On the Quantitative Structure of Δ^0_2

Sebastiaan A. Terwijn*
Mathematisches Institut
Ludwig-Maximilians-Universität München
Theresienstraße 39
D-80333 München
Germany
terwijn@rz.mathematik.uni-muenchen.de

July 21, 2000

Abstract

We analyze the quantitative structure of Δ^0_2. Among other things, we prove that a set is Turing complete if and only if its lower cone is nonnegligible, and that the sets of r.e.-degree form a small subset of Δ^0_2.

Mathematical Subject Classification: 03D15, 03D30, 28E15
Keywords: Computable measure theory, Turing degrees, completeness.

1 Introduction

We study an effective measure theory suited for the study of Δ^0_2, the second level of the arithmetical hierarchy (alternatively, the sets computable relative to the halting problem K). This work may be seen as part of the constructivist tradition in mathematics as documented in [6]. The framework for effectivizing measure theory that we employ uses martingales. Martingales were first applied to the study of random sequences by J. Ville [23]. Recursive martingales were studied in Schnorr [20], and became popular in complexity theory in more recent years through the work of Lutz [15, 16]. Lutz

*Research supported by a Marie Curie fellowship of the European Union under grant no. ERB-FMBI-CT98-3248.

used recursive and more efficient martingales to analyze the quantitative structure of complexity classes like EXP, the exponential time computable sets (see [1] for a survey). Recursive martingales define a notion of null set which is suited for the study of the recursive sets [16]. By relativization, K-recursive martingales can be used to analyze the quantitative structure of Δ^0_2. In particular, we can study classes defined by Turing reducibility (which is trivial on the recursive sets) in a quantitative way. This is what we do below. We first prove that the lower cones of incomplete sets are small, and use this to prove that the sets of r.e.-degree form a small subset of Δ^0_2. In Section 5 we will make some remarks on other definitions of effective measure. In particular, we consider the stronger notion of recursive null set introduced by Schnorr [20] which is directly based on the intuitionistic notion of null set. (This notion was also used in Freidzon [7].) We show that the results of Sections 3 and 4 also hold for this stronger measure. Finally, in Section 6 we motivate the question whether the low sets have measure zero in Δ_2.

2 Measure in Δ_2

We proceed by giving the relevant definitions and by fixing notation. Since we will only consider arithmetical sets in this paper we will write Δ_2 instead of Δ^0_2.

Our recursion theoretic notation follows Soare [21]. We identify a subset of ω (the natural numbers) with its characteristic string. So 2^ω is the power set of ω. $2^{<\omega}$ is the set of finite binary strings, and λ is the empty string. $A\restriction n$ denotes the initial segment of the set A of length n. K denotes the standard Turing complete r.e. set.

We denote the upper cone of a set A by $A^{\leq_T} = \{B : A \leq_T B\}$ and the lower cone by ${}^{\leq_T}A = \{B : B \leq_T A\}$.

Definition 2.1 A function $d : 2^{<\omega} \to \mathbb{Q}^+$ is a *martingale* if for every $\sigma \in 2^{<\omega}$, d satisfies the averaging condition

$$2d(\sigma) = d(\sigma 0) + d(\sigma 1). \qquad (1)$$

A martingale d *succeeds on* a set A if

$$\limsup_{n\to\infty} d(A\restriction n) = \infty.$$

We say that d succeeds on, or *covers*, a class $\mathcal{A} \subseteq 2^\omega$ if d succeeds on every $A \in \mathcal{A}$. The *success set* $S[d]$ of d is the class of all sets on which d succeeds.

A class $\mathcal{A}$ has Δ_2-measure 0, denoted $\mu_{\Delta_2}(\mathcal{A}) = 0$, if there is a martingale $d \in \Delta_2$ such that $\mathcal{A} \subseteq S[d]$. $\mathcal{A}$ has *measure 0 in* Δ_2 if $\mathcal{A} \cap \Delta_2$ has Δ_2-measure 0. $\mathcal{A}$ has *measure 1 in* Δ_2 if the complement of $\mathcal{A}$ has measure 0 in Δ_2.

The above definition is robust in several respects. For example, for our purposes it does not matter whether in (1) we use "=" instead of "$\leq$". Also, normally one would start to define martingales to be real-valued functions and then proceed to study computability issues by using computable approximations. Again, for the resulting measure μ_{Δ_2} this does not make any difference (see e.g. [16, 22]).

