Arithmetical Measure

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Abstract. We develop arithmetical measure theory along the lines of Lutz [10]. This yields the same notion of "measure 0 set" as considered before by Martin-Löf, Schnorr, and others. We prove that the class of sets constructible by r.e.-constructors, a direct analogue of the classes Lutz devised his resource bounded measures for in [10], is not equal to RE, the class of r.e. sets, and we locate this class exactly in terms of the common recursion-theoretic reducibilities below K. We note that the class of sets that bounded truth-table reduce to K has r.e.-measure 0, and show that this cannot be improved to "truth-table." For Δ_2-measure the borderline between measure zero and measure nonzero lies between weak truth-table reducibility and Turing reducibility to K. It follows that there exists a Martin-Löf random set that is tt-reducible to K, and that no such set is btt-reducible to K. In fact, by a result of Kautz, a much more general result holds.
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1 Introduction

Restricted versions of the classical theory of Lebesgue measure have been used by a large number of authors for the study of the notion of "random infinite sequence." More recently, Lutz [10] systematically developed the theory of "resource bounded measure." His measure theory allows one to make quantitative assertions about countable classes, in the style of the "almost all" statements referring to Lebesgue measure. In particular, Lutz developed a theory of measure in exponential time computable classes, using polynomial time computable martingales. This measure theory quickly developed into a full-blown research area within structural complexity theory with many interesting results (see [11], [13] for a survey).

Following the approach of Lutz, in Section 3 we define the notion of r.e.-measure 0. An extension of the work of Schnorr quickly shows that this yields the same notion of measure studied before by Martin-Löf [12], Schnorr [15], and others. In his paper [10], Lutz uses the concept of constructor to define the classes that his various measure theories are for. Recursively enumerable constructors give rise to a class R(r.e.) of sets which lies between RE and Δ_2 in the arithmetical hierarchy (we show that both inclusions are proper). The r.e.-measure of RE is 0, hence r.e.-measure cannot be used to make quantitative assertions about RE. We prove that R(r.e.) does not have r.e.-measure 0. Of course, being countable, R(r.e.) is of Lebesgue measure 0. This holds for all the classes that are considered in this paper. From the fact that R(r.e.) does not have measure 0 it follows that the class ${}^{\leq_{tt}}(K)$ of sets that truth-table reduce to the universal r.e. set K does not have r.e.-measure 0. On the other hand, ${}^{\leq_{btt}}(K)$, the class of sets that bounded truth-table reduce to K, does have r.e.-measure 0.

In Section 4 we generalize the notions of Section 3. We thus obtain measures μ_{Σ_n}, μ_{Π_n}, and μ_{Δ_n} for every level of the arithmetical hierarchy. We indicate an exact border between "Δ_n-measure zero" and "Δ_n-measure nonzero" by proving that the class of sets that weakly truth-table reduce to $\emptyset^{(n-1)}$ has Δ_n-measure 0 and that $\Delta_n = {}^{\leq_T}(\emptyset^{(n-1)})$ itself does not have Δ_n-measure 0. We prove that μ_{Π_n} and μ_{Δ_n} are equal and that they are different from μ_{Σ_n}. This at first sight surprising situation is caused by the asymmetry of the supermartingale property.

Finally we make some notes on generalization of the results from Section 3 and on related results. It follows from the results in the previous sections that there is a Martin-Löf random set in ${}^{\leq_{tt}}(K)$, but not in ${}^{\leq_{btt}}(K)$. We have thus obtained an optimal placement of the Martin-Löf random sets in Δ_2 in terms of the well-known reducibilities below K. In fact, using a result by Kautz we have that for any set C there exists a C-n-random set in ${}^{\leq_{tt}}(C^{(n)})$, but not in ${}^{\leq_{btt}}(C^{(n)})$.

