( \rho \) is irreducible. The contragredient of \( \text{ind}^L_U \rho \) is isomorphic to \( \text{ind}^L_U \tilde{\rho} \), and a similar result holds for induction from \( LU \) (see [BH06, 3.5]). Thus \( \iota_L^J \rho \) is a common irreducible component of \( \text{ind}^L_U \rho \) and \( \text{ind}^L_U \tilde{\rho} \), so part (3) of Lemma 5.1.7 gives the asserted isomorphism.

We now construct an adjoint for \( \iota_L^J \). Let \( e_U \in \mathcal{H}(U) \) denote the function with constant value 1; thus \( e_U \) is the idempotent associated to the trivial representation of \( U \). The image \( e_U \pi \) of this idempotent in the representation \( \pi \) is precisely the space \( \pi^U \) of \( U \)-fixed vectors.

**Definition 5.1.11.** Let \( \pi \) be a finite-dimensional smooth representation of \( J \). Define \( r_L^J \pi \) to be the image of the map

\[
e_U : \pi^U \to \pi^U.
\]

One shows, as in Lemma 5.1.7, that \( r_L^J \) is an exact functor from \( \text{Mod}(J) \) to \( \text{Mod}(L) \), commuting with direct sums.

**Lemma 5.1.12.** [Dat09, Corollaire 2.9] There is a natural isomorphism \( r_L^J \iota_L^J \rho \cong \rho \) for each representation \( \rho \) of \( L \).

**Proof.** Consider the natural \( L \)-equivariant embedding

\[
(\iota_L^J \rho)^U \hookrightarrow (\text{ind}^L_U \rho)^U \cong_{f \mapsto f(1)} \rho.
\]

One shows, as in part (1) of Lemma 5.1.7, that \( (\iota_L^J \rho)^U \neq 0 \). If \( \rho \) is irreducible, then Schur’s lemma implies that the composition (5.1.13) is an isomorphism; by naturality, the same is true for all \( \rho \).

Reversing the roles of \( U \) and \( \overline{U} \) gives a natural \( L \)-equivariant isomorphism \( (\iota_L^J \rho)^\overline{U} \cong (\text{ind}^L_U \rho)^\overline{U} \cong \rho \). The map \( e_U : (\iota_L^J \rho)^\overline{U} \to (\iota_L^J \rho)^U \) is nonzero (by another variation on Lemma 5.1.7 part (1)). As above, Schur’s lemma and the naturality of the construction ensure that \( e_U : (\iota_L^J \rho)^\overline{U} \to (\iota_L^J \rho)^U \) is an isomorphism. The range \( r_L^J \iota_L^J \rho \) of \( e_U \) is therefore \( (\iota_L^J \rho)^U \cong \rho \).

For irreducible \( \pi \), there is a characterization of \( r_L^J \pi \) analogous to part (3) of Lemma 5.1.7:
Lemma 5.1.14. Let $\pi$ be an irreducible representation of $J$. There are two possibilities:

Either: $\text{Hom}_L(\pi^\mathcal{U}, \pi^U) = 0$; in this case, $r_L^J \pi = 0$.

Or: $\pi^U$ and $\pi^\mathcal{U}$ are irreducible representations of $L$, and $e_U$ is an isomorphism between them. In this case, $\pi \cong \nu_L^J(\pi^U) \cong \nu_L^J(\pi^\mathcal{U})$, and $r_L^J \pi \cong \pi^U \cong \pi^\mathcal{U}$.

Proof. If $\pi^U$ and $\pi^\mathcal{U}$ do not intertwine over $L$, then in particular the map $e_U : \pi^\mathcal{U} \to \pi^U$ is zero, and so $r_L^J \pi = 0$.

Suppose, on the other hand, that $\rho$ is an irreducible representation of $L$ appearing in both $\pi^U$ and $\pi^\mathcal{U}$. Frobenius reciprocity implies that the irreducible representation $\pi$ of $J$ is contained in $\text{ind}_{LU}^J \rho$ and $\text{ind}_{LU}^J \rho$, and so Lemma 5.1.7 part (3) gives $\pi \cong \nu_L^J \rho$. Thus $r_L^J \pi \cong \rho$, by Lemma 5.1.12, whose proof also showed that $\pi^U \cong \pi^\mathcal{U} \cong \rho$, and that $e_U$ is an isomorphism between these spaces.

Proposition 5.1.15. [Dat09, Proposition 2.2] There is a unique invertible element $z \in Z(L)$ such that $z^{-1} e_U e_\mathcal{U}$ acts as an idempotent on every smooth representation of $J$.

Proof. Let $\rho$ be an irreducible representation of $L$. Lemma 5.1.14 implies that there is a nonzero scalar $z(\rho) \in \mathbb{C}^\times$ such that $e_U e_\mathcal{U} e_U = z(\rho) e_U$ as operators on $\nu_L^J \rho$. The function $\hat{L} \to \mathbb{C}^\times$, $\rho \mapsto z(\rho)$ defines an invertible element $z \in Z(L)$ (see Example 3.1.1). Then the element $z^{-1} e_U e_\mathcal{U}$ acts as an idempotent on every irreducible representation of $J$ having the form $\nu_L^J \rho$ for some $\rho$. If $\pi$ is an irreducible representation of $J$ not having this form, then Lemma 5.1.14 implies that $e_U e_\mathcal{U} = 0$ on $\pi$, and so $z e_U e_\mathcal{U}$ is also an idempotent in this case.

In the next section (Proposition 5.2.3), we show that

$$z(\rho) = \frac{\dim \rho}{\dim (\nu_L^J \rho)}.$$ 

Corollary 5.3.2 computes $z$ explicitly, for the Iwahori subgroup of $\text{SL}_2(F)$.

We now come to the most important relationship between $\iota$ and $r$:
Proposition 5.1.16. [Dat09, Corollaire 2.9] The functors $\iota_L^J$ and $r_L^J$ are mutual two-sided adjoints.

Proof. Using the natural embeddings $\iota_L^J \rho \hookrightarrow \text{ind}_{LU}^J \rho$ and $r_L^J \pi \hookrightarrow \pi^U$, along with Frobenius reciprocity, we obtain a natural composite map

$$\text{Hom}_J(\pi, \iota_L^J \rho) \rightarrow \text{Hom}_J(\pi, \text{ind}_{LU}^J \rho) \rightarrow \text{Hom}_L(\pi^U, \rho) \rightarrow \text{Hom}_L(r_L^J \pi, \rho).$$

To show that this is an isomorphism for arbitrary $\pi$ and $\rho$, it is enough to consider irreducible $\pi$ and $\rho$.