The following basic facts are the relativized versions of the same facts for recursive measure. Part (i) shows the consistency of the definition of "measure 1 in Δ_2." Proofs can be found e.g. in [16, 22].

THEOREM 2.2 (Lutz)

(i) Δ_2 *does not have* Δ_2*-measure zero.*

(ii) *For every* $A \in \Delta_2$ *the singleton* $\{A\}$ *has* Δ_2*-measure zero.*

(iii) μ_{Δ_2} *is closed under* Δ_2*-unions (and is in particular finitely additive) i.e. if* $\mathcal{A}_i \subseteq 2^\omega$*,* $i \in \omega$*, is a sequence of classes and* d_i *is a sequence of martingales uniformly computable in* K *such that* d_i *succeeds on* $\mathcal{A}_i$ *for every* i*, then* $\bigcup_i \mathcal{A}_i$ *has* Δ_2*-measure zero.*

As an example of (iii) we may note that the r.e. sets have Δ_2-measure 0 since they are uniformly computable in K. We will later use the following result, saying that upper cones are small.

THEOREM 2.3 (Lutz and Terwijn) *For every nonrecursive set* $A \in \Delta_2$ *the upper cone*

$$A^{\leq_T} = \{B : A \leq_T B\}$$

has Δ_2*-measure zero.*

Proof. This is an effectivization of Sacks's theorem that $A^{\leq_T}$ has measure 0 for every nonrecursive set. A proof is in Terwijn [22, Thm 6.2.1]. □

COROLLARY 2.4 *The Turing-incomplete sets in* Δ_2 *do not have measure 0 in* Δ_2*.*

Proof. By Theorem 2.3 the degree of K has measure 0 in Δ_2, whereas by Theorem 2.2 Δ_2 itself does not have Δ_2-measure 0. The corollary follows from the additivity of Δ_2-measure. □

3 Weak completeness

Informally, a set is weakly complete for some class if its lower cone is not of measure zero in that class. We can consider various classes and reducibilities for which this definition makes sense. For our purposes let us make the following definition.

Definition 3.1 A set $A \in \Delta_2$ is *weakly complete* if ${}^{\leq_T}A$ does not have measure 0 in Δ_2.

The halting set K is weakly complete since Δ_2 is not of measure zero in itself by Theorem 2.2.

First note that if we define a set to be weakly complete if its lower cone has measure 1 in Δ_2,[1] then we immediately have that every weakly complete set is complete. This can be seen using the following argument of Regan et al. [18]:

Theorem 3.2 *If $\mathcal{C} \subseteq \Delta_2$ is closed under symmetric difference $\triangle$ and has measure 1 in Δ_2 then $\mathcal{C} = \Delta_2$.*

Proof. Suppose $\mathcal{C} \neq \Delta_2$, say $A \in \Delta_2 \setminus \mathcal{C}$. It is easy to see that also $A \triangle \mathcal{C} = \{A \triangle C : C \in \mathcal{C}\}$ has measure 1 in Δ_2. But $\mathcal{C}$ and $A \triangle \mathcal{C}$ are disjunct, because if $A \triangle C \in \mathcal{C}$ then also $A = (A \triangle C) \triangle C \in \mathcal{C}$, hence we have reached a contradiction. □

So if A is a set such that almost every set in Δ_2 reduces to it, then A is Turing complete because ${}^{\leq_T}A$ is closed under $\triangle$. The following theorem shows that this is true already when the lower cone of A is not small in Δ_2. So there are no nontrivial examples of weakly complete sets in Δ_2. This still leaves the possibility that some set in Δ_2 (necessarily a Turing complete one) would have a big lower cone with respect to some strong reducibility, like m- or tt- or wtt-reducibility. Since Δ_2 has no complete sets with respect to the strong reducibilities we could call such sets proper weakly complete with respect to these reducibilities. However, this cannot happen since for every $A \in \Delta_2$ its wtt-lower cone $\{B : B \leq_{wtt} A\}$ has measure zero in Δ_2 [22, Theorem 4.4.4].

N.B. In complexity theory it is known that the sets that are weakly p-m-complete for EXP, the class of exponential time computable sets, have measure 1 in EXP, whereas the p-m-complete sets have measure 0 (see the

[1] Such sets are called *almost complete* in complexity theory. They were studied recently in Ambos-Spies et al. [3]

survey [1] for references). The question whether there are p-T-incomplete weakly T-complete sets in EXP is open.