2 Preliminaries

We assume that the reader is familiar with the basic notions of computability theory. Our basic recursion-theoretic notation is as in Soare [16]. So ω is the set of natural numbers, 2^ω is Cantor space, the power set of ω, and $2^{<\omega}$ is the set of finite strings of zero's and one's, viewed as the set of initial segments of characteristic strings of sets in 2^ω. We fix a recursive pairing function $\langle .\,, . \rangle : \omega \times \omega \to \omega$ and denote the set $\{\langle x, n\rangle : x \in \omega\}$ by $\omega^{[n]}$ (the n-th "row" of ω). The empty set is denoted by $\emptyset$, and the empty string in $2^{<\omega}$ is denoted by ϵ. For $A \in 2^\omega$, $\overline{A}$ denotes the complement, $\omega - A$, of A. Sets are identified with their characteristic strings, and $A\restriction n$ denotes the initial segment of length n of A. For elements $v, w \in 2^{<\omega}$, vw denotes the concatenation of v and w, and $v \sqsubset w$, $v \sqsubseteq w$ denotes that v is a (proper) prefix of w. Subsets of 2^ω are called *classes*, and for a class $\mathcal{A}$, co$-\mathcal{A} = \{\overline{A} : A \in \mathcal{A}\}$. K is the universal r.e. set. The jump of a set C is denoted by C', and the n-th iterated jump of C by $C^{(n)}$. The e-th partial recursive function is φ_e, and $\varphi_{e,s}(x)$ is the result of running φ_e on x using less than s computation steps. We denote by $\varphi_e(x)\downarrow$ ($\varphi_e(x)\uparrow$) that $\varphi_e(x)$ is defined (undefined). For a function $f : \omega \to \omega$, $f\restriction n$ denotes the string $f(0)f(1)\dots f(n-1)$, and if f is partial then $(f\restriction n)\downarrow$ denotes that this string is defined. The sets of nonzero rationals and reals are denoted by $\mathcal{Q}^+$ and $\mathcal{R}^+$, respectively. We will use the standard recursion-theoretic reducibilities $\leq_m$, $\leq_{btt}$, $\leq_{tt}$, $\leq_{wtt}$, and $\leq_T$. We will make use of the following characterization of $\leq_{tt}$.

Lemma 2.1 (Carstens [3]) *For every $A \in 2^\omega$, $A \leq_{tt} K$ if and only if there exist recursive functions g and h such that for every $x \in \omega$, $\lim_s g(s,x) = A(x)$ with $|\{s : g(s,x) \neq g(s+1,x)\}| \leq h(x)$.*

For $r \in \{m, btt, tt, wtt, T\}$, ${}^{\leq_r}(A)$ denotes the class of sets that are r-reducible to the set A.

Throughout this paper, the standard notion of a recursively enumerable function will play an important role:

Definition 2.2 *Let A be a countable set recursively isomorphic to ω. Let B be a countable set totally ordered by $\leq_B$ such that $(B, \leq_B)$ is recursively isomorphic to $(\omega, \leq)$. A function $f : A \to B$ is* recursively enumerable (r.e.) *if the set $\{(x,y) : y \leq_B f(x)\}$ is an r.e. set. Similarly, f is* co-r.e. *or* Π_1 *if $\{(x,y) : f(x) \leq_B y\}$ is r.e. The class of (total) r.e.-functions is denoted by r.e.*

Note that a function f is r.e. if and only if there exists a recursive approximation f_s such that for all x, $f_s(x) \leq f_{s+1}(x)$ and $(\exists s)(\forall t \geq s)[f_t(x) = f(x)]$. We will often make use of this last fact, namely that the recursive approximation f_s actually attains its limit (see Section 3 for further discussion on this point). Clearly a function which is both r.e. and co-r.e. is recursive. The class of (total) recursive functions is denoted by rec.

Σ_n, Π_n, and Δ_n denote the classes from Kleene's arithmetical hierarchy. They are also used to denote the corresponding function classes, defined in

Section 4. We also write REC for Δ_1 and RE for Σ_1, the classes of recursive and recursively enumerable sets, respectively. For a measurable class $\mathcal{A}$ the Lebesgue measure of $\mathcal{A}$ is denoted by $\lambda(\mathcal{A})$.

Definition 2.3 *A* supermartingale *d is a function from $2^{<\omega}$ to $\mathcal{R}^+$ with the property*

$$d(w0) + d(w1) \leq 2d(w)$$

for every $w \in 2^{<\omega}$.

Functions with this property are called *super*martingales, as opposed to martingales with the property $d(w0)+d(w1) = 2d(w)$. Lutz [10, p239] remarks that for the classes that he considers it makes no difference whether one uses supermartingales or martingales. In our setting it will make a huge difference. We will consider supermartingales which are (approximable by) r.e.-functions, and one can easily check that r.e.-martingales with the property $d(w0)+d(w1) = 2d(w)$ are always recursive. The difference then is clear from Corollary 3.8.

A supermartingale d *succeeds* on a set A if $\limsup_n d(A\restriction n) = \infty$. The class of all sets on which d succeeds is denoted by $S[d]$. A supermartingale succeeds on a class if and only if it succeeds on every member of it. One can prove that a class has Lebesgue measure 0 if and only if there is a (super)martingale that succeeds on it. By imposing restrictions on the complexity of the martingales Lutz obtained his resource bounded generalization of the classical theory of Lebesgue measure.

