PARTIAL COMBINATORY ALGEBRA AND GENERALIZED NUMBERINGS

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ABSTRACT. Generalized numberings are an extension of Ershov's notion of numbering, based on partial combinatory algebra (pca) instead of the natural numbers. We study various algebraic properties of generalized numberings, relating properties of the numbering to properties of the pca. As in the lambda calculus, extensionality is a key notion here.

1. Introduction

A numbering is a surjective mapping $\gamma : \omega \to S$ from the natural numbers ω to a set S. The theory of numberings was started by Ershov in a series of papers, beginning with [8] and [9]. Ershov studied the computability-theoretic properties of numberings, as generalizations of numberings of the partial computable functions. In particular, he called a numbering *precomplete* if for every partial computable unary function ψ there exists a computable unary function f such that for every n

$$\psi(n)\downarrow \Longrightarrow \gamma(f(n)) = \gamma(\psi(n)). \tag{1}$$

Following Visser, in this case we say that f *totalizes* ψ *modulo* γ. Ershov showed that Kleene's recursion theorem holds for arbitrary precomplete numberings. Visser [29] extended this to his so-called "anti diagonal normalization theorem" (ADN theorem). Another generalization of the recursion theorem is the famous Arslanov completeness criterion [2], that extends the recursion theorem from computable functions to all functions bounded by an incomplete computably enumerable (c.e.) Turing degree. Selivanov [26] showed that Arslanov's result also holds for any precomplete numbering. In Terwijn [28] a joint generalization of Arslanov's completeness criterion and Visser's ADN theorem was proved. It is currently open whether this joint generalization also holds for every precomplete numbering.

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Classic examples are numberings of the partial computable (p.c.) functions. Such a numbering is *acceptable* if it can be effectively translated back and forth into some standard numbering of the p.c. functions φ_e. Rogers [24] proved that a numbering is acceptable if and only if it satisfies both the enumeration theorem and parametrization (also known as the S-m-n–theorem). It follows from this that every acceptable numbering is precomplete. On the other hand, Friedberg [12] showed that there exists an effective numbering of the p.c. functions without repetitions. Friedberg's 1-1 numbering is not precomplete, as can be seen as follows. Suppose that $\gamma : \omega \to \mathcal{P}$ is a 1-1 numbering of the (unary) p.c. functions that is precomplete. By (1), we then have for every p.c. function ψ a computable function f such that

$$\psi(n)\downarrow \Longrightarrow \gamma(f(n)) = \gamma(\psi(n)) \implies f(n) = \psi(n).$$

The second implication follows because γ is 1-1. So we see that in fact f is a total extension of ψ. But it is well-known that there exist p.c. functions that do not have total computable extensions. So we see that 1-1 numberings of the p.c. functions are never precomplete. For more about 1-1 numberings see Kummer [15].

A topic closely related to numberings is that of computably enumerable equivalence relations (ceers). For every numbering γ we have the associated equivalence relation defined by $n \sim_\gamma m$ if $\gamma(n) = \gamma(m)$. Conversely, for every countable equivalence relation, we have the numbering of its equivalence classes. So the study of numberings is essentially equivalent to the study of countable equivalence relations, a topic with an extended literature that has also been studied extensively in descriptive set theory. In particular we see that the above terminology about numberings also applies to ceers, and all the concepts in this paper can also be studied from that perspective. Lachlan [16], following work of Bernardi and Sorbi [6], proved that all precomplete ceers are computably isomorphic. For a recent survey about ceers, see Andrews, Badaev, and Sorbi [1], in which the reader can find a long list of references about this topic, including those of other complexities.

In the examples of numberings given above, the set ω is not merely a set used to number the elements of a set S, but it carries extra structure as the domain of the partial computable functions, making it into a so-called partial combinatory algebra (pca). Combinatory completeness is the characteristic property that makes a structure with an application operator a pca. This is the analog of the S-m-n–theorem (parametrization) for the p.c. functions. In section 2 below we review the basic definitions of pca.