If $\pi \not\cong \iota_L^J \rho$, then $\text{Hom}_J(\pi, \iota_L^J \rho) = 0$ by Schur’s lemma, while $\text{Hom}_L(r_L^J \pi, \rho) = 0$ by Lemma 5.1.14. By the same token, if $\pi = \iota_L^J \rho$ then both the domain and the codomain of the composition (5.1.17) are one-dimensional, and so it will suffice to show that the composition is nonzero. Frobenius reciprocity sends the embedding $\iota_L^J \rho \hookrightarrow \text{ind}_{LU}^J \rho$ to the isomorphism $(\iota_L^J \rho)^U \rightarrow \rho, f \mapsto f(1)$. This implies that the composition (5.1.17) is nonzero on the identity map $\iota_L^J \rho \rightarrow \iota_L^J \rho$, and we conclude that (5.1.17) exhibits $\iota_L^J$ as right-adjoint to $r_L^J$.

The adjunction

$$\text{Hom}_J(\iota_L^J \rho, \pi) \cong \text{Hom}_L(\rho, r_L^J \pi)$$

is proved by a similar argument. \qed

We conclude this section with a technical lemma that will be used below.

Lemma 5.1.18. Let $J = ULU$ and $J' = VLV$ be two compact totally disconnected groups with Iwahori decompositions, such that $V \subseteq U$ and $\overline{U} \subseteq \overline{V}$. For all representations $\rho$ and $\tau$ of $L$, one has

$$\text{Hom}_{J \cap J'}(\iota_L^J \rho, \iota_L^J \tau) \cong \text{Hom}_L(\rho, \tau).$$

Example 5.1.19. Let $J = I$ be the Iwahori in $\text{SL}_2(F)$, with its usual decomposition $I = ULU$. Let $J' = I^w$, where $w = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and take $V = \overline{U}^w$ and $\overline{V} = U^w$. More generally, the hypotheses of the lemma are satisfied when $J, J' \subset \text{SL}_n(F)$ are the isotropy groups of two adjacent chambers in the building of $\text{SL}_n(F)$. 

Proof of Lemma 5.1.18. We may assume that $\rho$ and $\tau$ are irreducible. Write $H = J \cap J' = VLU$. Given $T \in \text{Hom}_H(\iota^J_L \rho, \iota^J_L \tau)$, extend by zero to obtain an $H$-equivariant map

$$\tilde{T} : \text{ind}^J_{LU} \rho \to \text{ind}^J_{LV} \tau.$$ 

Restriction of functions from $J$ to $H$ gives an $H$-equivariant isomorphism $\text{ind}^J_{LU} \rho \cong \text{ind}^H_{LV} \rho$, and similarly $\text{ind}^J_{LV} \tau \cong \text{ind}^H_{LV} \tau$. So we have

$$\tilde{T} \in \text{Hom}_H(\text{ind}^H_{LV} \rho, \text{ind}^H_{LV} \tau) \cong \text{Hom}_L(\rho, \tau),$$

and the map $T \mapsto \tilde{T}$ is an injection from $\text{Hom}_H(\iota^J_L \rho, \iota^J_L \tau)$ to $\text{Hom}_L(\rho, \tau)$. The latter space is either zero- or one-dimensional, so to complete the proof it will be enough to show that $\text{Hom}_H(\iota^J_L \rho, \iota^J_L \rho)$ is nonzero.

We showed above that $\text{ind}^J_{LU} \rho \cong \text{ind}^H_{LV} \rho$ over $H$. Similarly, restriction of functions from $J$ to $H$ defines a surjective $H$-equivariant map $\text{ind}^J_{LV} \rho \to \text{ind}^H_{LV} \rho$, and the diagram

$$\begin{array}{ccc}
\text{ind}^J_{LU} \rho & \xrightarrow{\iota^J_L} & \text{ind}^J_{LV} \rho \\
\downarrow \text{restrict} & & \downarrow \text{restrict} \\
\text{ind}^H_{LV} \rho & \xrightarrow{\iota^H_L} & \text{ind}^H_{LV} \rho
\end{array}$$

commutes. This implies that $\iota^H_L \rho$ is a quotient, and therefore a subrepresentation, of $\iota^J_L \rho$. A similar argument works with $J'$ in place of $J$. Thus $\iota^J_L \rho$ and $\iota^J_L \rho$ intertwine over $H$. \qed

5.2 The Maps $\iota$ and $r$ on Characters

Let $J = ULU$ be a compact totally disconnected group with an Iwahori decomposition. The groups $L$ and $J$ will be fixed throughout this section, and we write $\iota = \iota^J_L$ and $r = r^J_L$.

The functors $\iota$ and $r$ induce maps between the vector spaces $R_C(J)$ and $R_C(L)$ of virtual representations. Passing from representations $\pi$ to their characters $\text{ch}_\pi$ (see Example 2.1.43),
we obtain maps
\[ \text{Cl}^\infty(J) \xrightarrow{\iota} \text{Cl}^\infty(L) \]

between the spaces of locally constant class functions on \( J \) and \( L \). In this section, we compute these maps \( \iota \) and \( r \).

**Example 5.2.1.** Suppose that \( J = UL \) (i.e., \( \overline{U} = 1 \)). The functor \( \iota \) is then the usual inflation of representations, while \( r \) is the functor \( \pi \mapsto \pi^U \). The action on characters is easily computed from these identifications, and one finds

\[ \iota(\psi)(ul) = \psi(l) \quad \text{and} \quad r(\varphi)(l) = \int_u \varphi(ul) \, du, \]

for \( \psi \in \text{Cl}^\infty(L) \) and \( \varphi \in \text{Cl}^\infty(J) \). In other words, \( \iota \) is given by pulling functions back along the quotient map \( J \to L \), while \( r \) is given by integrating along the fibers of this map.

**Definition 5.2.2.** Let \( J = ULU \) be a group with Iwahori decomposition, and consider the map

\[ \lambda : J \to L, \quad \lambda(ul\overline{u}) = l. \]

Define maps
\[ \text{Cl}^\infty(J) \xrightarrow{\lambda^*} \text{Cl}^\infty(L) \]

by

\[ (\lambda^*\psi)(j) = \int_J \psi(\lambda^{-1}jk) \, dk \quad \text{and} \quad (\lambda^*\varphi)(l) = \int_U \int_U \varphi(ul\overline{u}) \, d\overline{u} \, du, \]

for \( \psi \in \text{Cl}^\infty(L) \) and \( \varphi \in \text{Cl}^\infty(J) \).