Since supermartingales are real-valued functions, we need a notion of computability for them.

Definition 2.4 *A supermartingale d is an* r.e.-supermartingale *if there is an r.e.-function $\hat{d} : \omega \times 2^{<\omega} \to \mathcal{Q}^+$ such that*

$$(\forall k \in \omega)(\forall w \in 2^{<\omega})[|d(w) - \hat{d}(k, w)| \leq 2^{-k}].$$

The function $\hat{d}$ is called an r.e.-computation *of d.*

Now a class $\mathcal{A}$ has *r.e.-measure 0*, denoted $\mu_{\text{r.e.}}(\mathcal{A}) = 0$, if there exists an r.e.-supermartingale that succeeds on $\mathcal{A}$. Similarly, $\mathcal{A}$ has *rec-measure 0* ($\mu_{\text{rec}}(\mathcal{A}) = 0$) if there is a rec-martingale (i.e. a martingale with a *recursive* computation) that succeeds on $\mathcal{A}$. $\mathcal{A}$ has *r.e.-(rec-)measure 1* if $\mathcal{A}^c = \{X : X \notin \mathcal{A}\}$ has r.e.-(rec-)measure 0.

We will also use the notion of constructor. A *constructor* is a function $\delta : 2^{<\omega} \to 2^{<\omega}$ with the property that $\delta(x) \sqsupset x$ for every $x \in 2^{<\omega}$. The *set constructed by δ* is the unique set $R(\delta)$ with $\delta^n(\epsilon) \sqsubset R(\delta)$, where δ^n denotes the n-th iterate of δ.

Definition 2.5 *For a class of functions* Δ, $R(\Delta)$ *denotes the class*

$$\{R(\delta) :\ \delta \text{ is a constructor and } \delta \in \Delta\}.$$

3 R.e.-measure

We will use the following fact, which is a version of a folklore fact obtained by many people independently (e.g. [1, Lemma 2.1], [13]).

Lemma 3.1 *Let* d *be an r.e.-supermartingale. Then there is a supermartingale* $\tilde{d} : 2^{<\omega} \to \mathcal{Q}^+$ *which is r.e. such that* $S[\tilde{d}] \supseteq S[d]$.

Proof. Let $f : \omega \times 2^{<\omega} \to \mathcal{Q}^+$ be an r.e.-computation of d:

$$(\forall k \in \omega)(\forall w \in 2^{<\omega})[|d(w) - f_k(w)| \leq 2^{-k}].$$

Define $\tilde{d}(w) = f_{|w|}(w) + 4 \cdot 2^{-|w|}$. Then $\tilde{d}(w) \geq d(w) + 3 \cdot 2^{-|w|}$ and $\tilde{d}(w) \leq d(w) + 5 \cdot 2^{-|w|}$. Furthermore,

$$\begin{aligned}
\tilde{d}(w0) + \tilde{d}(w1) &\leq d(w0) + 5 \cdot 2^{-|w|-1} + d(w1) + 5 \cdot 2^{-|w|-1} \\
&\leq 2(d(w) + 5/2 \cdot 2^{-|w|}) \\
&\leq 2(d(w) + 3 \cdot 2^{-|w|}) \\
&\leq 2\tilde{d}(w),
\end{aligned}$$

so $\tilde{d}$ is a supermartingale, and $\tilde{d}$ is r.e. because f is. Finally, $S[\tilde{d}] \supseteq S[d]$ since $\tilde{d}(w) \geq d(w)$. □

Having defined r.e.-measure using the approach of Lutz [10] we now prove that this yields the same notion of measure as considered before by Martin-Löf, Schnorr, and others. The proof is a simple extension of the work of Schnorr [15].

Definition 3.2 (Martin-Löf [12], Kautz [5]) *A class* $\mathcal{A}$ *of Lebesgue measure 0 is* Σ_n^C-approximable *if there is a recursive sequence of* Σ_n^C*-classes* $\{\mathcal{S}_i\}_{i\in\omega}$ *with* $\lambda(\mathcal{S}_i) \leq 2^{-i}$ *and* $\mathcal{A} \subseteq \bigcap_i \mathcal{S}_i$. *A set* A *is* n-random *if* $\{A\}$ *is not* Σ_n*-approximable. The 1-random sets are also called Martin-Löf random.*