We can extend the notion of numbering from ω to arbitrary pcas as follows. A *generalized numbering* is a surjective mapping $\gamma\colon \mathcal{A} \to S$, where $\mathcal{A}$ is a pca and S is a set. This notion was introduced in Barendregt and Terwijn [5]. We also have a notion of precompleteness for generalized numberings, analogous to Ershov's notion (Definition 3.1). It was shown in [5] that the fixed point theorem also holds for precomplete generalized numberings. Precompleteness of generalized numberings is related to the topic of complete extensions. For example, the identity on a pca $\mathcal{A}$ is precomplete if and only if every element of $\mathcal{A}$ (seen as a function on $\mathcal{A}$) has a total extension in $\mathcal{A}$.

In section 4 we show that the numbering of functions of a pca is precomplete, which is the analog of the precompleteness of the standard numbering of the p.c. functions. In general the functions modulo extensional equivalence do not form a pca. This prompts the definition of the notion of *algebraic* numbering, which is a generalized numbering that preserves the algebraic structure of the pca.

Below we study generalized numberings in relation to the algebraic structure of the pca $\mathcal{A}$. Just as in the lambda-calculus, the notion of extensionality is central here. A pca is called *extensional* if $f = g$ whenever $fx \simeq gx$ for every x. We have a similar notion of extensionality based on generalized numberings (Definitions 5.1). In section 5 we show that there is a relation between extensionality (an algebraic property) and precompleteness (a computability theoretic property).

In section 6 we introduce strong extensionality, (Definition 6.1), and in section 7 we introduce some auxiliary equivalence relations to aid the comparison between extensional and algebraic.

In section 8 we investigate the relations between various notions of extensionality and algebraic numberings. We will see that neither notion implies the other, and that they are in a sense complementary notions.

Our notation is mostly standard. In the following, ω denotes the natural numbers. φ_e denotes the e-th partial computable (p.c.) function, in the standard numbering of the p.c. functions. We write $\varphi_e(n)\downarrow$ if the result of this computation is defined, and $\varphi_e(n)\uparrow$ otherwise. $W_e = \mathrm{dom}(\varphi_e)$ denotes the e-th computably enumerable (c.e.) set. For unexplained notions from computability theory we refer to Odifreddi [20] or Soare [27]. For background on lambda-calculus we refer to Barendregt [4].

2. Partial combinatory algebra

Combinatory algebra predates the lambda-calculus, and was introduced by Schönfinkel [25]. It has close connections with the lambda-calculus, and played an important role in its development. Partial combinatory algebra (pca) was first studied in Feferman [11]. To fix notation and terminology, we will briefly recall the definition of a pca, and for a more elaborate treatment refer to van Oosten [22].

Definition 2.1. A *partial applicative structure* (pas) is a set $\mathcal{A}$ together with a partial map from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. We denote the image of (a, b), if it is defined, by ab, and think of this as 'a applied to b'. If this is defined we denote this by $ab\downarrow$. By convention, application associates to the left. We write abc instead of $(ab)c$. *Terms* over $\mathcal{A}$ are built from elements of $\mathcal{A}$, variables, and application. If t_1 and t_2 are terms then so is t_1t_2. If $t(x_1, \dots, x_n)$ is a term with variables x_i, and $a_1, \dots, a_n \in \mathcal{A}$, then $t(a_1, \dots, a_n)$ is the term obtained by substituting the a_i for the x_i. For closed terms (i.e. terms without variables) t and s, we write $t \simeq s$ if either both are undefined, or both are defined and equal. Here application is *strict* in the sense that for t_1t_2 to be defined, it is required that both t_1, t_2 are defined. We say that an element $f \in \mathcal{A}$ is *total* if $fa\downarrow$ for every $a \in \mathcal{A}$.

A pas $\mathcal{A}$ is *combinatory complete* if for any term $t(x_1, \dots, x_n, x)$, $0 \leqslant n$, with free variables among $x_1, \dots, x_n, x$, there exists a $b \in \mathcal{A}$ such that for all $a_1, \dots, a_n, a \in \mathcal{A}$,

(i) $ba_1 \cdots a_n\downarrow$,
(ii) $ba_1 \cdots a_n a \simeq t(a_1, \dots, a_n, a)$.

A pas $\mathcal{A}$ is a *partial combinatory algebra* (pca) if it is combinatory complete.