Notice that \( (\lambda^*\psi)(1) = \psi(1) \) for all \( \psi \). This implies that \( \lambda^* \neq \iota \), since it is not true in general that \( \dim(\iota \rho) = \dim \rho \) (see Lemma 5.3.1, for instance). The element \( z \in \mathfrak{Z}(L) \) of Proposition 5.1.15, whose action on \( \mathcal{H}(L) \) restricts to an action on \( \text{Cl}^\infty(L) \), provides the necessary correction.

**Proposition 5.2.3.** Let \( J = ULU \) be a group with Iwahori decomposition, and let \( z \in \mathfrak{Z}(L) \) be as in Proposition 5.1.15.
(1) The maps

$$\begin{align*}
\text{Cl}^\infty(J) & \xrightarrow{r} \text{Cl}^\infty(L) \\
\text{Cl}^\infty(J) & \xleftarrow{\iota} \text{Cl}^\infty(J)
\end{align*}$$

are given by

$$r = z^{-1}\lambda_\pi \quad \text{and} \quad \iota = \lambda^* z^{-1}.$$

(2) For each $\rho \in \hat{L}$, one has

$$z(\rho) = \frac{\dim \rho}{\dim(\iota \rho)}.$$

Proof. (1) We first consider the map $r$. For each $\pi \in \hat{J}$ and $l \in L$,

$$\lambda_\pi(\text{ch}_\pi)(l) = \int_{U} \int_{U} \text{Trace}(\pi(l u \bar{u})) \, d\bar{u} \, du = \text{Trace}(\pi(l) \pi(e_U) \pi(e_{\bar{U}})).$$

If $r \pi = 0$, then $\pi(e_U) \pi(e_{\bar{U}}) = 0$, and so $r(\text{ch}_\pi) = z^{-1}\lambda_\pi(\pi) = 0$. On the other hand, suppose that $r \pi = \rho \in \hat{L}$. Then

$$\lambda_\pi(\text{ch}_\pi)(l) = \text{Trace}(\pi(l) \pi(e_U) \pi(e_{\bar{U}})) = z(\rho) \text{Trace}(\pi(l) z(\rho)^{-1} \pi(e_U) \pi(e_{\bar{U}})),$$

and $z(\rho)^{-1} \pi(e_U) \pi(e_{\bar{U}})$ is the projection of $\pi$ onto $r \pi$ (Proposition 5.1.15). So

$$\lambda_\pi(\text{ch}_\pi)(l) = z(\rho) \text{Trace}(\pi(l) \big|_{r \pi}) = z(r \pi) \text{ch}_{r \pi}(l) = z(r(\text{ch}_\pi))(l),$$

giving $r = z^{-1}\lambda_\pi$.

Now turn to the map $\iota$. We consider the $L^2$ inner products on $\text{Cl}^\infty(L)$ and $\text{Cl}^\infty(J)$:

$$\langle \psi_1, \psi_2 \rangle_L = \int_L \psi_1(l) \overline{\psi_2(l)} \, dl$$

for $\psi_1, \psi_2 \in \text{Cl}^\infty(L)$, and similarly for $J$. The character bases $\{\text{ch}_{\rho} \mid \rho \in \hat{L}\}$ and $\{\text{ch}_{\pi} \mid \pi \in \hat{J}\}$ are orthonormal (see [Ser77, 2.3 Theorem 3]).
For each $\psi \in \mathrm{Cl}^\omega(L)$ and $\varphi \in \mathrm{Cl}^\omega(J)$, we have

$$\langle \lambda^* \psi, \varphi \rangle_J = \int_J (\lambda^* \psi)(j) \overline{\varphi(j)} \, dj = \int_J \int_J \psi(\lambda(k^{-1} j k)) \overline{\varphi(j)} \, dk \, dj$$

$$= \int_J \psi(\lambda(h)) \overline{\varphi(h)} \, dh \quad (h = k^{-1} j k)$$

$$= \int_L \psi(l) \left( \int_{\mathcal{U}} \varphi(u \overline{n}) \, d\overline{n} \, du \right) \, dl \quad \text{(Lemma 5.1.4)}$$

$$= \langle \psi, \lambda \varphi \rangle_L.$$ 

Now, $\langle \iota \psi, \varphi \rangle_J = \langle \psi, r \varphi \rangle_L$, by Proposition 5.1.16, and so for all $\rho \in \hat{L}$ and $\pi \in \hat{J}$ we have

$$\langle \iota \, \text{ch}_\rho, \text{ch}_\pi \rangle_J = \langle \text{ch}_\rho, r \text{ch}_\pi \rangle_L = \langle \text{ch}_\rho, z^{-1} \lambda \text{ch}_\pi \rangle_L = \langle \lambda^* (z^{-1})^* \text{ch}_\rho, \text{ch}_\pi \rangle_J,$$

where $(z^{-1})^* (\text{ch}_\rho) = \overline{z(\rho)^{-1}} \text{ch}_\rho$. Part (2), proved below, implies that $z(\rho)$ is real, and so $(z^{-1})^* = z^{-1}$ and $\iota = \lambda^* z^{-1}$.

(2) We have seen that

$$\lambda^* (\text{ch}_\rho) = \overline{z(\rho)^{-1}} \lambda^* \left( \frac{1}{z(\rho)^{-1}} \text{ch}_\rho \right) = \overline{z(\rho)} \text{ch}_{\iota \rho}.$$ 

Recalling that $(\lambda^* \psi)(1) = \psi(1)$ for all $\psi \in \mathrm{Cl}^\omega(L)$, we find that

$$\dim \rho = \text{ch}_\rho(1) = (\lambda^* \text{ch}_\rho)(1) = \overline{z(\rho)} \text{ch}_{\iota \rho}(1) = \overline{z(\rho)} \dim(\iota \rho).$$

Since $\dim \rho$ and $\dim(\iota \rho)$ are real, the result follows.

Remark 5.2.4. The number $z(\rho)$ may be interpreted as measuring the relative position of the idempotents $e_U$ and $e_{\overline{T}}$ in the representation $\iota \rho$, as we shall now explain.