Schnorr [15, Satz 5.3] has shown that class $\mathcal{A}$ is Σ_1-approximable if and only if there is a *subcomputable* ("subberechenbare") martingale that succeeds on $\mathcal{A}$. Here a martingale g is subcomputable if it has a recursive approximation g_s satisfying $g_s(w) \leq g_{s+1}(w)$ for every $s \in \omega$ and $w \in 2^{<\omega}$, and such that $\lim_s g_s(w) = g(w)$ (note that in this definition it is not required that there is an $s \in \omega$ such that $g_s(w) = g(w)$!). Using Lemma 3.1 it is easy to see that a class

$\mathcal{A}$ has r.e.-measure 0 if an only if there is a subcomputable martingale that succeeds on $\mathcal{A}$ with the additional requirement that the recursive approximation g_s of g reaches its limit So clearly every class of r.e.-measure 0 has subcomputable measure 0. But the converse holds too: Given a subcomputable martingale g with recursive approximation g_s, we define an r.e.-supermartingale d with $S[d] \supseteq S[g]$ as follows. Define d through a recursive approximation d_s: For every $w \in 2^{<\omega}$, $d_0(w) = 0$, and if $g_{s+1}(w) > d_s(w) - 2^{-|w|-1}$ then put $d_{s+1}(w) = g_{s+1}(w) + 2^{-|w|}$, and put $d_{s+1}(w) = d_s(w)$ otherwise. It is immediate from the definition of d_s that

$$(\forall w \in 2^{<\omega})(\exists s)(\forall t \geq s)[d_t(w) = d_s(w)],$$

hence $d_s(w)$ reaches a limit $d(w)$ after a finite number of steps. It holds for every $w \in 2^{<\omega}$ that $g(w) + 2^{-|w|-1} \leq d(w) \leq g(w) + 2^{-|w|}$, hence it follows from the martingale property of g that

$$\begin{aligned} d(w0) + d(w1) &\leq g(w0) + g(w1) + 2 \cdot 2^{-(|w|+1)} \\ &\leq 2(g(w) + 2^{-|w|-1}) \\ &\leq 2d(w), \end{aligned}$$

whence d is indeed an r.e.-supermartingale. So we have arrived at

Theorem 3.3 *For every class $\mathcal{A} \subseteq 2^\omega$, $\mathcal{A}$ is Σ_1-approximable if and only if $\mu_{\text{r.e.}}(\mathcal{A}) = 0$.*

Corollary 3.4 *A class $\mathcal{A}$ has r.e.-measure 0 if and only if $\mathcal{A}$ does not contain a Martin-Löf random set.*

Proof. From the existence of a universal Martin-Löf-test (Martin-Löf [12]) it follows that

$$\mu_{\text{r.e.}}(\{A \in 2^\omega : A \text{ is Martin-Löf random}\}) = 1.$$

Note that this is stronger than merely saying that the class of Martin-Löf random sets has Lebesgue measure 1 (Schnorr [15, Korollar 4.7]). Hence if $\mathcal{A}$ contains no Martin-Löf random set then $\mu_{\text{r.e.}}(\mathcal{A}) = 0$. The converse is true by the definition of Martin-Löf random set. □

In particular, we can use all the known facts about Martin-Löf randomness in the study of r.e.-measure.

Recall the definition of $R(\Delta)$ from Definition 2.5. Lutz [10] observed that

$$R(\text{rec}) = \text{REC}$$

$$\mu_{\text{rec}}(\text{REC}) \neq 0.$$

R(r.e.) is the set of all $R(\delta)$ for δ r.e., the results of constructors $\delta : 2^{<\omega} \to 2^{<\omega}$ for which the set $\{(x,y) : y \leq \delta(x)\}$ is r.e., where $\leq$ denotes the usual lexicographic ordering on $2^{<\omega}$.

Definition 3.5 *We say that $A \in 2^\omega$ is* right-limit *of the infinite set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \sigma \in X)[\sigma \leq A \restriction |\sigma|]$ and $(\forall n)(\exists m \geq n)[A \restriction m \in X]$.*
We say that $\sigma \in 2^{<\omega}$ is right-limit *of the (possibly infinite) set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \tau \in X)[\tau \sqsubseteq \sigma \vee \exists n(\tau \restriction n < \sigma \restriction n)]$.*

It will be useful to have the following characterization. R(r.e.) is the class of sets that are right-limits of r.e. sets of initial segments in $2^{<\omega}$. Equivalently, $A \in R$(r.e.) iff there is a recursive function $\phi : \omega \times \omega \to \{0,1\}$ such that $\lim_k \phi(k,n) = A(n)$, and for every $k, n \in \omega$, $\phi(k) \restriction n \leq \phi(k+1) \restriction n$ and

$$(\forall m < n)[\phi(k,m) = A(m)] \wedge \phi(k,n) = 1 \Rightarrow A(n) = 1 \tag{1}$$

(if A is the right-limit of an r.e. set $X \subseteq 2^{<\omega}$, with recursive enumeration $\{X_s\}_{s \in \omega}$, we get ϕ as in (1) by putting $\phi(k,n) = \tau_k(n)$, where τ_k is the right-limit of X_s).

