Kripke Models, Distributive Lattices, and Medvedev Degrees

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Abstract

We define a variant of the standard Kripke semantics for intuitionistic logic, motivated by the connection between constructive logic and the Medvedev lattice. We show that while the new semantics is still complete, it gives a simple and direct correspondence between Kripke models and algebraic structures such as factors of the Medvedev lattice.

Keywords: Kripke semantics, Medvedev degrees, intuitionistic propositional logic.

1 Introduction

Immediately after Heyting had isolated the axioms of intuitionistic logic in [1] people began to wonder what the precise interpretation of these axioms was. In the course of history many (complete) interpretations have been given, one of the most famous being Kripke semantics [6]. Also several interpretations were suggested that later turned out to be incomplete. Among these are Kleene's realizability [4] and the approach by Kolmogorov and Medvedev that will be discussed below. Early on certain complete algebraic interpretations were given, for example in Jaśkowski [3]. It is to be noted that these algebraic interpretations, as well as the later Kripke semantics, have little or nothing to do with the basic intuitions surrounding intuitionistic logic, namely those of effectivity, constructivity, or computability (cf. the informal Brouwer-Heyting-Kolmogorov interpretation [16]). Kolmogorov [5]

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suggested to interpret the propositional connectives constructively using a "calculus of problems". This idea was later implemented in various ways by Medvedev [7, 8]. Although this approach initially failed to give a complete semantics for the intuitionistic propositional logic IPC, the structures introduced by Medvedev turned out to be interesting for different reasons as well. In particular, there are interesting connections with the Turing degrees and other structures from computability theory, cf. Sorbi [13]. Moreover, Skvortsova [10] showed that the main idea of the Kolmogorov/Medvedev approach could be used to give a complete computational semantics for IPC after all (see below). The results in this area are a particularly attractive blend of results and methods from computability theory (such as lattice embedding results) with proof-theoretic ones (coming from the study of intuitionistic and intermediate logics).

In this note we present a variant of Kripke semantics in which the interpretation of "or" is different from the usual one. We show that this alternative interpretation is sound and complete for "distributive" structures, where the notion of distributivity is a rather liberal one, applying to partial orders in general instead of just lattices. The new notion of forcing will be denoted by $\Vdash^*$, and the new kind of Kripke models under this forcing notion will be called Kripke* models. The variation is motivated by the observation that every configuration in the Medvedev lattice can be interpreted as a Kripke* model. This gives a direct relation between Kripke* models on the one hand and the Medvedev lattice on the other.

We briefly recall the definition of the Medvedev lattice $\mathfrak{M}$. Let ω denote the natural numbers and let ω^ω be the set of all functions from ω to ω (Baire space). A *mass problem* is a subset of ω^ω. One can think of such subsets as a "problem", namely the problem of producing an element of it, and so we can think of the elements of the mass problem as its set of solutions. A mass problem $\mathcal{A}$ *Medvedev reduces* to mass problem $\mathcal{B}$ if there is an effective procedure of transforming solutions to $\mathcal{B}$ into solutions to $\mathcal{A}$: $\mathcal{A} \leq \mathcal{B}$ if there is a partial computable functional $\Psi : \omega^\omega \to \omega^\omega$ such that for all $f \in \mathcal{B}$, $\Psi(f)$ is defined and $\Psi(f) \in \mathcal{A}$.[1] This can be seen as an implementation of Kolmogorov's idea of a calculus of problems. The relation $\leq$ induces an

[1]Note that although Medvedev reducibility is designed especially for sets of functions, it is close in spirit to ordinary Turing reducibility $\leq_T$ for individual functions: $g \leq_T f$ if there is a partial computable functional Φ that is defined at least on f such that $\Phi(f) = g$. One may also compare Medvedev reducibility to Wadge reducibility $\leq_W$ from descriptive set theory: For sets of reals (or mass problems) $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \leq_W \mathcal{B}$ if there is a continuous functional Φ such that $\Phi^{-1}(\mathcal{B}) = \mathcal{A}$. Note that this notion is quite different: the continuity is in the other direction, there is no 'if and only if', and no effectivity.