Theorem 2.2. (Feferman [11]) *A pas $\mathcal{A}$ is a pca if and only if it has elements k and s with the following properties for all $a, b, c \in \mathcal{A}$:*

- *k is total and $kab = a$,*
- *$sab\downarrow$ and $sabc \simeq ac(bc)$.*

Note that k and s are nothing but partial versions of the familiar combinators from combinatory algebra. As noted in [11, p95], Theorem 2.2 has the consequence that in any pca we can define lambda-terms in the usual way (cf. Barendregt [4, p152]):[1] For every term $t(x_1, \dots, x_n, x)$, $0 \leqslant n$, with free variables among $x_1, \dots, x_n, x$, there

[1]Because the lambda-terms in combinatory algebra do not have the same substitution properties as in the lambda calculus, we use the notation λ^* rather than λ. Curry used the notation $[x]$ to distinguish the two (cf. [7]).

exists a term $\lambda^* x.t$ with variables among $x_1, \dots, x_n$, with the property that for all $a_1, \dots, a_n, a \in \mathcal{A}$,

- $(\lambda^* x.t)(a_1, \dots, a_n)\downarrow$,
- $(\lambda^* x.t)(a_1, \dots, a_n)a \simeq t(a_1, \dots, a_n, a)$.

The most important example of a pca is Kleene's *first model* $\mathcal{K}_1$, consisting of ω with application defined as $nm = \varphi_n(m)$. Kleene's *second model* $\mathcal{K}_2$ [13] consists of the reals (or more conveniently Baire space ω^ω), with application $\alpha\beta$ defined as applying the continuous functional with code α to the real β. See Longley and Normann [18] for a more detailed definition.

In the axiomatic approach to the theory of computation, there is the notion of a basic recursive function theory (BRFT). Since this is supposed to model basic computability theory, it will come as no surprise that every BRFT gives rise to a pca. In case the domain of the BRFT is ω, it actually contains a copy of the p.c. functions. See Odifreddi [20] for a discussion of this, and references to the literature, including the work of Wagner, Strong, and Moschovakis.

Other pcas can be obtained by relativizing $\mathcal{K}_1$, or by generalizing $\mathcal{K}_2$ to larger cardinals. Also, every model of Peano arithmetic gives rise to a pca by considering $\mathcal{K}_1$ inside the model. Further constructions of pcas are discussed in van Oosten and Voorneveld [23], and in van Oosten [22] even more examples of pcas are listed. Finally, it is possible to define a combination of Kleene's models $\mathcal{K}_1$ and $\mathcal{K}_2$ (due to Plotkin and Scott) using enumeration operators, cf. Odifreddi [21, p857ff].

3. Generalized numberings

A *generalized numbering* [5] is a surjective mapping $\gamma\colon \mathcal{A} \to S$, where $\mathcal{A}$ is a pca and S is a set. As in the case of ordinary numberings, we have an equivalence relation on $\mathcal{A}$ defined by $a \sim_\gamma b$ if $\gamma(a) = \gamma(b)$.

As for ordinary numberings, in principle every generalized numbering corresponds to an equivalence relation on $\mathcal{A}$, and conversely. However, below we will mainly be interested in numberings that also preserve the algebraic structure of the pca, making this correspondence less relevant.

The notion of precompleteness for generalized numberings was defined in [5]. By [5, Lemma 6.4], the following definition is equivalent to it.

Definition 3.1. A generalized numbering $\gamma\colon \mathcal{A} \to S$ is *precomplete*[2] if for every $b{\in}\mathcal{A}$ there exists a total element $f{\in}\mathcal{A}$ such that for all $a{\in}\mathcal{A}$,

$$ba{\downarrow} \Longrightarrow\ fa \sim_\gamma ba. \tag{2}$$

In this case, we say that f *totalizes* b *modulo* $\sim_\gamma$.

In [5], generalized numberings were used to prove a combination of a fixed point theorem for pcas (due to Feferman [11]), and Ershov's recursion theorem [9] for precomplete numberings on ω.

Every pca $\mathcal{A}$ has an associated generalized numbering, namely the identity $\gamma_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. In section 5 we will see examples of when the numbering $\gamma_{\mathcal{A}}$ is or is not precomplete.

4. Algebraic numberings

Definition 4.1. Let $\mathcal{A}$ be a pca. Define an equivalence on $\mathcal{A}$ by $a \sim_e b$ if

$$\forall x \in \mathcal{A}(ax \simeq bx).$$

The following result generalizes the precompleteness of the numbering $n \mapsto \varphi_n$ of the partial computable functions.