Let $\pi$ be an irreducible representation of $J$, and choose a $J$-invariant inner product on $\pi$. The self-adjoint idempotents $P = \pi(e_U)$ and $Q = \pi(e_{\overline{T}})$ determine a finite-dimensional unitary representation of the infinite dihedral group $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$: the generating involutions $s_1, s_2 \in \Gamma$ map to the self-adjoint unitary operators $2P - 1$ and $2Q - 1$, respectively. This representation $\pi|_{\Gamma}$ of $\Gamma$ decomposes into a direct sum of isotypical components,
and each isotypical component is stable under the action of $L$.

Recall the list of irreducible unitary representations of $\Gamma$: for each angle $\alpha \in [0, \pi/2]$ one forms the two-dimensional representation $\tau_\alpha$ in which $P$ and $Q$ are represented by the matrices

$$
\tau_\alpha(P) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau_\alpha(Q) = \begin{bmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & \sin^2 \alpha \end{bmatrix}.
$$

For $\alpha \in (0, \pi/2)$, these $\tau_\alpha$ are irreducible and mutually inequivalent. The representations $\tau_0$ and $\tau_{\pi/2}$ each decompose into one-dimensional summands:

$$
\tau_0 = \begin{bmatrix} \tau'_0 & 0 \\ 0 & \tau'_0 \end{bmatrix} \quad \text{and} \quad \tau_{\pi/2} = \begin{bmatrix} \tau'_{\pi/2} & 0 \\ 0 & \tau'_{\pi/2} \end{bmatrix}.
$$

These four one-dimensional representations, together with the irreducible $\tau_\alpha$, form a complete list of the irreducible unitary representations of $\Gamma$. (The list is obtained by expressing $\Gamma$ as a semidirect product $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathbb{Z}$, and applying Mackey theory [Mac52, Section 14].)

Now, $r \pi$ is the range of $PQ$, and $PQ$ is nonzero only in $\tau'_0$ and in the $\tau_\alpha$ components for $\alpha \in (0, \pi/2)$. So we have

$$
r \pi \neq 0 \iff \pi|_{\hat{L}} \text{ contains } \tau'_0 \text{ or } \tau_\alpha \text{ for some } \alpha \in (0, \pi/2).
$$

Suppose $r \pi \neq 0$, so that $\pi = \iota \rho$ for some $\rho \in \hat{L}$. Since $r \pi$ is an irreducible representation of $L$, and $L$ preserves the isotypical decomposition of $\pi|_{\Gamma}$, it follows that $\pi|_{\Gamma}$ contains exactly one of the representations $\tau'_0$ or $\tau_\alpha$ (possibly with multiplicity $> 1$). We then have $PQP = \cos^2(\alpha)P$ (setting $\alpha = 0$ if $\pi|_{\Gamma}$ contains $\tau'_0$), which by the definition of $z$ implies that

$$
(5.2.5) \quad z(\rho) = \cos^2(\alpha).
$$

Thus the formula $z(\rho) = \frac{\dim \rho}{\dim(\iota \rho)}$ imposes a restriction on the irreducible representations of $\Gamma$ that may occur in irreducible representations of $J$. For example, if $L$ is commutative, so that $\dim \rho = 1$ for every $\rho \in \hat{L}$, then the representation $\tau_\alpha$ of $\Gamma$ may occur only in those
\[ \pi \in \hat{\mathcal{J}} \text{ having dim } \pi = \frac{1}{\cos^2(\alpha)}. \]

### 5.3 The Iwahori Subgroup of SL\(_2(F)\)

Let \( G = \text{SL}_2(F) \), with \( F \) a \( p \)-adic field, and consider the standard Iwahori subgroup

\[
I = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} \end{bmatrix}.
\]

As noted above, this group admits an Iwahori decomposition \( I = ULU \), where

\[
U = \begin{bmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{bmatrix}, \quad \overline{U} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}.
\]

We shall give an explicit description of the functor \( \iota^I_L \), and relate the functor \( r^I_L \) to asymptotics on the building of \( G \). Since \( I \) and \( L \) will remain fixed, we write \( \iota = \iota^I_I \) and \( r = r^I_L \) throughout this section.

Let \( \chi \) be an irreducible representation of \( L \), i.e., a smooth character \( \chi : \mathcal{O}^\times \to \mathbb{C}^\times \). If \( \chi \) is the trivial character, then \( \iota \chi \) is the trivial representation of \( I \) (Example 5.1.8). So we assume that \( \chi \) is a nontrivial character of \( \mathcal{O}^\times \), and denote by \( \mathfrak{c} \) the \textit{conductor} of \( \chi \):

\[
\mathfrak{c} = \min\{n \geq 1 \mid \chi \text{ is trivial on } 1 + \mathfrak{p}^n\}.
\]

Then define

\[
I_\mathfrak{c} = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ p^\mathfrak{c} & \mathcal{O} \end{bmatrix},
\]

and \( \chi : I_\mathfrak{c} \to \mathbb{C}^\times, \chi \left[ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right] := \chi(a) \).

**Lemma 5.3.1.** As representations of \( I \), \( \iota \chi \cong \text{ind}^I_{I_\mathfrak{c}} \chi \).

**Proof.** A straightforward matrix computation shows that the image of

\[
I_U : \text{ind}^I_{UL} \chi \to \text{ind}^I_{LU} \chi
\]
lies in the subspace \( \text{ind}_{I_c}^I \chi \). So \( \iota \chi \subseteq \text{ind}_{I_c}^I \chi \). Using Mackey’s induction-restriction formula, and the minimality of \( c \), one can show that \( \text{ind}_{I_c}^I \chi \) is irreducible: see [BHP93, Lemma 9.2].

**Corollary 5.3.2.** The element \( z \in \mathfrak{Z}(L) \) from Proposition 5.1.15 is given by

\[
z(\chi) = \begin{cases} 
1 & \text{if } \chi \text{ is trivial}, \\
q^{1-c} & \text{if } \chi \text{ is nontrivial with conductor } c,
\end{cases}
\]

where \( q = |\mathcal{O}/p| \) is the cardinality of the residue field of \( F \).