equivalence relation on the mass problems: $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$. The equivalence class of $\mathcal{A}$ is denoted by $[\mathcal{A}]$ and is called the *Medvedev degree* of $\mathcal{A}$. We denote Medvedev degrees by boldface symbols. Note that there is a smallest Medvedev degree, denoted by $\mathbf{0}$, namely the degree of any mass problem containing a computable function. There is also a largest degree $\mathbf{1}$, the degree of the empty mass problem, of which it is impossible to produce an element by whatever means. Finally, it is possible to define a meet operator $\times$ and a join operator $+$ on mass problems: For functions f and g, as usual define the function $f \oplus g$ by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. Let $n\frown\mathcal{A} = \{n\frown f : f \in \mathcal{A}\}$, where $\frown$ denotes concatenation. Define

$$\mathcal{A} + \mathcal{B} = \big\{f \oplus g : f \in \mathcal{A} \wedge g \in B\big\}$$

and

$$\mathcal{A} \times \mathcal{B} = 0\frown\mathcal{A} \cup 1\frown\mathcal{B}.$$

It is not hard to show that $\times$ and $+$ indeed define a greatest lower bound and a least upper bound operator on the Medvedev degrees. For more information and discussion on $\mathfrak{M}$ we refer to Sorbi [13], Terwijn [14, 15].

A distributive lattice $\mathfrak{L}$ with 0, 1 is called a *Brouwer algebra* if for any elements a and b it holds that the element $a \to b$ defined by

$$a \to b := \text{least}\big\{c \in \mathfrak{L} : b \leq a + c\big\}$$

always exists. $\mathfrak{L}$ is called a *Heyting algebra* if its dual is a Brouwer algebra. Medvedev [7] showed that $\mathfrak{M}$ is a Brouwer algebra, and Sorbi [11] showed that $\mathfrak{M}$ is not a Heyting algebra.

Above we have already defined the operations $\times$, $+$, and $\to$ on $\mathfrak{M}$. We can also define a negation operator $\neg$ by defining $\neg\mathbf{A} = \mathbf{A} \to \mathbf{1}$ for any Medvedev degree $\mathbf{A}$.

Given any Brouwer algebra $\mathfrak{L}$ (such as $\mathfrak{M}$) with join denoted by $+$ and meet by $\times$, we can evaluate formulas as follows. An $\mathfrak{L}$-*valuation* is a function σ : Form $\to \mathfrak{L}$ from propositional formulas to $\mathfrak{L}$ such that for all formulas A and B, $\sigma(A \vee B) = \sigma(A) \times \sigma(B)$, $\sigma(A \wedge B) = \sigma(A) + \sigma(B)$, $\sigma(A \to B) = \sigma(A) \to \sigma(B)$, $\sigma(\neg A) = \sigma(A) \to 1$. (Note the upside-down reading of $\wedge$ and $\vee$ when compared to the usual lattice theoretic interpretation. For more remarks regarding notation see [14].) Write $\mathfrak{L} \models A$ if $\sigma(A) = 0$ for any $\mathfrak{L}$-valuation σ. Finally, define

$$\text{Th}(\mathfrak{L}) = \big\{\alpha : \mathfrak{L} \models \alpha\big\}.$$

A *B-homomorphism* is a mapping between Brouwer algebras preserving $+$, $\times$, $\to$, and 0 and 1. A *B-embedding* is an injective B-homomorphism.

Note that if $\mathfrak{L}_1$ is B-embeddable into $\mathfrak{L}_2$ then $\mathrm{Th}(\mathfrak{L}_2) \subseteq \mathrm{Th}(\mathfrak{L}_1)$ because every $\mathfrak{L}_1$-valuation is also a $\mathfrak{L}_2$-valuation. If there is a (not necessarily injective) B-homomorphism from $\mathfrak{L}_1$ onto $\mathfrak{L}_2$ we obtain the converse $\mathrm{Th}(\mathfrak{L}_1) \subseteq \mathrm{Th}(\mathfrak{L}_2)$. Note that the top element has to be preserved in order to fix the meaning of negation.