Proposition 4.2. *The natural map $\gamma_e : \mathcal{A} \to \mathcal{A}/{\sim_e}$ is precomplete.*

Proof. Let $b \in \mathcal{A}$. We have to prove that there is a total $f \in \mathcal{A}$ such that when $ba\downarrow$ then $fa \sim_e ba$, i.e. $\forall c{\in}\mathcal{A}(fac \simeq bac)$. This follows from the combinatory completeness of $\mathcal{A}$: Consider the term bxy. By combinatory completeness there exists $f \in A$ such that for all $a, c \in \mathcal{A}$, $fa{\downarrow}$ and $fac \simeq bac$. □

Remark 4.3. Note that $\mathcal{A}/{\sim_e}$ is in general *not* a pca, at least not with the natural application defined by $\overline{a} \cdot \overline{b} = \overline{a \cdot b}$. For example, in Kleene's first model $\mathcal{K}_1$ we have for $n, m \in \omega$ that $n \sim_e m$ if $\varphi_n = \varphi_m$. Now we can certainly have that $m \sim_e m'$, but $\varphi_n(m) \neq \varphi_n(m')$, so we see that the natural definition of application $\overline{n} \cdot \overline{m} = \overline{n \cdot m}$ in $\omega/{\sim_e}$ is not independent of the choice of representative m.

The previous considerations prompt the following definition. First we extend the definition of $\sim_\gamma$ from $\mathcal{A}$ to the set of closed terms over $\mathcal{A}$ as follows:

Definition 4.4. $a \sim_\gamma b$ if either a and b are terms that are both undefined, or $a, b \in \mathcal{A}$ and $\gamma(a) = \gamma(b)$.

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[2]There is also a notion of completeness for numberings, that we will however have no use for in this paper. A precomplete generalized numbering γ is *complete* if there is a special element $s{\in}S$ (not depending on b) such that in addition to (2), $\gamma(fa) = s$ for every a with $ba{\uparrow}$.

This extended notion $\sim_\gamma$ is the analog of the Kleene equality $\simeq$.

Definition 4.5. Call a generalized numbering $\gamma : \mathcal{A} \to S$ *algebraic* if $\sim_\gamma$ is a congruence, i.e.

$$a \sim_\gamma a' \wedge b \sim_\gamma b' \Longrightarrow ab \sim_\gamma a'b'.$$

In this case, we also call the pca $\mathcal{A}$ γ-algebraic.

If γ is algebraic, we can factor out by $\sim_\gamma$, as in algebra:

Proposition 4.6. *Suppose that $\mathcal{A}$ is γ-algebraic. Then $\mathcal{A}/{\sim_\gamma}$ is again a pca.*

Proof. Define application in $\mathcal{A}/{\sim_\gamma}$ by $\overline{a} \cdot \overline{b} = \overline{a \cdot b}$. By algebraicity this is well-defined. Combinatory completeness follows because we have the combinators $\overline{s}$ and $\overline{k}$. □

We note that for a generalized numbering $\gamma : \mathcal{A} \to S$ there is in general no relation between the notion of precompleteness and algebraicity. This follows from results in the following sections.

- algebraic does not imply precomplete. Namely, let $\gamma_{\mathcal{K}_2}$ be the identity on Kleene's second model $\mathcal{K}_2$. Then $\gamma_{\mathcal{K}_2}$ is trivially algebraic. However, by [5], the numbering $\gamma_{\mathcal{K}_2}$ is not precomplete.
- precomplete does not imply algebraic. Namely, the canonical map $\gamma_e : \mathcal{A} \to \mathcal{A}/{\sim_e}$ is precomplete by Proposition 4.2. However, it is not algebraic, since otherwise we would have by Proposition 4.6 that $\mathcal{A}/{\sim_e}$ is a pca, which in general it is not by Remark 4.3. Alternatively: If precomplete implied algebraic, by Proposition 5.2 we would have that $\mathcal{A}$ is γ-extensional implies that $\mathcal{A}$ is γ-algebraic, contradicting Proposition 8.2.