N.B. Ambos-Spies and Terwijn [22, Chapter 3] defined a measure tailored for the recursively enumerable sets, and studied the resulting weak completeness notions. They proved that in the class of r.e. sets the weak completeness notions for truth-table, weak truth-table, and Turing reducibility coincide with the ordinary completeness notions, whereas those for many-one and bounded truth-table reducibility differ from their ordinary counterparts.

THEOREM 3.3 *Every weakly complete set is Turing complete.*

Proof. Suppose that $A <_T K$. We define uniformly in K for every $e \in \omega$ a martingale d_e such that

$$R_e: \quad \{e\}^A \text{ total and 0,1-valued} \Longrightarrow \{e\}^A \in S[d_e].$$

By Theorem 2.2 (iii) this suffices to prove the theorem. Let $f \leq_T K$ be a function that is not dominated by any function $g <_T K$, for example, $f(x) = \mu s(K_s{\restriction}x = K{\restriction}x)$ (the smallest s such that all the $y \in K$ smaller than x are enumerated into K within s steps).

We now define d_e in stages s. On stage s we define d_e on all strings $\sigma \in 2^{<\omega}$ of length s. The value $d_e(\sigma)$ will depend only on $|\sigma|$. (Such martingales were called 'oblivious' in Ambos-Spies et al. [2].)

Stage $s = 0$. Define $d_e(\lambda) = 1$.

Stage $s+1$. Given $d_e(\sigma)$, $|\sigma| = s$, use the oracle K to search for a string $\tau \sqsubset A$ with $|\tau| \leq f(s)$ such that $\{e\}^{\tau}_{|\tau|}(s)\downarrow$. If such τ does not exist, or if $\{e\}^{\tau}_{|\tau|}(s)\downarrow \notin \{0,1\}$, we do not make a bet, namely we let $d_e(\sigma i) = d_e(\sigma)$ for $i \in \{0,1\}$. If τ exists and $\{e\}^{\tau}_{|\tau|}(s)\downarrow = i \in \{0,1\}$ we define $d_e(\sigma i) = 2d_e(\sigma)$, that is, we bet all our money on $\{e\}^A(|\sigma|) = i$. This concludes the definition of d_e.

It is clear that d_e is defined on all strings for every e, uniformly in K. We check that R_e is satisfied. Suppose that $\{e\}^A$ is total and computes a set. Then the function

$$f_e(n) = \mu t(\exists \tau \sqsubset A\,[\,|\tau| = t \wedge \{e\}^{\tau}_t(n)\downarrow])$$

is an A-computable function. By choice of f we have that there are infinitely many s such that $f(s) \geq f_e(s)$. But this means that in the definition of d_e infinitely often the string τ is found and a bet is placed successfully. (Note that we never make a wrong bet.) Hence $\{e\}^A \in S[d_e]$. □

Corollary 3.4 *For every $\emptyset <_T A <_T K$ the sets incomparable with A have measure one in Δ_2.*

Proof. The sets incomparable with A are the complement of the union ${}^{\leq_T}A \cup A^{\leq_T}$ of the lower and the upper cone of A. The former has measure 0 in Δ_2 by Theorem 3.3 and the latter by Theorem 2.3. $\square$

4 Smallness of the r.e.-degrees

After Theorem 2.2 we observed that the r.e. sets form a small subset of Δ_2. We now apply Theorem 2.3 to obtain

Theorem 4.1 *The class of sets that have r.e.-degree has measure 0 in Δ_2.*

Proof. Let W_e be the e-th r.e. set. For every W_e we define a martingale d_e as in the proof of Theorem 3.3 such that

$$W_e <_T K \implies d_e \text{ succeeds on } {}^{\leq_T}W_e.$$

Note that the definition of d_e does not depend on the assumption $W_e <_T K$, only the success of d_e does. So we have a recursive sequence of Δ_2-martingales d_e, and Theorem 2.2 (iii) gives that $\bigcup\{{}^{\leq_T}W_e : W_e <_T K\}$ has Δ_2-measure 0. Finally, Theorem 2.3 and the finite additivity of μ_{Δ_2} give that

$$\{A : \exists e(A \equiv_T W_e)\} \subseteq \{A : \exists e(A \leq_T W_e <_T K)\} \cup \{A : K \leq_T A\}$$