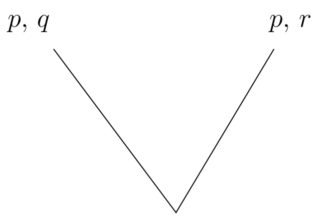
It follows from results of Medvedev [8] and Jankov [2] that $\mathrm{Th}(\mathfrak{M})$ is the deductive closure of IPC and the weak law of the excluded middle $\neg\alpha \vee \neg\neg\alpha$ (also known as De Morgan, or Jankov logic). Although this shows that the Kolmogorov/Medvedev approach for providing a semantics for IPC does not work directly, the ideas can still be used to interpret IPC. Namely, one can consider *factors* of $\mathfrak{M}$, i.e. study $\mathfrak{M}$ modulo a filter or an ideal. Given a Brouwer algebra $\mathfrak{L}$ and an ideal I in $\mathfrak{L}$, $\mathfrak{L}/I$ is still a Brouwer algebra. If G is a filter in $\mathfrak{L}$ then $\mathfrak{L}/G$ is not necessarily a Brouwer algebra, but if G is principal then $\mathfrak{L}/G$ is again a Brouwer algebra. In such a factorized lattice G plays the role of 1. E.g. if G is the principal filter in $\mathfrak{M}$ generated by the degree $\mathbf{D}$ then negation in $\mathfrak{M}/G$ can be defined by $\neg\mathbf{A} = \mathbf{A} \rightarrow \mathbf{D}$. The algebraic properties of $\mathfrak{M}/G$ are directly related to the theories $\mathrm{Th}(\mathfrak{M}/G)$. For example, whether the weak law of the excluded middle holds in $\mathrm{Th}(\mathfrak{M}/G)$ is related to whether the element that generates G is join-reducible in $\mathfrak{M}$. (For more information see [13, 14].) Skvortsova [10] showed that there are indeed factors that capture IPC: There exists a principal filter G such that $\mathrm{Th}(\mathfrak{M}/G) = \mathrm{IPC}$.

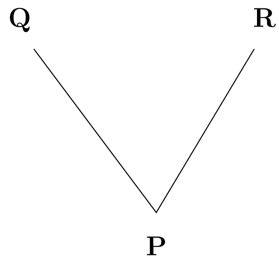
2 A variant of Kripke semantics

The reader may observe that a configuration in the Medvedev degrees bears much resemblance to a Kripke model. We have already seen that negation in $\mathfrak{M}$ causes problems, but this can be remedied by considering suitable factors $\mathfrak{M}/G$. Let us for the moment consider only formulas without negation, so that Medvedev's theorem (that the positive fragments of $\mathrm{Th}(\mathfrak{M})$ and IPC coincide [8]) applies. Consider for example the following formula φ:

$$(p \rightarrow q \vee r) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$$

This formula is not derivable in IPC. The left hand side of the following figure shows a Kripke countermodel for φ, and the right hand side is a configuration of Medvedev degrees that shows that $\varphi \notin \mathrm{Th}(\mathfrak{M})$.





If $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are interpreted by Medvedev degrees as in the configuration on the right, where $\mathbf{P} = \mathbf{Q} \times \mathbf{R}$, then φ does not evaluate to $\mathbf{0}$ because $\mathbf{P} \geq \mathbf{Q} \times \mathbf{R}$ but neither $\mathbf{P} \geq \mathbf{Q}$ nor $\mathbf{P} \geq \mathbf{R}$. This example shows that there is a difference in interpretation between $\vee$ in Kripke models and $\times$ in the Medvedev lattice: $A \vee B$ holds in a node k in a Kripke model only if A or B holds in k, but in the above configuration $\mathbf{Q} \times \mathbf{R}$ "holds" in $\mathbf{P}$ but neither $\mathbf{Q}$ nor $\mathbf{R}$ does. We now define a variant of the standard Kripke semantics, where the interpretation of $\vee$ is changed, and that will give a precise connection between Kripke models on the one hand and configurations in $\mathfrak{M}$ on the other. Given a partial order M, let $k = i \sqcap j$ denote that the greatest lower bound of i and j exists in M and is equal to k.

Let $\Vdash$ denote the usual forcing relation in Kripke models, cf. [16]. We define a variant $\Vdash^*$ where $\vee$ has a somewhat different interpretation. Namely, for a propositional Kripke model M, a node $k \in M$, and propositional formulas A and B, we let $k \Vdash^* A \vee B$ if any one of the following holds:

1. $k \Vdash^* A$ or $k \Vdash^* B$,
2. $(\exists l \leq k)[l \Vdash^* A \vee B]$
3. There exist $i, j \in M$ with $k = i \sqcap j$ such that $i \Vdash^* A$ and $j \Vdash^* B$.