**Proof.** Proposition 5.2.3 gives \( z(\chi) = \dim(\iota \chi)^{-1} \). For nontrivial \( \chi \), Lemma 5.3.1 implies that

\[
dim(\iota \chi) = [I : I_c] = [p : p^c] = q^{c-1}.
\]

**Remark 5.3.3.** The pair \( (L, \chi) \) is a type for the inertial class \( [M, \chi]_M \in \mathfrak{B}(M) \), where \( M \) is the diagonal subgroup of \( G = \text{SL}_2(F) \). (See [BK98] for the language and theory of types.) The pair \( (I_c, \chi) \) is a cover of \( (L, \chi) \), and so \( (I_c, \chi) \) is a type for the component \( [M, \chi]_G \in \mathfrak{B}(G) \) (see [Kut04, 2 Proposition]). Since the induced representation \( \iota \chi = \text{ind}_{I_c}^I \chi \) remains irreducible, the pair \( (I, \iota \chi) \) is also a type for \( [M, \chi]_G \in \mathfrak{B}(G) \).

We now turn to the map \( r : \hat{I} \to \hat{L} \cup \{0\} \). Let \( t \in G \) be the element

\[
t = \begin{bmatrix}
\varpi^{-1} & 0 \\
0 & \varpi
\end{bmatrix}.
\]

In terms of the building of \( G \), \( t \) acts by a translation by two vertices along the apartment associated to the diagonal subgroup \( M \).

Recall the following notation: if \( \pi \) is a representation of a subgroup \( H \) of a group \( G \), and if \( g \in G \), then \( \pi^g \) denotes the representation \( x \mapsto \pi(gxg^{-1}) \) of the group \( H^g := g^{-1}Hg \).

**Lemma 5.3.4.** (1) Let \( \pi \) be a smooth, finite-dimensional representation of \( I \). Then

\[
\text{Hom}_L(\pi^U, \pi^U) \cong \text{Hom}_{I_c, I^c}(\pi, \pi^c).
\]
for all sufficiently large $n$.

(2) Suppose that $\pi$ is irreducible. Then

$$r\pi = 0 \iff \text{Hom}_{I \cap I^n}(\pi, \pi^t_n) = 0 \quad \text{for all } n > 0.$$ 

(3) Suppose that $\pi$ is irreducible, and that $r\pi \neq 0$. Then, for large $n$, $\pi$ and $\pi^t_n$ contain a unique common irreducible representation of $I \cap I^n$. The restriction to $L$ of that common irreducible is isomorphic to $r\pi$.

Proof. To simplify the notation, let $I^n = I \cap I^n$. Explicitly,

$$I^n = \begin{bmatrix} \mathcal{O} & p^{2n} \\ p & \mathcal{O} \end{bmatrix}.$$ 

This group has an Iwahori decomposition $I^n = U^n L U$, where $U^n \simeq U$.

Since $t^n$ centralizes $L$, we have an isomorphism $\pi^U \simeq (\pi^t_n)^{U^n}$ of representations of $L$, and so

$$\text{Hom}_L(\pi^U, \pi^U) \simeq \text{Hom}_L(\pi^U, (\pi^t_n)^{U^n}).$$

Because $\pi$ is smooth and finite-dimensional, the kernel of $\pi$ contains some congruence subgroup $\left[ \frac{1+p^r}{p}, \frac{p^r}{1+p^r} \right]$. Clearly $U^n$ lies in this subgroup for sufficiently large $n$, as does $U^{t^n}$. So, for sufficiently large $n$, $\pi$ is trivial on $U^n$, while $\pi^t_n$ is trivial on $U$. We thus have for large $n$ that

$$\text{Hom}_L(\pi^U, (\pi^t_n)^{U^n}) \simeq \text{Hom}_{U^n L U}(\pi, (\pi^t_n)^{U^n}) \simeq \text{Hom}_{U^n L U}(\pi, \pi^t_n).$$

This proves part (1).

Now suppose that $\pi$ is irreducible. Lemma 5.1.14 implies that $r\pi = 0$ if and only if $\text{Hom}_L(\pi^U, \pi^U) = 0$. Thus part (2) follows from part (1).

Finally, suppose that $r\pi = \chi \neq 0$. Then, again by Lemma 5.1.14, $r\pi$ is equal to the image of any nonzero element of $\text{Hom}_L(\pi^U, \pi^U)$. The isomorphism in part (1) identifies this
image with the unique common irreducible \( I^n \)-representation of \( \pi \) and \( \pi^{t_n} \), for sufficiently large \( n \).

\[ \square \]

**Proposition 5.3.5.** Let \( \pi \) be an irreducible representation of \( I \). Then

\[ r\pi = 0 \iff \dim (\text{End}_G(\text{ind}^G_I \pi)) < \infty. \]

**Proof.** The Mackey induction-restriction formula gives

\[ \text{End}_G(\text{ind}^G_I \pi) \cong \bigoplus_{g \in I \backslash G/I} \text{Hom}_{I \cap I^g}(\pi, \pi^g). \]

Now let \( w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). According to the Bruhat decomposition [Ser03, II.1.7],

\[ \{ t^n, t^n w \mid n \in \mathbb{Z} \} \]

is a set of representatives for the double-coset space \( I \backslash G/I \).

If \( r\pi \neq 0 \), then part (3) of Lemma 5.3.4 ensures that the space \( \text{Hom}_{I \cap I^n}(\pi, \pi^{t^n}) \) is nonzero for all \( n >> 0 \). Thus \( \text{End}_G(\text{ind}^G_I \pi) \) is infinite-dimensional in this case.

For the converse, suppose that \( r\pi = 0 \). Lemma 5.3.4 implies that the double cosets \( It^nI \), for \( n \geq 0 \), contribute only finitely many dimensions to \( \text{End}_G(\text{ind}^G_I \pi) \). Moreover, since

\[ \text{Hom}_{I \cap I^n}(\pi, \pi^{t^n}) \cong \text{Hom}_{I \cap I^{-n}}(\pi^{t^{-n}}, \pi), \]

the same is true for \( n \leq 0 \). Mimicking the proof of Lemma 5.3.4 one finds that

\[ \text{Hom}_{I \cap I^n w}(\pi, \pi^{t^n w}) \cong \text{Hom}_L(\pi^U, (\pi^U)^w) = 0 \quad \text{for} \ n >> 0, \]

and

\[ \text{Hom}_{I \cap I^n w}(\pi, \pi^{t^n w}) \cong \text{Hom}_L(\pi^U, (\pi^U)^w) = 0 \quad \text{for} \ n << 0. \]

Thus the contribution of the double cosets \( It^n w I \) to \( \text{End}_G(\text{ind}^G_I \pi) \) is also finite-dimensional. \( \square \)
Remark 5.3.6. Let $\pi$ be an irreducible representation of $I$. If $r\pi = \chi \neq 0$, then $\text{ind}_I^G \pi$ contains no supercuspidal component: indeed, we have seen that $(I, \pi)$ is a type for $[M, \chi]_G \in \mathfrak{B}(G)$, and so $\text{ind}_I^G \pi$ lies in $\text{Mod}(G)[M, \chi]_G$.