From this characterization it follows immediately that RE $\subseteq R$(r.e.) $\subseteq \Delta_2$ (the second inclusion follows from the Limit Lemma [16, III.3.3]). In the next theorem we locate the class R(r.e.) more precisely. We denote by DRE the class of differences $W_e - W_d$ of r.e. sets (the *d.r.e.* sets, see [16, p57]). This is precisely the class of sets with a recursive approximation that changes at most two times for every argument (starting with 0).

Theorem 3.6 *The inclusions* RE $\subseteq R$(r.e.) $\subseteq {}^{\leq_{tt}}(K)$ *are both proper. Furthermore,* R(r.e.) $\not\subseteq {}^{\leq_{btt}}(K)$ *and* DRE $\not\subseteq R$(r.e.).

DRE $\subsetneq$ ${}^{\leq_{btt}}(K)$

$\subsetneq$ $\subsetneq$

RE ${}^{\leq_{tt}}(K)$

$\subsetneq$ $\subsetneq$

R(r.e.)

Proof. Note that indeed we have the inclusion R(r.e.) $\subseteq {}^{\leq_{tt}}(K)$ by the characterization in Lemma 2.1, where we can take $h(x) = 2^x$. That the two inclusions RE $\subseteq R$(r.e.) $\subseteq {}^{\leq_{tt}}(K)$ are both proper will follow from R(r.e.) $\not\subseteq {}^{\leq_{btt}}(K)$ and DRE $\not\subseteq R$(r.e.), since RE $\subsetneq$ DRE $\subsetneq {}^{\leq_{btt}}(K) \subsetneq {}^{\leq_{tt}}(K)$.

One can prove $R(\text{r.e.}) \not\subseteq {\leq_{btt}}(K)$ directly by a diagonalization construction, but the result will also follow from Theorem 3.7 (iii). So we do not give a proof here.

That DRE $\not\subseteq R(\text{r.e.})$ can be proved directly using a finite injury argument, but, as pointed out to us by an anonymous referee, it also follows from results in [16]: There exist d.r.e. sets that do not have r.e. degree [16, VII.2.4] (this also requires a finite injury priority argument), but on the other hand every set in $R(\text{r.e.})$ *does* have r.e. degree. This last fact follows from the Limit Lemma, see [16, Corollary 3.4]. Hence there exist sets in DRE that are not in $R(\text{r.e.})$. □

One may now wonder what the r.e.-measure is of classes such as REC, RE, and $R(\text{r.e.})$. One can easily prove that $\mu_{\text{rec}}(\text{REC}) \neq 0$ by constructing for each recursive martingale d a recursive set A such that d does not succeed on A: Given $A{\restriction}n$ define $A(n) = 1$ iff $d((A{\restriction}n)1) \leq d((A{\restriction}n)0)$. However, this argument fails if d is an r.e.-supermartingale since the set A constructed as above will only be Δ_2. The following theorem shows that indeed the analogous result is not true at all.

Let $\mathcal{B}$ be the smallest Boolean algebra containing all the r.e. sets, i.e. $\mathcal{B}$ is the closure of the class of r.e. sets under complementation, union, and intersection ($\mathcal{B}$ is Ershov's *Boolean*, or *difference hierarchy*, see Odifreddi [14]). $\mathcal{B}$ is exactly the "cone" of sets that are bounded truth-table reducible to K ([14, Prop. III.8.7]).

Theorem 3.7 (i) $\mu_{\text{r.e.}}(\text{RE}) = 0$.
(ii) $\mu_{\text{r.e.}}({\leq_{btt}}(K)) = 0$.
(iii) $\mu_{\text{r.e.}}(R(\text{r.e.})) \neq 0$.