We also postulate for any propositional formula A the following contraction property:

$$\text{if } k \Vdash^* A \vee A \text{ then } k \Vdash^* A. \tag{1}$$

The clauses for $\wedge$, $\rightarrow$, and $\neg$ for $\Vdash^*$ are identical to those for $\Vdash$. For example, $k \Vdash^* A \rightarrow B$ if in every $k' \geq k$ with $k' \Vdash^* A$ it holds that $k' \Vdash^* B$. Thus, in the new semantics, if a node k is the meet of a node where A holds and another node where B holds, then $A \vee B$ holds in k. In this way we allow the truth in Kripke models to "run ahead" in certain situations. We will call a Kripke model under this new semantics a *Kripke* model.*[2]

The above definition is to be understood as follows. Since the rule (1) contracts formulas, we have to clarify how it combines with the inductive character of the other clauses. We also have to take some care that the applications of the contraction property (1) are well-founded in order to exclude applications of it where any formula can be made true by relying on an infinite amount of successor nodes without a proper atomic foundation. We thus give a more formal definition as follows:

Definition 2.1 Let the *logical complexity* of a formula A be defined as usual.[3]

1. For atomic A, let $k \Vdash^*_0 A$ if A holds in k by the valuation of M.

2. Suppose that $k \Vdash^*_0 A$ is defined for all A of complexity $\leq n$. For every A of complexity $\leq n$ and every ordinal α let $k \Vdash^*_{\alpha+1} A$ if either

   (i) $k \Vdash^*_\alpha A$ or if

   (ii) $k = i \sqcap j$ for some i, j with $i \Vdash^*_\alpha A$ and $j \Vdash^*_\alpha A$, or if

   (iii) $l \Vdash^*_\alpha A$ for some $l \leq k$.

3. For all A of complexity $\leq n$, define $k \Vdash^* A$ if $k \Vdash^*_\alpha A$ for some α.

4. Now define $\Vdash^*_0$ for formulas of complexity $n+1$ as follows:
   $k \Vdash^*_0 B \wedge C$ if both $k \Vdash^* B$ and $k \Vdash^* C$,
   $k \Vdash^*_0 B \to C$ if $\forall l \geq k\ (l \Vdash^* B \to l \Vdash^* C)$,
   $k \Vdash^*_0 B \vee C$ if $k \Vdash^* B$ or $k \Vdash^* C$ or $k = i \sqcap j$ and $i \Vdash^* B$ and $j \Vdash^* C$.

This inductively defines $k \Vdash^* A$ for formulas A of every complexity.

Thus $k \Vdash^*_0 A$ holds if A is forced using the rule for the main connective in A, and before defining the forcing relation for formulas of higher complexity we first complete the definition for formulas of the complexity of A by using repeated applications of the $\vee$-rule and the contraction rule (1). In fact, taken over the whole model M we need at most ω steps to complete this:

[2]As a remark on methodology we note that it is not possible to require the property (1) only for atomic A and derive the general form from that: It is possible to produce a distributive Kripke* model in which (1) holds for the atoms but not in general.

[3]That is, atomic formulas have complexity 1, the complexity of $\neg A$ is that of A plus 1, and the complexity of $A \vee B$, $A \wedge B$, and $A \to B$ is the maximum of the complexities of A and B plus 1. Below we will treat $\neg A$ as a special case of implication by defining $\neg A = A \to \bot$.

Lemma 2.2 *For any given k and A, the definition of $k \Vdash^*_\alpha A$ stabilizes after finitely many steps. That is, $k \Vdash^* A$ if and only if $k \Vdash^*_m A$ for some finite m.*

Proof. The use of ordinals in item 2 of Definition 2.1 ensures that the applications of the contraction rule are well-founded, and hence define a well-founded binary tree. But every well-founded binary tree is finite by Königs lemma. □

The definition of $\Vdash^*$ has an impredicative feature, namely the contraction rule (1), that we clarified using the formal Definition 2.1. To shed some more light on it we introduce an auxiliary notion and prove a lemma that will be useful in what follows.[4]

Definition 2.3 Given a partial order M, we inductively define a binary relation $X \parallel k$ between finite subsets $X \subseteq M$ and elements $k \in M$ as follows:

- $\{k\} \parallel k$
- $X \parallel i \ \wedge \ Y \parallel j \ \wedge \ k = i \sqcap j \implies X \cup Y \parallel k.$

Lemma 2.4 *If $k \Vdash^* A \vee B$ then $\exists X \parallel k \forall l \in X \ \big(l \Vdash^* A \vee l \Vdash^* B\big)$.*