has measure 0 in Δ_2. $\square$

We could have proved this theorem in a slightly different way, namely by defining for every r.e. set W_e two martingales d_e^0 and d_e^1, the first succeeding on ${}^{\leq_T}W_e$ if $W_e <_T K$, and the second succeeding on ${W_e}^{\leq_T}$ if $\emptyset <_T W_e$. This needs, however, that in the proof of Theorem 2.3 the definition of the martingale is independent of the hypothesis $\emptyset <_T A$, and only the success of the martingale depends on this. As this is indeed the case, this second proof now gives that

$$\{A : \exists e(A \leq_T W_e <_T K)\} \cup \{A : \exists e(\emptyset <_T W_e \leq_T A)\}$$

has Δ_2-measure 0. This yields

COROLLARY 4.2 *Almost every set (in the sense of Δ_2-measure) in Δ_2 is Turing-incomparable to every r.e. set, except the recursive and the complete ones.*

The existence of sets as in Corollary 4.2 can be proved directly (Kleene-Post style arguments [9, 21]). Corollary 4.2 can be seen as a probabilistic version of this result, just as the main result of Kučera and Terwijn [12] can be seen as a probabilistic solution to Post's problem.

5 Modulated measure

In this section we make some remarks about other measures and strengthen the results of the previous sections. Instead of Δ_2-measure one could use a weaker notion of null set like Σ_1-measure (as defined in Martin-Löf [17][2]) or recursive measure to study Δ_2. These two examples, however, are unnatural for this purpose because Δ_2 contains elements that are Martin-Löf random (Σ_1-random), i.e. elements A such that the Σ_1-measure of $\{A\}$ is not zero.[3] This trivially makes results like Theorem 2.3 and Theorem 3.3 untrue for these measures, and is an indication that they are to weak for the study of Δ_2. Instead we consider another definition of measure, which is more in the spirit of constructive mathematics.

Schnorr [20] pointed out a nonconstructive aspect of Martin-Löfs definition of measure [17], and he introduced the following stronger notion of null set, based on the intuitionistic notion of null set in Brouwer [4]. The definition given here is not his original one, but one that is equivalent to it [20, Satz 9.7].

DEFINITION 5.1 A class $\mathcal{A}$ has *modulated recursive measure 0* if and only if there is a recursive martingale d and a nondecreasing unbounded recursive function h such that $\mathcal{A} \subseteq S_h[d]$, where

$$S_h[d] = \{X : \limsup_{n\to\infty} \frac{d(X\restriction n)}{h(n)} \geq 1\}.$$

The pair (d, h) is also called a *total recursive sequential test.*

[2] One can define Σ_1-measure by replacing in Definition 2.1 Δ_2 by Σ_1. (A function f is in Σ_1 if the set $\{\langle x, y\rangle : y \leq f(x)\}$ is r.e.)

[3] Martin-Löf random sets play an important role in the area of Kolmogorov complexity, see Li and Vitányi [14].

Like the recursive measure, we can relativize this definition to K. Thus we obtain the notion of *modulated* Δ_2*-measure 0* (mod-Δ_2-measure 0 for short), which is defined exactly as modulated recursive measure 0 except that in the definition we replace recursive everywhere by K-recursive. Note that indeed we have that no set in Δ_2 is mod-Δ_2-random. (Every single $A \in \Delta_2$ can be covered by a martingale that computes A, thus growing with rate $h(n) = 2^n$.)

Let us summarize the various notions of null set that we have discussed:

(i) modulated recursive null

(ii) recursive null

(iii) Σ_1-null

Relativized to K these give

(iv) modulated Δ_2-null

(v) Δ_2-null

(vi) Σ_2-null

From the definitions it is immediate that (i)$\Rightarrow$(ii)$\Rightarrow$(iii). (ii)$\not\Rightarrow$(i) was proved by Wang [24] and (iii)$\not\Rightarrow$(ii) follows since the recursive sets have Σ_1-measure zero. The strict implications (iv)$\Rightarrow$(v)$\Rightarrow$(vi) follow from relativizing this. Finally, (iii)$\Rightarrow$(iv) follows from the analysis in Schnorr [20],[4] and (iv)$\not\Rightarrow$(iii) because, as noted above, there are $A \in \Delta_2$ such that $\{A\}$ is not Σ_1-null, whereas for every $A \in \Delta_2$ we have that $\{A\}$ is mod-Δ_2-null. So (i)$-$(vi) is indeed a sequence of notions of measure 0 increasing in strength. As discussed above, (iii) is too weak for the study of Δ_2 and (vi) is too strong, since Δ_2 has Σ_2-measure 0. The results from the previous sections used (v). In this section we will prove that these results also hold for (iv). This yields stronger statements, since Terwijn [22] proved that mod-Δ_2-measure is strictly weaker (there are less null sets) on Δ_2 than Δ_2-measure.[5] We will use the following notion.