5. Precompleteness and extensionality

We can think of combinatory completeness of a pca as an analog of the S-m-n–theorem (also called the parametrization theorem) from computability theory [20]. Suppose $\mathcal{A}$ is a pca, and $\gamma : \mathcal{A} \to S$ is a generalized numbering. Every element $a \in \mathcal{A}$ represents a partial function on $\mathcal{A}$, namely $x \mapsto ax$. In analogy to $\mathcal{K}_1$, one could call these the partial $\mathcal{A}$-computable functions. Note that the precompleteness of the numbering $n \mapsto \varphi_n$ of the partial computable functions follows from the S-m-n–theorem, and that the precompleteness of the numbering of the partial $\mathcal{A}$-computable functions follows likewise from the combinatory completeness of $\mathcal{A}$ (see Proposition 4.2).

Note that the identity on $\mathcal{K}_1$ is not precomplete, as there exist p.c. functions that do not have a total computable extension. In [5] it was shown that the identity on Kleene's second model $\mathcal{K}_2$ is also not precomplete. In general, every pca $\mathcal{A}$ has the identity $\gamma_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ as an associated generalized numbering, and $\gamma_{\mathcal{A}}$ is precomplete if and only if every element $b \in \mathcal{A}$ has a total extension $f \in \mathcal{A}$. Faber and van Oosten [10] showed that the latter is equivalent to the statement that $\mathcal{A}$ is isomorphic to a total pca. Here "isomorphic" refers to isomorphism in the category of pcas introduced in Longley [17].

Definition 5.1. Let $\mathcal{A}$ be a pca, and $\gamma : \mathcal{A} \to S$ a generalized numbering. We say that $\mathcal{A}$ is γ-*extensional* if

$$\forall a \in \mathcal{A}(fa \simeq ga) \Longrightarrow f \sim_\gamma g \tag{3}$$

for all $f, g \in \mathcal{A}$.

In other words, $\mathcal{A}$ is γ-extensional if the relation $\sim_\gamma$ extends the relation $\sim_e$ from Definition 4.1. For the special case where $\gamma : \mathcal{A} \to \mathcal{A}$ is the identity, this is called *extensionality* of $\mathcal{A}$, cf. Barendregt [3, p1094]. Note, however, that (3) can equally be seen as a property of γ, rather than of $\mathcal{A}$.

Proposition 5.2. *Suppose $\mathcal{A}$ is γ-extensional. Then γ is precomplete.*

Proof. This is similar to Proposition 4.2. Given $b \in \mathcal{A}$, we have to prove that there exists a total $f \in \mathcal{A}$ such that for every $a \in \mathcal{A}$, $fa \sim_\gamma ba$ whenever $ba\downarrow$. Consider the term bxy. By combinatory completeness of $\mathcal{A}$ there exists a total $f \in \mathcal{A}$ such that $fac \simeq bac$ for all $a, c \in \mathcal{A}$. Now suppose $ba\downarrow$. It follows from γ-extensionality of $\mathcal{A}$ that $fa \sim_\gamma ba$. Hence γ is precomplete.[3] □

In particular, we see from Proposition 5.2 that the identity $\gamma_{\mathcal{A}}$ on $\mathcal{A}$ is precomplete if $\mathcal{A}$ is extensional.

It is possible that a generalized numbering $\gamma : \mathcal{A} \to S$ is precomplete for some other reason than $\mathcal{A}$ being γ-extensional, as we now show.

Proposition 5.3. *There exists a generalized numbering $\gamma : \mathcal{A} \to S$ that is precomplete, but such that $\mathcal{A}$ is not γ-extensional.*

Proof. Let $\mathcal{A}$ be a total pca. Then γ is trivially precomplete (this is immediate from Definition 3.1), but a total pca $\mathcal{A}$ need not be extensional. For example, let $\mathcal{A}$ be a model of the lambda calculus. This certainly does not have to be extensional, for example the graph model

[3] Alternatively, we could derive Proposition 5.2 from Proposition 4.2 by noticing that if $\gamma, \gamma' : \mathcal{A} \to S$ are generalized numberings such that $\sim_\gamma$ extends $\sim'_\gamma$, and γ' is precomplete, then also γ is precomplete. By Proposition 4.2 we have that $\gamma' = \gamma_e$ is precomplete, and if $\mathcal{A}$ is γ-extensional then $\sim_\gamma$ extends $\sim_e$, so it follows that γ is precomplete.