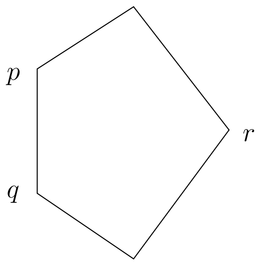
Proof. If $k \Vdash^* A \vee B$ then by Lemma 2.2 we may assume that $k \Vdash^*_m A \vee B$ for some finite m. We prove the lemma by induction on m. If $k \Vdash^*_0 A \vee B$ this is clear by item 4 in Definition 2.1. If $k \Vdash^*_{m+1} A \vee B$ because (item 2) $k = i \sqcap j$, $i \Vdash^*_m A \vee B$, and $j \Vdash^*_m A \vee B$, then by induction hypothesis we have

$$\exists X \parallel i \, \forall l \in X \ \big(l \Vdash^* A \vee l \Vdash^* B\big),$$
$$\exists Y \parallel j \, \forall l \in Y \ \big(l \Vdash^* A \vee l \Vdash^* B\big),$$

hence $X \cup Y \parallel k$ satisfies the induction step. □

After thus having defined Kripke* semantics, we first note that it is sound only for *distributive* structures: Consider the nondistributive lattice N_5, interpreted as a Kripke* model with the atoms p, q and r holding in the nodes as in the following figure:

[4]The notion $X \parallel k$ we introduce here was suggested to us by the referee. Lemma 2.4 was suggested by the referee as an alternative to the definition of $\Vdash^*$. We have opted instead of keeping the original definition and proving the lemma as an auxiliary result. The referees definition is conceptually simpler than ours, but the treatment above shows in addition that our original definition (if interpreted in the right way) amounts to the same.



Then for the distributive law $A = (q \vee p) \wedge (r \vee p) \to (q \wedge r) \vee p$ it holds that in the "q-node" k we have that $k \Vdash A$ but not $k \Vdash^* A$: $k \Vdash^* (q \vee p) \wedge (r \vee p)$ because of the new interpretation of $\vee$, but still $k \nVdash^* (q \wedge r) \vee p$. Since A is of course intuitionistically valid, we see that the Kripke* semantics is in general not sound for IPC. It is so, however, if we restrict it to *distributive* structures:

Definition 2.5 We call a partial order $\langle P, \leq \rangle$ *distributive* if for all $k_0, k_1, l \in P$, if k_0 and k_1 have a greatest lower bound k in P such that $k \leq l$, then there are $l_0 \geq k_0$ and $l_1 \geq k_1$ in P such that l is the greatest lower bound of l_0 and l_1.

One can easily check that for lattices the "butterfly" property from Definition 2.5 is equivalent to distributivity. I.e. a lattice is distributive (satisfies the distributive laws) if and only if it is distributive as a partial order (in the sense of Definition 2.5). But note that the property from Definition 2.5 is more general since it may also hold on structures that are not lattices. Thus we will call a Kripke* model that is distributive as a partial order a *distributive Kripke* model*.

The definition of $\Vdash^*$ has upward persistency built into it (cf. item 2 (iii) in Definition 2.1). The next proposition shows that if we restrict ourselves to distributive structures then this is not really necessary:

Proposition 2.6 (Upward persistency) *Let L be a distributive Kripke* model, and suppose that item 2 (iii) in Definition 2.1 is dropped. Then for any formula A and nodes k and l of L, if $k \Vdash^* A$ and $k \leq l$ then also $l \Vdash^* A$.*

Proof. By Lemma 2.2, if $k \Vdash^* A$ then $k \Vdash^*_m A$ for some m. We prove upward persistency by double induction on m and the complexity of A, following Definition 2.1.

If A is atomic and $k \Vdash^*_0 A$ then the valuation of L makes A true in k. In this case $l \Vdash^*_0 A$ for all $l \geq k$ because by definition valuations are upwards persistent.

Suppose that $k \Vdash^*_{m+1} A$ because of item 2 (ii), i.e. because $k = k_0 \sqcap k_1$ with $k_0 \Vdash^*_m A$ and $k_1 \Vdash^*_m A$, and suppose that $l \geq k$. By distributivity there are $l_0 \geq k_0$ and $l_1 \geq k_1$ such that $l = l_0 \sqcap l_1$. By induction hypothesis we have $l_0 \Vdash^*_m A$ and $l_1 \Vdash^*_m A$, hence $l \Vdash^*_{m+1} A$.

If $k \Vdash^*_0 A$ and $A = B \vee C$, and either $k \Vdash^* B$ or $k \Vdash^* C$ then by the induction hypothesis we are done. If $k = k_0 \sqcap k_1$ such that $k_0 \Vdash^* B$ and $k_1 \Vdash^* C$ then by distributivity there are $l_0 \geq k_0$ and $l_1 \geq k_1$ such that $l = l_0 \sqcap l_1$. By induction hypothesis we have $l_0 \Vdash^* B$ and $l_1 \Vdash^* C$, and hence that $l \Vdash^*_0 B \vee C$.