[4] Satz 9.5 in [20] says that every null set defined by a total recursive sequential test has modulated recursive measure 0. Since every Σ_1-null set is covered by a total K-recursive sequential test the claim follows by relativization.

[5] Theorem 6.4.2 of [22] says that modulated recursive measure is strictly weaker than recursive measure even when restricted to REC. This answered a question of Lutz. The proof of this fact relativizes. That there are (necessarily nonrecursive) sets that are not in any modulated recursive null set but that can be covered by a recursive martingale was proved earlier by Wang [24].

Definition 5.2 For functions f and g, we say that g *densely dominates* f if

$$\liminf_{n\to\infty} \frac{\|\{i \le n : g(i) \ge f(i)\}\|}{n} \ge \frac{1}{2}. \tag{2}$$

Lemma 5.3 *There is a function $f \le_T K$ such that no function $g <_T K$ densely dominates f.*

Proof. Let $h \le_T K$ be a function not dominated by any $g <_T K$ (cf. the proof of Theorem 3.3). Define $f(x) = h(\lfloor \log x \rfloor)$. If g satisfies (2) then for almost every e there is a natural number $x \in [2^e, 2^{e+1})$ such that $g(x) \ge f(x)$. But then $g'(e) = \max\{g(x) : x \in [2^e, 2^{e+1})\}$ dominates h, contradiction. □

We will also need effective unions:

Lemma 5.4 *If $\mathcal{A}_i \subseteq 2^\omega$, $i \in \omega$, is a sequence of classes and (d_i, h_i) is a sequence of sequential tests uniformly computable in K such that $\mathcal{A}_i \subseteq S_{h_i}[d_i]$ for every i, then $\bigcup_i \mathcal{A}_i$ has mod-Δ_2-measure zero.*

Proof. We will not prove this in detail. The proof is almost the same as that of Theorem 2.2 (iii). The only extra ingredient that one needs is that for a uniformly computable sequence h_i as above one can define a computable monotone unbounded h growing slower than any of the h_i, which is easy to see. (Such constructions were studied by P. du Bois-Reymond as early as 1877, see Hardy [8, Theorem 3].) □

Theorem 5.5 *The lower cone of every $A <_T K$ has mod-Δ_2-measure zero.*

Proof. We build on the proof of Theorem 3.3. There we argued that if $\{e\}^A$ is total, then infinitely often we doubled the value of the martingale. Now if we use the f from Lemma 5.3 we can do this at more than half of the stages rather than just infinitely often. Namely, for f_e as in the proof of Theorem 3.3, we now have that infinitely often

$$\|\{i \le n : f(i) > f_e(i)\}\| \ge \frac{1}{2}n.$$

So if $B = \{e\}^A$ is total then, for $h(n) = 2^{\frac{1}{2}n}$, we have

$$\limsup_{n\to\infty} \frac{d(B\restriction n)}{h(n)} \ge 1,$$

that is, $B \in S_h[d]$. Now we can apply Lemma 5.4 to finish the proof. (In fact we need less than Lemma 5.4, since all the h_i's are the same here.) $\square$

An inspection of the proof of Theorem 2.3 in [22] shows that in fact we have

THEOREM 5.6 *For every nonrecursive $A \in \Delta_2$ the upper cone $A^{\leq_T}$ has mod-Δ_2-measure 0.*

Now the stronger versions of Theorem 4.1 and Corollary 4.2 follow in the same way as these results follow from the proofs of Theorems 2.3 and 3.3:

THEOREM 5.7 *The sets of r.e. degree have mod-Δ_2-measure 0.*

COROLLARY 5.8 *Almost every set (in the sense of modulated Δ_2-measure) in Δ_2 is Turing-incomparable to every r.e. set, except the recursive and the complete ones.*

We end this section by remarking that the previous results cannot be strengthened from mod-Δ_2-measure to Σ_1-measure. For Theorems 5.6 and 5.7 and Corollary 5.8 this is immediate by the existence of Σ_1-random sets X with $X \equiv_T K$. For Theorem 5.5 this follows from the existence of low Σ_1-random sets X, i.e. X with $K^X \equiv_T K$. The failure of Corollary 5.8 for Σ_1-measure is also illustrated by the result of Kučera [11] that there exist a nonrecursive r.e. set A and a low Σ_1^A-random set X such that $A \leq_T X$. That the converse of this, namely that an incomplete r.e. set bounds a Σ_1-random set, cannot happen was proved by Kučera [10].