Proof. (i) follows from the well-known fact that no Martin-Löf random set is r.e. (In fact, every Martin-Löf random set is bi-immune [6].) (ii) follows from (i) and the fact that for every set A in $\mathcal{B}$ either A or $\overline{A}$ contains an infinite r.e. set (Jockusch and others [16, III.3.10]).
(iii) Let d be any r.e.-supermartingale. We have to show that there is an element of $R(\text{r.e.})$ on which d does not succeed. Let A be the leftmost path in 2^ω such that $d(A{\restriction}n) \leq 1$ for every n. Note that A exists since for any (super)martingale d with $d(\epsilon) = 1$, $\{B \subseteq \omega : \forall n(d(B{\restriction}n) \leq 1\}$ is nonempty, in fact, has positive Lebesgue measure. It is easy to see, using that d is r.e., that A is the right-limit of an r.e. set of initial segments, and hence that $A \in R(\text{r.e.})$. □

Corollary 3.8 *The measures* μ_{rec} *and* $\mu_{\text{r.e.}}$ *are unequal.*

Proof. $\mu_{\text{r.e.}}(\text{REC}) = 0$ by Theorem 3.7 (i), but Lutz proved that $\mu_{\text{rec}}(\text{REC}) \neq 0$, cf. the discussion preceding Theorem 3.7. □

Theorem 3.7 shows that r.e.-measure is not suited for the quantitative study of RE, hence Lutz's approach [10], that worked for classes like 2^ω, REC, and the linear and polynomial deterministic exponential time and space classes, does not work here.

It follows from Theorem 3.7 that if A is r.e. then $\mu_{\text{r.e.}}(\{A\}) = 0$. The converse is certainly not true: it is easy to construct a martingale which succeeds on all nondense sets (i.e. sets with a characteristic string that contains significantly more zero's than one's), and among those are sets of arbitrary high complexity.

Corollary 3.8 shows that the approach using martingales d with the property $d(w0) + d(w1) = 2d(w)$ instead of our supermartingales with the weaker property $d(w0) + d(w1) \leq 2d(w)$ does indeed make a difference (cf. the discussion after Definition 2.3).

It follows from Theorem 4.4 that $\mu_{\text{r.e.}}(\Delta_2) \neq 0$. Δ_2 is exactly the "cone" of sets that are Turing reducible to K, the halting set. In our notation: $\Delta_2 = {}^{\leq_T}(K)$. Results on the measure of cones form a classical topic in the intersection of measure theory and computability theory. As a corollary to Theorem 3.7 we have a stronger result than $\mu_{\text{r.e.}}({}^{\leq_T}(K)) \neq 0$, saying that for truth-table reducibility the cone ${}^{\leq_{tt}}(K)$ does not have r.e.-measure 0.

Corollary 3.9 $\mu_{\text{r.e.}}({}^{\leq_{tt}}(K)) \neq 0$.

Proof. In Theorem 3.6 we saw that $R(\text{r.e.}) \subset {}^{\leq_{tt}}(K)$, so the result follows from Theorem 3.7 (iii). □

Theorem 3.7 (ii), in conjunction with Corollary 3.9, gives a precise border between r.e.-measure zero and r.e.-measure nonzero in terms of the common recursion-theoretic reducibilities below K.

4 Arithmetical measure

The results from Section 3 generalize to all the levels of the arithmetical hierarchy. We have the following function classes corresponding to the various levels of the arithmetical hierarchy.

Definition 4.1 *The class of (total) Σ_n-functions, $n \geq 1$, is defined to be*

$$\{f : 2^{<\omega} \to \mathcal{Q}^+ : \{(x,y) : f(x) \geq y\} \text{ is } \Sigma_n\}.$$

Similarly, the class of (total) Π_n-functions, $n \geq 1$, consists of $\{f : 2^{<\omega} \to \mathcal{Q}^+ : \{(x,y) : f(x) \leq y\}$ is $\Sigma_n\}$. The function classes Δ_n are defined as $\Delta_n = \Sigma_n \cap \Pi_n$.

Note that the Σ_n-functions are those that are Δ_n-approximable from below, by a Δ_n-function that attains its limit value. Similarly Π_n-functions can be approximated from above in the same manner. Therefore, the Δ_n-functions coincide with the functions computable recursively in $\emptyset^{(n)}$.

The measures μ_{Σ_n}, μ_{Π_n}, and μ_{Δ_n}, with $n \geq 1$, are defined exactly as the measures $\mu_{\text{r.e.}}$ and μ_{rec}. So, for example, $\mu_{\Pi_n}(\mathcal{A}) = 0$ if there is no supermartingale with a computation in Π_n that succeeds on $\mathcal{A}$. As in the case of rec-measure, in the definition of μ_{Δ_n} we may use martingales instead of supermartingales

Now Lemma 3.1 is proved exactly as before, and the proof of Theorem 3.7 relativizes.