The cases for the other connectives are all straightforward and do not need distributivity. □

Proposition 2.7 (Meet-persistency) *If $X \parallel k$ and $l \Vdash^* A$ for every $l \in X$ then $k \Vdash^* A$.*

Proof. By definition of $\Vdash^*$ we have that if $k = i \sqcap j$, $i \Vdash^* A$, and $j \Vdash^* A$, then $k \Vdash^* A$. Now if $X \parallel k$, then X is generated by finitely many meets, so by applying the previous statement finitely many times we obtain the statement of the lemma. □

Proposition 2.8 (Soundness) *For all propositional formulas A, $\vdash A$ (i.e. A is derivable in* IPC*) implies that $L \Vdash^* A$ for all distributive Kripke* models L.*

Proof. We prove that if $\Gamma \vdash A$ for a set of formulas Γ and $l \Vdash^* \Gamma$ for some $l \in L$, then $l \Vdash^* A$. We prove this by induction on derivations in the natural deduction presentation of IPC, cf. [16]. Let L be a distributive Kripke* model. Suppose that the last inference rule used in the derivation is the elimination rule $\vee$E:

$$\frac{\Gamma_1 \vdash A \vee B \qquad \Gamma_2, A \vdash C \qquad \Gamma_3, B \vdash C}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash C}$$

The induction hypothesis is that for all $l \in L$, $l \Vdash^* \Gamma_1 \Rightarrow l \Vdash^* A \vee B$, $l \Vdash^* \Gamma_2, A \Rightarrow l \Vdash^* C$, and $l \Vdash^* \Gamma_3, B \Rightarrow l \Vdash^* C$. Suppose that for $k \in L$ we have $k \Vdash^* \Gamma_1, \Gamma_2, \Gamma_3$. We have to prove that $k \Vdash^* C$. By induction hypothesis we have $k \Vdash^* A \vee B$. If $k \Vdash^* A$ or $k \Vdash^* B$ we are done. Otherwise, by Lemma 2.4 we have $\exists X \parallel k \forall l \in X \; \big(l \Vdash^* A \vee l \Vdash^* B\big)$. By upward persistency (Proposition 2.6) every $l \in X$ has $l \Vdash^* \Gamma_2, \Gamma_3$, hence by induction hypothesis $l \Vdash^* C$ for every $l \in X$. Now it follows from meet-persistency (Proposition 2.7) that $k \Vdash^* C$.

The induction steps for the other IPC-rules are all straightforward, and do not need distributivity. □

The following correspondence between Kripke* models and Brouwer algebras will be useful, both for the proof of completeness (Theorem 2.10) and in Section 3. Let $\mathfrak{L}$ be any Brouwer algebra with largest element 1 and let $\mathfrak{K} = \mathfrak{L} - \{1\}$. Then we can interpret $\mathfrak{K}$ as a Kripke* model by stipulating that for every node $P \in \mathfrak{K}$, P (as a propositional atom) holds in every node $Q \in \mathfrak{K}$ with $Q \geq P$. So the elements of $\mathfrak{K}$ play the role of the nodes as well as of the propositional atoms. Let A be a propositional formula, with atoms from $\mathfrak{K} \cup \{\bot\}$. Then we can interpret A as a point of $\mathfrak{L}$ as in Section 1, where $\vee$ is interpreted by $\times$ and $\wedge$ by $+$, and $\bot$ by 1. The reason that 1 is left out of the subalgebras that we consider is to guarantee that no node forces $\bot$. We now have the following simple observation about such formulas A and the forcing relation $\Vdash^*$.

Proposition 2.9 *Let $\mathfrak{K}$ be as above. Then for any formula A and $P \in \mathfrak{K}$, $P \Vdash^* A$ (in the Kripke* model $\mathfrak{K}$) if and only if $P \geq A$ (in the partial order $\mathfrak{K}$). In particular we have that $\mathfrak{K} \Vdash^* A$ if and only if $\mathfrak{K} \models A$.*

Proof. We prove that $P \Vdash^* A \Rightarrow P \geq A$ by induction on $\Vdash^*$, as in Proposition 2.6.

For $Q \in \mathfrak{K}$ atomic we have $P \Vdash^*_0 Q \Leftrightarrow P \geq Q$ by definition of the interpretation of $\mathfrak{K}$ as a Kripke* model. For $A = \bot$ we have $P \nVdash^* \bot$ by definition of $\Vdash^*$, and $P \not\geq 1$ because $1 \notin \mathfrak{K}$.