6 Questions

As an example of the well-known incompatibility of measure and category one can prove that the 1-generic sets have Δ_2-measure zero. Much more interestingly, one can prove that the sets that have 1-generic *degree* have measure zero in Δ_2. This follows from the idea in the proof of Kurtz [13, Theorem 4.2]. This gives in fact that the downward closure of the 1-generic sets has Σ_1-measure zero. Demuth and Kučera [5] proved the slightly more general statement that no diagonally nonrecursive function is recursive in a 1-generic set.

THEOREM 6.1 (Kurtz [13], Demuth and Kučera [5]) *The class of sets that have 1-generic degree has Σ_1-measure zero. Equivalently, no Martin-Löf random set has 1-generic degree.*

Proof. The proof given here is a rather straightforward effectivization of the proof of Theorem 4.2 in Kurtz [13]. For every e,$i \in \omega$ define the Σ^0_1-class $\mathcal{S}^e_i$ as follows. Let $\{\sigma_j\}_{j\in\omega}$ be a recursive enumeration of $2^{<\omega}$. Let τ_j be the least τ extending σ_j such that $\Phi_e(\tau)$ is defined on at least $i+j+1$ arguments. (As usual Φ_e denotes the e-th Turing reduction.) Define $\mathcal{S}^e_i$ to be the open set defined by the range of the partial recursive function $\sigma_j \mapsto \tau_j$.

First we note that

$$\mu(\Phi_e(\mathcal{S}^e_i)) \leq \sum_{j\in\omega} \mu(\Phi_e(\tau_j)) \leq \sum_{i\in\omega} 2^{-(i+j+1)} = 2^{-i}. \tag{3}$$

Second, since the classes $\mathcal{S}^e_i$ are uniformly r.e. in e and i, it follows that the classes $\Phi_e(\mathcal{S}^e_i)$ are uniformly r.e. in e and i. So by (3), for every e the class $\bigcap_{i\in\omega} \Phi_e(\mathcal{S}^e_i)$ is a Σ_1-null set, and by a standard sum argument we have that $\bigcup_{e\in\omega}\bigcap_{i\in\omega} \Phi_e(\mathcal{S}^e_i)$ is Σ_1-null.

Now if G is 1-generic and G does not meet $\mathcal{S}^e_i$, then by 1-genericity there is an initial segment $\sigma \sqsubseteq G$ such that no $\tau \sqsupseteq \sigma$ is in $\mathcal{S}^e_i$. Hence for every $\tau \sqsupseteq \sigma$, $\Phi_e(\tau)$ is never defined on at least $i+j+1$ arguments, where $\sigma = \sigma_j$. Therefore if $\Phi_e(G)$ is total then G meets every $\mathcal{S}^e_i$. So we have

$$\begin{aligned}
\{A : A \leq_T G \text{ for some 1-generic } G\} &= \\
\bigcup_{e\in\omega} \{A : A = \Phi_e(G) \text{ for some 1-generic } G\} &\subseteq \\
\bigcup_{e\in\omega} \{A : A = \Phi_e(G) \text{ for some } G \text{ that meets } \mathcal{S}^e_i \text{ for every } i\} &\subseteq \\
\bigcup_{e\in\omega}\bigcap_{i\in\omega} \Phi_e(\mathcal{S}^e_i),&
\end{aligned}$$

and since this last class has Σ_1-measure 0 the theorem follows. □

Since all sets in Δ_2 that have 1-generic degree are low (A is low whenever it holds that $K^A \leq_T K$) one can ask whether the low sets have measure zero in Δ_2. One would expect this to be the case, since the low sets behave in many respects like the recursive sets, which have measure zero in Δ_2.