It is not true, on the other hand, that $r\pi = 0$ implies that $\text{ind}_I^G \pi$ is supercuspidal. For example, letting $f = \mathcal{O}/\mathfrak{p}$ denote the residue field of $F$, we consider the groups

$$B(f) = \left[ \begin{array}{cc} f & f \\ 0 & f \end{array} \right] \cap \text{SL}_2(f), \quad \text{and} \quad N(f) = \left[ \begin{array}{cc} 1 & f \\ 0 & 1 \end{array} \right].$$

Let $\psi : f \to \mathbb{C}^\times$ be a nontrivial additive character, considered as a representation of $N(f)$.

Since $B(f)$ is a quotient of $I$, the representation $\pi = \text{ind}_{N(f)}^{B(f)} \psi$ may be inflated to a representation of $I$.

We have $\pi^U = \pi^{N(f)} = 0$: indeed,

$$\pi|_{N(f)} \cong \bigoplus_{m \in M(f)} \psi^m,$$

where $M(f) \cong B(f)/N(f) = \left[ \begin{array}{cc} f & 0 \\ 0 & f \end{array} \right]$,

and no $\psi^m$ is trivial. Therefore $r\pi = 0$.

Now, $\pi$ does contain a nonzero vector fixed by the diagonal subgroup $M(f)$: namely, the function

$$f \left( \left[ \begin{array}{cc} x & y \\ 0 & x^{-1} \end{array} \right] \right) = \psi(xy).$$

The quotient map $I \to B(f)$ sends $I \cap I^w$ onto $M(f)$, and so we have $\pi^{I \cap I^w} \neq 0$. An application of the Mackey formula then gives $(\text{ind}_I^G \pi)^I \neq 0$, which by results of Borel and Casselman implies that $\text{ind}_I^G \pi$ is not supercuspidal. (See [Bor76, Lemma 4.7] and [Cas80, Proposition 2.4].)
5.4 Parahoric Induction

We continue to consider the Iwahori subgroup \( I \subset \text{SL}_2(F) \), with its decomposition \( I = ULU \). Let \( K = \text{SL}_2(\mathcal{O}) \), and define a functor \( \iota^K_L : \text{Mod}(L) \to \text{Mod}(K) \) by

\[
\iota^K_L(\rho) = \text{ind}^K_L(\iota^I_L(\rho)).
\]

This is an example of parahoric induction; see [Dat09] for the general definition.

The family of representations \( \{ \iota^K_L \chi \mid \chi \in \hat{L} \} \) may be considered a kind of “principal series” for \( \text{SL}_2(\mathcal{O}) \). We will show that the irreducibility and intertwining properties of these representations are exactly analogous to those of the principal series for \( \text{SL}_2(F) \) (as explained in [GGPS69, Chapter 2 §3]).

**Lemma 5.4.1.** Let \( \rho \) and \( \tau \) be representations of \( L \). Then

\[
\text{Hom}_K(\iota^K_L \rho, \iota^K_L \tau) \cong \text{Hom}_L(\rho, \tau) \oplus \text{Hom}_L(\rho, \tau^w).
\]

**Proof.** Using the Mackey formula and the Bruhat decomposition \( K = I \sqcup IwI \), we find

\[
\text{Hom}_K(\iota^K_L \rho, \iota^K_L \tau) \cong \text{Hom}_I(\iota^I_L \rho, \iota^I_L \tau) \oplus \text{Hom}_{I \cap Iw}(\iota^I_L \rho, (\iota^I_L \tau)^w).
\]

The first summand is isomorphic to \( \text{Hom}_L(\rho, \tau) \), by Proposition 5.1.16 and Lemma 5.1.12. We have \( (\iota^I_L \tau)^w \cong (\iota^I_L \tau)^w \), and so Lemma 5.1.18 implies that the second summand is isomorphic to \( \text{Hom}_L(\rho, \tau^w) \).

Applying Lemma 5.4.1 to irreducible representations, we obtain:

**Proposition 5.4.2.** Let \( \chi \) and \( \chi' \) be irreducible representations of \( L \).

1. If \( \chi \not\cong \chi^w \), then \( \iota^K_L \chi \) is irreducible.

2. If \( \chi \cong \chi^w \), then \( \iota^K_L \chi \) is a sum of two inequivalent irreducibles.

3. \( \iota^K_L \chi \cong \iota^K_L \chi^w \).
(4) $\text{Hom}_K(i^K_L \chi, i^K_L \chi') = 0$ if $\chi' \not\sim \chi$ or $\chi''$.

The same discussion applies when $K$ is replaced by the group

$$K' = \begin{bmatrix} \mathcal{O} & p^{-1} \\ p & \mathcal{O} \end{bmatrix},$$

and $w$ is replaced by $w' = \begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}$. Note that the action of $w'$ on $L$ is the same as the action of $w$.

5.5 Parahoric Induction and Chamber Homology for $\text{SL}_2(F)$

Let $G = \text{SL}_2(F)$, acting on its Bruhat-Tits building $X_G$, and let $M \subset G$ denote the diagonal subgroup, acting on its building $X_M$. We shall use parahoric induction to construct a map in chamber homology, $H^M_*(X_M) \to H^G_*(X_G)$, and show that this map is the one induced by parabolic induction from $\text{Mod}(M)$ to $\text{Mod}(G)$. The argument is similar to that of Section 4.5.

The building $X_M$ may be identified, $M$-equivariantly, with an apartment in $X_G$:

$$X_M : \quad \cdots \quad e_2 \quad \bullet \quad e_1 \quad \circ \quad K \quad I \quad K' \quad e_1 \quad \circ \quad e_2 \quad \bullet \quad \cdots$$

We have $M \cong L \times \mathbb{Z}$, where $L$ acts trivially on $X_M$, and $\mathbb{Z}$ acts by $v_i \mapsto v_{i+2}$.