If $P \Vdash^*_{m+1} A$ because $P = Q \times R$ and $Q \Vdash^*_m A$, $R \Vdash^*_m A$, then by induction hypothesis $Q \geq A$ and $R \geq A$, hence $P = Q \times R \geq A \times B$.

If $P \Vdash^*_0 A \vee B$ because $P \Vdash^* A$ or $P \Vdash^* B$ then we immediately obtain $P \geq A \times B$. If $P \Vdash^*_0 A \vee B$ because $P = Q \times R$ such that $Q \Vdash^* A$ and $R \Vdash^* B$ then we have $Q \geq A$ and $R \geq B$, and hence $P \geq A$.

For $A \to B$ we have

$$
\begin{aligned}
P \Vdash^*_0 A \to B &\iff (\forall Q \geq P)[Q \Vdash^* A \Rightarrow Q \Vdash^* B] \\
&\iff (\forall Q \geq P)[Q \geq A \Rightarrow Q \geq B] \\
&\iff (\forall Q \geq P)[Q + A \geq B] \\
&\iff P + A \geq B \\
&\iff P \geq A \to B.
\end{aligned}
$$

The case $\neg A$ is a special case of the previous case, and the case $A \wedge B$ is straightforward.

We prove the converse $P \geq A \Rightarrow P \Vdash^* A$ by formula induction on A. The atomic case holds by definition of the valuation in $\mathfrak{K}$. If $P \geq A \times B$, then by distributivity of $\mathfrak{L}$ there are $Q \geq A$ and $R \geq B$ such that $P = Q \times R$. Since in the Kripke* model $\mathfrak{K}$ it holds that $Q \Vdash^* A$ and $R \Vdash^* B$, we have $P \Vdash^* A \vee B$ by definition of $\Vdash^*$. If $P \geq A \to B$, then we can reverse all the implications in the first part of the proof to obtain $P \Vdash^* A \to B$. Finally, the case of negation is a special case of the one for implication, and the case of conjunction is again straightforward. □

The next theorem shows that under the restriction to distributive models we still have completeness for IPC, so we see that our variation does not change anything essential. In fact, we can apply an old result of Jaśkowski [3] to show completeness with respect to the smaller class of Kripke* models whose frames are finite Brouwer algebras. Note that in such Kripke* models, by definition of $\Vdash^*$, every atom occurs essentially only once. (Namely an atom P is forced in the meet of all nodes where P is forced, so that modulo upward persistency P occurs only once.) The name "completeness theorem" for the next result is of course justified by the classical completeness theorem of Kripke [6] saying that $\Vdash A$ (A is valid on all Kripke models) if and only if $\vdash A$ (A is provable in IPC).

Theorem 2.10 (Completeness) *For any propositional formula A the following are equivalent:*

(i) $\Vdash A$ *(A is valid on all Kripke models)*

(ii) $L \Vdash^* A$ *for all distributive Kripke* models* L.

(iii) $L \Vdash^* A$ *for all Kripke* models* L *that are also distributive lattices.*

(iv) $L \Vdash^* A$ *for all finite Kripke* models* L *that are also Brouwer algebras.*

Proof. (i)$\Longrightarrow$(ii).[5] This is immediate from soundness for $\Vdash^*$ (Proposition 2.8) and the ordinary completeness theorem for $\Vdash$ just quoted.

(ii)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(iv) are immediate.

(iv)$\Longrightarrow$(i). This follows from Proposition 2.9 together with the result of Jaśkowski [3] that IPC $= \bigcap \big\{\mathrm{Th}(\mathfrak{L}) : \mathfrak{L}$ is a finite Brouwer algebra$\big\}$. (A

[5] In a previous version of this paper it was falsely claimed that $k \Vdash A \Rightarrow k \Vdash^* A$ for every element k of a distributive Kripke* model K, so that every Kripke* countermodel would automatically be a Kripke countermodel. This is of course not true because $\Vdash^*$ sometimes really forces more in a node than $\Vdash$, hence inevitably sometimes also less. Note that the argument via Proposition 2.8 filters through derivations in IPC and does not give a direct insight in the relation between Kripke countermodels and Kripke* countermodels.

proof of this result is in Rose [9].[6]) Namely, if L is a finite Brouwer algebra and $L \Vdash^* A$, then L is valuated with the atoms p occurring in A, and for every such atom p there is a minimal node in the Brouwer algebra L that forces p. So A can be interpreted as a propositional combination of certain elements of L. By Proposition 2.9 we have that $L \models A$ as a Brouwer algebra. □

Although less interesting for our purposes, we note that we also have the equivalence $\Vdash A$ if and only if $L \Vdash A$ for all distributive Kripke models L. That is, we also have completeness of the classical forcing $\Vdash$ with respect to distributive Kripke models. To see this, note that the Lindenbaum-Tarski algebra $\mathfrak{L}_{\mathrm{IPC}}$ of IPC is distributive and that we have $\mathfrak{L}_{\mathrm{IPC}} \Vdash A$ if and only if $\vdash A$.