QUESTION 6.2 *Do the low sets have measure zero in* Δ_2*?*

Still more general, we have:

QUESTION 6.3 *Does every jump class* $\{B : B' \equiv_T S\}$, $S \in [\emptyset', \emptyset'']$ *have measure zero in* Δ_2*?*

ACKNOWLEDGMENT. I would like to thank Klaus Ambos-Spies for helpful discussions and for his support.

References

[1] K. Ambos-Spies and E. Mayordomo, *Resource-bounded measure and randomness*, in: A. Sorbi (ed.), Complexity, logic, and recursion theory, 1–47, Dekker, New York, 1997.

[2] K. Ambos-Spies, E. Mayordomo, Y. Wang, X. Zheng, *Resource-bounded balanced genericity, stochasticity and weak randomness*, Proc. 13th Symp. on Theoret. Aspects of Comp. Sci., Lect. Notes in Comp. Sci. 1046 (1996) 63–74, Springer-Verlag.

[3] K. Ambos-Spies, W. Merkle, J. Reimann, and S. A. Terwijn, *Almost complete sets.* Proc. 17th Symp. on Theoret. Aspects of Comp. Sci., Lect. Notes in Comp. Sci. 1770 (2000) 419–430, Springer-Verlag.

[4] L. E. J. Brouwer, *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten*, Zweiter Teil, Verhandelingen Nederlandse Akademie van Wetenschappen, afd. Natuurk. Sect. 12,7 (1919).

[5] O. Demuth and A. Kučera, *Remarks on 1-genericity, semigenericity and related concepts*, Comment. Math. Univ. Carolinae, 28 (1987), 85–94.

[6] Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel (eds.), *Handbook of Recursive Mathematics*, Vol. 1 and 2, Elsevier, 1998.

[7] R. I. Freidzon, *Families of recursive predicates of measure zero*, J. Soviet Math. 6 (1976), 449–455. Translated from Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 32, (1972) 121–128.

[8] G. H. Hardy, *Orders of infinity: the 'Infinitärcalcül' of Paul du Bois-Reymond*, Cambridge University Press, 1910 (reprinted 1954).

[9] S. C. Kleene and E. L. Post, *The upper semi-lattice of degrees of recursive unsolvability*, Ann. of Math. (2) 59 (1954) 379–407.

[10] A. Kučera, *Measure, Π^0_1-classes and complete extensions of PA*, in: H.-D. Ebbinghaus, G. H. Müller, and G. E. Sacks, eds., Recursion theory week 1984, Lect. Notes in Math. 1141 (1985) 245–259.

[11] A. Kučera, *On relative randomness*, Annals of Pure and Applied Logic 63 (1993) 61–67.

[12] A. Kučera and S. A. Terwijn, *Lowness for the class of random sets*, J. Symbolic Logic 64(4) (1999) 1396–1402.

[13] S. A. Kurtz, *Randomness and genericity in the degrees of unsolvability*, PhD Thesis, University of Illinois, 1981.

[14] M. Li and P. Vitányi, *An introduction to Kolmogorov complexity and its applications*, 2nd edition, Springer-Verlag, 1997.

[15] J. H. Lutz, *Category and measure in complexity classes*, SIAM J. Comput. 19 (1990) 1100–1131.

[16] J. H. Lutz, *Almost everywhere high nonuniform complexity*, J. Comput. System Sci. 44 (1992) 220–258.

[17] P. Martin-Löf, *The definition of random sequences*, Information and Control 9 (1966) 602–619.

[18] K. W. Regan, D. Sivakumar, and J.-Y. Cai, *Pseudorandom Generators, Measure Theory, and Natural Proofs*, 36th Annual Symp. on Foundations of Comp. Sci. (FOCS'95), 26–35, IEEE Computer Society Press, 1995.

[19] G. E. Sacks, *Degrees of unsolvability*, Annals of Mathematics Studies 55, Princeton University Press, 1963.

[20] C.-P. Schnorr, *Zufälligkeit und Wahrscheinlichkeit*, Lect. Notes in Math. 218, Springer-Verlag, 1971.

[21] R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, 1987.

[22] S. A. Terwijn, *Computability and Measure*, PhD Thesis, ILLC/University of Amsterdam, 1998.

[23] J. Ville, *Étude critique de la notion de collectif*, Gauthier-Villars, 1939.

[24] Y. Wang, *Randomness and complexity*, PhD Thesis, Universität Heidelberg, 1996.