The chamber homology $H^M_*(X_M)$ is the homology of the complex

$$0 \to R_C(L) \oplus R_C(L) \overset{\delta_M}{\longrightarrow} R_C(L) \oplus R_C(L) \to 0,$$

$$\delta_M(\rho_0, \rho_1) = (\rho_0 + \rho_1, -\rho_0 - \rho_1).$$

On the other hand, $H^G_*(X_G)$ is the homology of the complex

$$0 \to R_C(I) \overset{\delta_G}{\longrightarrow} R_C(K) \oplus R_C(K') \to 0,$$

$$\delta_G(\pi) = (\text{ind}_I^K \pi, -\text{ind}_I^{K'} \pi).$$
In pictures, showing the quotient complexes $X_M/M$ and $X_G/G$ labeled by their respective coefficient systems:

\[ X_M/M : \quad \begin{array}{c}
\xymatrix{
R_C(L) \ar@/^/[dr] & R_C(L) \ar@/^/[dl] \\
\cdot & R_C(L)
}
\end{array} \]

\[ X_G/G : \quad \begin{array}{c}
\xymatrix{
R_C(K) \ar@/^/[dr] & R_C(I) \ar@/^/[dl] \\
\cdot & R_C(K')
}
\end{array} \]

**Lemma 5.5.1.** The diagram

\[
\begin{array}{cccc}
(p_0, p_1) & R_C(L) \oplus R_C(L) & R_C(L) \oplus R_C(L) & (p_0, p_1) \\
\downarrow & \delta_M & \downarrow & \downarrow \\
\iota_L^I p_0 + \iota_L^I p_1^w & R_C(I) & R_C(K) \oplus R_C(K') & (\iota_L^K p_0, \iota_L^K p_1)
\end{array}
\]

is commutative, and so induces a map in chamber homology

\[ \iota_c : H^M_*(X_M) \to H^G_*(X_G). \]

**Proof.** The diagram commutes by virtue of the equality $\iota_L^K \rho \cong \iota_L^K \rho^w$ from Proposition 5.4.2 (along with the analogous formula for $K'$).

Consider the map $i_c = 1_{G_c} \circ i \circ 1_{M_c} : HH_0(H^0(M)) \to HH_0(H^0(G))$ induced by the Jacquet functor $i^G_M$, as in Chapter 4. Identifying chamber homology with the compact part of Hochschild homology (Theorem 2.4.13), we view $i_c$ as a map $H^M_*(X_M) \to H^G_*(X_G)$.

**Theorem 5.5.2.** One has $i_c = i_c$ as maps $H^M_*(X_M) \to H^G_*(X_G)$.

The proof of Theorem 5.5.2 occupies most of the remainder of this section. Possible generalizations of the theorem are discussed in Remark 5.5.7, below.
Consider the Weyl group $W = N_G(M)/M = \{1, w\}/ \pm 1$. The action of $W$ on the apartment $X_M$ extends to an $M$-equivariant action on the chain complex $C_*(X_M, \mathcal{R}_M)_M$, as in Lemma 4.5.12. The action is given in degree zero by

$$w_0 : R_C(L) \oplus R_C(L) \to R_C(L) \oplus R_C(L), \quad w(\rho_0, \rho_1) = (\rho_0^w, \rho_1^w),$$

and in degree one by

$$w_1 : R_C(L) \oplus R_C(L) \to R_C(L) \oplus R_C(L), \quad w(\rho_0, \rho_1) = (\rho_1^w, \rho_0^w).$$

In Example 4.4.21, we computed the map

$$r_c : H^G_\pi(X_G) \to H^M_\pi(X_M)$$

corresponding to the Jacquet functor $r^G_M$. We now consider the composition

$$r_c \circ t_c : H^M_\pi(X_M) \to H^M_\pi(X_M).$$

**Proposition 5.5.3.** One has $r_c \circ t_c = r_c \circ i_c = id + w$ as endomorphisms of $H^M_\pi(X_M)$.

**Proof.** The equality $r_c \circ t_c = id + w$ follows from Theorem 4.5.10. We will show that the equality $r_c \circ t_c = id + w$ holds at the level of chain complexes.

In degree zero, $r_c \circ t_c$ is the map induced on homology by the composition

$$R_C(L) \oplus R_C(L) \xrightarrow{t_c} R_C(K) \oplus R_C(K') \xrightarrow{r_c} R_C(L) \oplus R_C(L)$$

$$(\rho_0, \rho_1) \xrightarrow{t_c} (\iota^K_L \rho_0, \iota^K_L \rho_1) \xrightarrow{r_c} \left( (\iota^K_L \rho_0)^{N_K}, (\iota^K_L \rho_1)^{N_{K'}} \right)$$

where $N_K = \left[ \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right]$ and $N_{K'} = \left[ \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right]^{-1}$.

The Mackey formula and the Bruhat decomposition $K = I \sqcup IwLN_K$ imply that, for
each representation \( \pi \) of \( I \), one has

\[
(\text{ind}_I^K \pi)^{N_K} \cong \pi^{N_K} \oplus \left( \text{ind}_{L, I^\pi}^{L, I^\pi} \pi^w \right)^{N_K} \cong \pi^{N_I} \oplus \left( \pi^N \right)^w.
\]

Therefore

\[
(i_L^I \rho_0)^{N_K} = (\text{ind}_I^K i_L^I \rho_0)^{N_K} \cong (i_L^I \rho_0)^{N_I} \oplus \left( (i_L^I \rho_0)^{N_I} \right)^w \cong \rho_0 \oplus \rho_0^w.
\]

(The final equality holds by Lemma 5.1.14.) One finds similarly that

\[
(i_L^I \rho_1)^{N_K} \cong \rho_1 \oplus \rho_1^w,
\]

and therefore

\[
r_c \circ \iota_c (\rho_0, \rho_1) = (\rho_0 + \rho_0^w, \rho_1 + \rho_1^w) = (\text{id} + w_0)(\rho_0, \rho_1).
\]

In degree one, \( r_c \circ \iota_c \) is the composition

\[
\begin{array}{ccc}
R_C(L) \oplus R_C(L) & \xrightarrow{\iota_c} & R_C(I) \\
\downarrow \rho_0, \rho_1 & \mapsto & \rho_0^w, \rho_1^w \\
\end{array}
\]

Lemma 5.1.14 implies that \( (i_L^I \rho_1)^{N_I} \cong (i_L^I \rho_0)^{N_I} \cong \rho \), and so

\[
r_c \circ \iota_c (\rho_0, \rho_1) = (\rho_0 + \rho_0^w, \rho_1^w + \rho_1) = (\text{id} + w_1)(\rho_0, \rho_1).
\]

Proof of Theorem 5.5.2. We wish to show that \( \iota_c = \iota_c \). We do this separately in degree zero and in degree one. Note first that

\[
\iota_c = 1_{G_c} \circ \iota_c 1_{M_c} = \iota_c 1_{M_c},
\]

thanks to the upper-triangularity of \( i \) (i.e., \( i_{nc,c} = 0 \); see Corollary 4.5.11).