We noted above that in a semantics for IPC we can restrict ourselves to finite Kripke* models were every propositional atom occurs at most once, modulo upward persistency. (That is, for every atom p, the set of nodes where p holds is a principal cone in the poset.) This will enable us in the next section to obtain the desired correspondence between Kripke* models and sets of Medvedev degrees.

We note that in infinite distributive Kripke* models it is not necessarily the case that every propositional atom occurs at most once. (For a counterexample, consider a frame with an infinite descending chain of nodes without infimum, in each of which p is forced.) However, the property that every atom occurs essentially only once holds for all Kripke* models that are also κ-complete lattices, where κ is the cardinality of the lattice.

3 Kripke semantics and the Medvedev lattice

After the groundwork from the previous section we can now finally give the correspondence between Kripke semantics and configurations in the Medvedev lattice.

Consider any Brouwer subalgebra $\mathfrak{K}$ of $\mathfrak{M}$. Then we can interpret $\mathfrak{K} - \{\mathbf{1}\}$, ordered by the Medvedev reducibility $\leq$, as a Kripke* model as in Section 2. In particular, by Proposition 2.9 we have that for any $\mathbf{P} \in \mathfrak{K}$, $\mathbf{P} \Vdash^* A$ if and only if $\mathbf{P} \geq A$. This holds equally well for any factor $\mathfrak{M}/G$ of the Medvedev degrees, where $\mathbf{1}$ is interpreted by a principal filter G.

[6]Instead of algebras, Jaśkowski and Rose use the more general notion of matrix in the proof of this result. It can be checked that the sequence of matrices J_i in [9] that give IPC is in fact a sequence of Brouwer algebras. The order can be defined by $x \leq y :\equiv x \vee y = x$.

For a Kripke* model K and a subalgebra $\mathfrak{K} \subseteq \mathfrak{M} - \{\mathbf{1}\}$ let us write $K \cong \mathfrak{K}$ if $\mathfrak{K}$ is isomorphic to K as a Kripke* model.

Theorem 3.1 *For every Kripke* model K that is also a finite Brouwer algebra there is a finite set of degrees $\mathfrak{K} \subseteq \mathfrak{M} - \{\mathbf{1}\}$ such that $K \cong \mathfrak{K}$, and vice versa.*

Proof. We have just seen that every $\mathfrak{K} \subseteq \mathfrak{M} - \{\mathbf{1}\}$ can be seen as a Kripke* model. Conversely, if K is a Kripke* model whose frame is a finite Brouwer algebra then by Sorbi [12], K is embeddable (as a Brouwer algebra) into $\mathfrak{M}$ (maybe without preserving 1; If K has an irreducible 1 then the top can be preserved as well). □

Of course, Theorem 3.1 does not make it any easier to find examples of G with $\mathrm{Th}(\mathfrak{M}/G) = \mathrm{IPC}$. It merely points out the relation between Kripke semantics and validity in degree structures such as $\mathfrak{M}$.

Theorem 3.1 also holds for any of the factors $\mathrm{Th}(\mathfrak{M}/G)$ if G is generated by a degree different from $\mathbf{0}$ or $\mathbf{0}'$:

Theorem 3.2 *Let G be a principal filter generated by a degree $\mathbf{D} > \mathbf{0}'$. For every Kripke* model K that is also a finite Brouwer algebra there is a finite set of degrees $\mathfrak{K} \subseteq \mathfrak{M}/G - \{\mathbf{D}\}$ such that $K \cong \mathfrak{K}$, and vice versa.*

Proof. As noted above, every $\mathfrak{K} \subseteq \mathfrak{M}/G - \{\mathbf{D}\}$ can be interpreted as a Kripke* model. Conversely, if K is a Kripke* model whose frame is a finite Brouwer algebra then by the proof of [12, Theorem 2.11], K is embeddable (as a Brouwer algebra, but maybe without preserving $\mathbf{1}$) into $\mathfrak{M}/G$. □

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