Theorem 4.2 *For all $n \geq 1$, $\mu_{\Sigma_n}({}^{\leq_{btt}}(\emptyset^{(n)})) = 0$.*

Proof. Relativize the proof of Theorem 3.7 (ii) to the oracle $\emptyset^{(n)}$. □

Corollary 4.3 *For all $n \geq 1$, $\mu_{\Sigma_n} \neq \mu_{\Delta_n}$.*

In the next theorem we find an exact border between Δ_2-measure zero and Δ_2-measure nonzero in terms of the reducibilities $\leq_{wtt}$ and $\leq_T$ below K.

Theorem 4.4 *For every $n \geq 2$, $\mu_{\Delta_n}({}^{\leq_{wtt}}(\emptyset^{(n-1)})) = 0$, and for every $n \geq 1$, $\mu_{\Delta_n}(\Delta_n) \neq 0$.*

Proof. For the first part, we prove that $\mu_{\Delta_2}({}^{\leq_{wtt}}(K)) = 0$, and note that the proof relativizes. By [14, Ex. III.8.14], ${}^{\leq_{wtt}}(K)$ equals ${}^{\leq_{tt}}(K)$, the class of sets that truth-table reduce to K. Whence it suffices to prove that $\mu_{\Delta_2}({}^{\leq_{tt}}(K)) = 0$. Given codes e and d for the recursive functions φ_e, φ_d, we assume that these represent a tt-reduction of a set A to K as in Lemma 2.1. We construct a Δ_2-martingale $d_{\langle e,d\rangle}$ that succeeds on A if this guess is correct. Put $d_{\langle e,d\rangle}(\epsilon) = 1$. Given $d_{\langle e,d\rangle}(w)$, use the oracle $\emptyset'$ to compute whether $\varphi_d(|w|)\downarrow$ and $\varphi_e(s, |w|)$ changes at most $\varphi_d(|w|)$ times. Note that in this case we can compute with $\emptyset'$ the "limit" $c \in \omega$, i.e. we can $\emptyset'$-compute s and c in ω such that for all $t \geq s$ either $\varphi_e(t, |w|) \downarrow= c$ or $\varphi_e(t, |w|) \uparrow$. If this limit exists and is $i \in \{0, 1\}$, put $d_{\langle e,d\rangle}(wi) = 2 \cdot d_{\langle e,d\rangle}(w)$ and $d_{\langle e,d\rangle}(w(1-i)) = 0$. Otherwise set $d_{\langle e,d\rangle}(w0) = d_{\langle e,d\rangle}(w1) = 0$. It is easy to check that if the pair φ_e, φ_d forms a tt-reduction, then $d_{\langle e,d\rangle}$ succeeds on the set that is reduced. The definition of $d_{\langle e,d\rangle}$ is uniform in e and d, so ${}^{\leq_{tt}}(K)$ is a Δ_2-union of Δ_2-measure 0 classes, hence has Δ_2-measure 0.

The second part, $\mu_{\Delta_n}(\Delta_n) \neq 0$, is proved exactly as $\mu_{\text{rec}}(\text{REC}) \neq 0$ (see the discussion preceding Theorem 3.7). □

Note that analogous to Theorem 3.6 we have that $\Sigma_n \subsetneq R(\Sigma_n) \subsetneq \Delta_{n+1}$ and $\Pi_n \subsetneq R(\Pi_n) \subsetneq \Delta_{n+1}$. From Theorem 4.4 it follows that for every $n \geq 1$, $\mu_{\Delta_{n+1}}(R(\Sigma_n)) = 0$ and $\mu_{\Delta_{n+1}}(R(\Pi_n)) = 0$. In contrast to this result we have the generalized version of Theorem 3.7 (iii) (the proof of the second part is completely symmetric):

Theorem 4.5 *For all* $n \geq 1$, $\mu_{\Sigma_n}(R(\Sigma_n)) \neq 0$ *and* $\mu_{\Sigma_n}(R(\Pi_n)) \neq 0$.

To complete the picture of inclusions, note that $\Delta_n = R(\Delta_n) = R(\Sigma_n) \cap R(\Pi_n)$. It follows that $\Sigma_n \not\subseteq R(\Pi_n)$ since otherwise $\Sigma_n \subseteq R(\Sigma_n) \cap R(\Pi_n) = \Delta_n$, a contradiction. In particular $R(\Sigma_n) \not\subseteq R(\Pi_n)$. So the only inclusion relations are $\Delta_n \subsetneq \Sigma_n \subsetneq R(\Sigma_n) \subsetneq \Delta_{n+1}$, $\Delta_n \subsetneq \Pi_n \subsetneq R(\Pi_n) \subsetneq \Delta_{n+1}$, and no other inclusions hold.