Degree zero: Every class in \( H^M_0(X_M) \) can be represented as a linear combination of cycles of the form \( (\chi, 0) \in R_C(L) \oplus R_C(L) \), where \( \chi \) is a character of \( L \). On these cycles,

\[
\iota_c(\chi, 0) = (i_L^K \chi, 0).
\]
Thus, under the isomorphism $H^G_\ast(X) \cong HH_\ast(\mathcal{H}(G))_c$, $\iota_c(\chi, 0)$ is sent to
\[
\frac{1}{\dim((\iota^K_L \chi)\sim)} e((\iota^K_L \chi)\sim),
\]
where $e((\iota^K_L \chi)\sim)$ is (the class in $HH_0(\mathcal{H}(G))$ of) the idempotent associated to the contra-gredient representation $(\iota^K_L \chi)\sim$. Note that $(\iota^K_L \chi)\sim \cong \iota^K_L \tilde{\chi}$, by Lemma 5.1.10 (and the corresponding property of $\text{ind}^G_K$, for which see [BH06, 3.5]).

For each admissible representation $\pi$ of $G$ we thus have
\[
\langle \text{ch}_\pi, \iota_c(\chi, 0) \rangle = \frac{1}{\dim((\iota^K_L \chi)\sim)} \text{Trace}\left( \pi(e_{\iota^K_L \tilde{\chi}}) \right) = \dim \left( \text{Hom}_K \left( \iota^K_L \tilde{\chi}, \pi \right) \right)
= \dim \left( \text{Hom}_G \left( \text{ind}_K^G \iota^K_L \tilde{\chi}, \pi \right) \right).
\]

(5.5.4)

On the other hand, recalling the equality $\langle \text{ch}_\pi, i_c(x) \rangle = \langle \text{ch}_{\pi\sigma}, x \rangle$ from Lemma 4.1.8, we find
\[
\langle \text{ch}_\pi, i_c(\chi, 0) \rangle = \langle \text{ch}_{\pi\sigma}, e_{\tilde{\chi}} \rangle = \dim \left( \text{Hom}_L \left( \tilde{\chi}, \text{ind}_M^G \pi \right) \right)
= \dim \left( \text{Hom}_G \left( \text{ind}_L^G \iota^K_L \tilde{\chi}, \pi \right) \right).
\]

(5.5.5)

The pair $(I_c, \tilde{\chi})$ is a cover for the $M$-type $(L, \tilde{\chi})$ (see [Kut04] for a review of the terminology and a proof of this fact). A theorem of Dat ([Dat99, 1.5], cf. [Blo05, Théorème 2]) then implies that
\[
\text{ind}_K^G \iota^K_L \tilde{\chi} \cong \text{ind}_L^G \iota^K_L \tilde{\chi} \cong \text{ind}_M^G \tilde{\chi}.
\]

(5.5.6)

The characters of admissible representations separate the points of $HH_0(\mathcal{H}(G))$ ([Kaz86, Theorem 0]), so the equations (5.5.4), (5.5.5), and (5.5.6) together give $\iota_c = i_c$, as claimed.

Degree one: The map $r_c$ is injective in degree one (Corollary 4.4.22). Thus the equality $r_c \circ \iota_c = r_c \circ i_c$ of Proposition 5.5.3 gives $\iota_c = i_c$ immediately. \hfill \Box

Remark 5.5.7. The definition of $\iota_c$ in Lemma 5.5.1 makes sense also for $G = \text{SL}_n(F)$, and
$M$ its diagonal subgroup. For example, in degree $n-1$ the map is

$$\iota_c : R_C(L)^n \to R_C(I), \quad (\rho_0, \ldots, \rho_{n-1}) \mapsto \sum_{w \in W} \iota_L^w \rho_w,$$

where $I \subset G$ is the standard Iwahori subgroup, $L \subset I$ is the diagonal, and $W = N_G(M)/M$ is the Weyl group, which acts simply transitively on the set of chambers in a fundamental domain for the action of $M$ on its apartment $X_M$. In degree zero, the map is

$$\iota_c : R_C(L)^n \to \bigoplus_{i=0}^{n-1} R_C(K_i), \quad (\rho_0, \ldots, \rho_{n-1}) \mapsto (\iota_L^{K_0} \rho_0, \ldots, \iota_L^{K_{n-1}} \rho_{n-1}),$$

where $K_0, \ldots, K_{n-1}$ are the isotropy groups of the vertices of the chamber stabilized by $I$, and $\iota_L^{K_i} \rho_i = \text{ind}_{I}^{K_i} \iota_L \rho_i$.

The fact that the resulting map commutes with the differentials follows from Lemma 5.1.18, just as it did for $SL_2(F)$. The analog of Proposition 5.5.3 holds for $SL_n(F)$, with the same proof: one has

$$r_c \circ \iota_c = r_c \circ i_c = \sum_{w \in W} w.$$

The proof of Theorem 5.5.2 in degree zero also carries over to this case:

**Proposition 5.5.8.** Let $G = SL_n(F)$, and let $M \subset G$ be the diagonal subgroup. Then $\iota_c = i_c$ as maps $H_0^M(X_M) \to H_0^G(X_G)$. \hfill $\Box$

The situation in higher degrees is no longer as simple as it was for $SL_2(F)$. Nevertheless, we conjecture that the equality $\iota_c = i_c$ holds in all degrees:

**Conjecture 5.5.9.** Proposition 5.5.8 also holds for higher-degree homology.

**Remark 5.5.10.** When one replaces the diagonal subgroup with a larger Levi subgroup, for example the block-diagonal subgroup

$$M = \begin{bmatrix} F & F & 0 \\ F & F & 0 \\ 0 & 0 & F \end{bmatrix} \subset SL_n(F),$$

The proof of Theorem 5.5.2 in degree zero also carries over to this case:
one can still use parahoric induction to define a candidate for the map $\iota_c$, but it is no longer clear that this gives a map of complexes. (The issue is closely related to Dat’s question of whether the parahoric functors depend on the choice of parahoric subgroup; see [Dat09, Question 2.14].) It is possible that new tools will be needed to understand this case.
Bibliography


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