In Corollary 4.3 we have seen that the measure induced by the function class Σ_n differs from the measure induced by Δ_n. We now prove that, surprisingly, the measure induced by Π_n equals the latter. Whence the measure μ_{Σ_n} is stronger (more sets have measure 0) than the measure μ_{Π_n}. The reason for this asymmetry lies in the asymmetry of the supermartingale property $d(w0) + d(w1) \leq 2d(w)$, which makes Σ_n-supermartingales more powerful than Π_n-supermartingales.

Theorem 4.6 *For all* $n \geq 1$, $\mu_{\Pi_n} = \mu_{\Delta_n}$.

Proof. We give the proof for $n = 1$. The proof for arbitrary $n \in \omega$ is obtained by relativizing the following proof to the oracle $\emptyset^{(n)}$. It suffices to prove that if a class has Π_1-measure 0 then it has rec-measure 0. Let d be a Π_1-supermartingale, with nonincreasing recursive approximation d_s say. We prove that there exists a rec-supermartingale d' with $S[d'] \supseteq S[d]$. W.l.o.g. $d(\epsilon) = 1$. Define $d'(\epsilon) = 1$. Suppose now that $d'(w)$ has been defined and that $d'(w) \geq d(w)$. Choose the least $s \in \omega$ such that $d_s(w0) + d_s(w1) \leq 2d_s(w)$. Note that s exists since d is a martingale and $(\exists s)(\forall t \geq s)[d_t(w) = d(w)]$. Define $d'(wi) = d_s(wi)$, for $i \in \{0, 1\}$. Then $d'(wi) \geq d(wi)$ because $d_s(wi)$ is nonincreasing in s. For d' thus defined we have that $d'(w0) + d'(w1) \leq 2d_s(w) \leq 2d'(w)$, so d' is a supermartingale. Clearly d' is recursive, and $S[d'] \supseteq S[d]$ because for every $w \in 2^{<\omega}$, $d'(w) \geq d(w)$. □

5 Notes

Kautz [5, p26] has shown that a class $\mathcal{A}$ is Σ_n^C-approximable if and only if there is a $C^{(n-1)}$-recursive sequence (rather than just recursive) of Σ_n^C-classes $\{\mathcal{S}_i\}_{i \in \omega}$

with $\lambda(\mathcal{S}_i) \leq 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{S}_i$. It follows that we may relativize the result of Theorem 3.3 to the oracle $\emptyset^{(n-1)}$ to obtain

Theorem 5.1 *For every class $\mathcal{A} \subseteq 2^\omega$ and every $n \geq 1$, $\mathcal{A}$ is Σ_n-approximable if and only if $\mu_{\Sigma_n}(\mathcal{A}) = 0$.*

Schnorr [15, Satz 7.6] proves that there is a 1-random (Martin-Löf random) set A in Δ_2. A natural example of such a set is Chaitins Halting Probability Ω [9, p187]. For more on Martin-Löf random sets in Δ_2 see M. van Lambalgen [8]. The results from the previous sections show

Theorem 5.2 (i) *There is a Martin-Löf random set in $^{\leq_{tt}}(K)$.*
(ii) *There is no Martin-Löf random set in $^{\leq_{btt}}(K)$.*

Proof. Immediate from Corollary 3.4, Corollary 3.9 and Theorem 3.7 (ii). □
With a similar proof one can actually show, using Theorem 5.1, that for any $C \in 2^\omega$, there exists a C-n-random set in $^{\leq_{tt}}(C^{(n)})$ (even in $R(\Sigma_n^C)$) but not in $^{\leq_{btt}}(C^{(n)})$.

It is known that the Martin-Löf random set of Theorem 5.2 (i) cannot have the same tt-degree as K (Bennett [2], Juedes, Lathrop, and Lutz [4]).

Note that there is not an analogue of the result $\mu_{\text{r.e.}}(\{A \in 2^\omega : A \text{ is Martin-Löf random}\}) = 1$ for the case of Δ_n-measure. In fact, it is easy to see that $\mu_{\Delta_n}(\{A \in 2^\omega : A \text{ is } \Delta_n\text{-random}\}) \neq 1$. Namely, for every Δ_n-martingale d there exists $A \in \Delta_n$ such that $A \notin S[d]$. Hence d does not succeed on all the non-Δ_n-random sets. However, the class of Δ_n-random sets does of course have Lebesgue measure 1.

It follows from results of Arslanov and Kučera [6] that if A is an r.e. set that is Turing-incomplete then $\mu_{\text{r.e.}}(^{\leq_T}(A)) = 0$.
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