$P\omega$ is not extensional, cf. [4, p474]. (An example of a model of the lambda calculus that *is* extensional is Scott's model D_∞.)

Another example is the set of terms $\mathfrak{M}(\beta\eta)$ in the lambda calculus. This combinatory algebra is extensional, by inclusion of the η-rule. However, the set of *closed* terms $\mathfrak{M}^0(\beta\eta)$ is not extensional by Plotkin [19]. □

Proposition 5.3 shows that the converse of Proposition 5.2 does not hold.

6. Strong extensionality

Given a generalized numbering $\gamma : \mathcal{A} \to S$, we have two kinds of equality on $\mathcal{A}$: $a \simeq b$ and $a \sim_\gamma b$. The notion of γ-extensionality is based on the former. We obtain a stronger notion if we use the latter, where we use the extended notion of $\sim_\gamma$ for closed terms from Definition 4.4.

Definition 6.1. We call $\mathcal{A}$ *strongly γ-extensional* if

$$\forall x(fx \sim_\gamma gx) \Longrightarrow f \sim_\gamma g$$

for every $f, g \in \mathcal{A}$.

Theorem 6.2. *Strong γ-extensionality implies γ-extensionality, but not conversely.*

Proof. First, strong γ-extensionality implies γ-extensionality because $\forall x\ fx \simeq gx$ implies $\forall x\ fx \sim_\gamma gx$, so the premiss of the first notion is weaker than that of the second.

To see that the implication is strict, we exhibit a pca $\mathcal{A}$ that is γ-extensional but not strongly γ-extensional. Take $\mathcal{A}$ to be Kleene's first model $\mathcal{K}_1$, and let $\gamma = \gamma_e$, the numbering of equivalence classes from Proposition 4.2. That $\mathcal{A}$ is γ_e-extensional is trivial, since $f \sim_e g$ means precisely $\forall x\ fx \simeq gx$.

To see that $\mathcal{A}$ is not strongly γ_e-extensional, let $d, e \in \mathcal{A}$ be such that $d \neq e$ and $\forall x\ dx \simeq ex$, and define $f = kd$ and $g = ke$, with k the combinator. Then $fx = d$ and $gx = e$, hence $\forall x\ fx \sim_e gx$ because $d \sim_e e$. But not $\forall x\ fx \simeq gx$ because $d \neq e$, so $f \not\sim_e g$. □

7. Left and right equivalences

For the discussion below (and also to aid our thinking), we introduce two equivalence relations. Let $\mathcal{A}$ be a pca, and $\gamma : \mathcal{A} \to S$ a generalized numbering.

Definition 7.1. We define two kinds of equivalence relations on $\mathcal{A}$, corresponding to right and left application:

- $f \sim_{R_\gamma} g$ if $\forall x\ fx \sim_\gamma gx$.
- $f \sim_{L_\gamma} g$ if $\forall z\ zf \sim_\gamma zg$.

Recall the relation $\sim_e$ from section 4. Note that $\sim_e$ is equal to $\sim_{R_\gamma}$ for γ the identity. With these equivalences we can succinctly express extensionality as follows:

$$\mathcal{A} \text{ is } \gamma\text{-extensional if } f \sim_e g \Longrightarrow f \sim_\gamma g,$$
$$\mathcal{A} \text{ is strongly } \gamma\text{-extensional if } f \sim_{R_\gamma} g \Longrightarrow f \sim_\gamma g.$$

Proposition 7.2. *For all $f, g \in \mathcal{A}$ we have $f \sim_{L_\gamma} g \Longrightarrow f \sim_{R_\gamma} g$.*

Proof. For every x, define $z_x = \lambda^* h.hx$. (Note that in every pca we can define such lambda terms, cf. section 2.) Then

$$\begin{aligned} f \sim_{L_\gamma} g &\Longrightarrow \forall x\ z_x f \sim_\gamma z_x g \\ &\Longrightarrow \forall x\ fx \sim_\gamma gx \\ &\Longrightarrow f \sim_{R_\gamma} g. \end{aligned}$$
□

8. Algebraic versus extensional

For a given pca $\mathcal{A}$ and a generalized numbering γ on $\mathcal{A}$, note that the following hold: If $\mathcal{A}$ is γ-algebraic, then for every $f, g \in \mathcal{A}$,

$$f \sim_\gamma g \Longrightarrow f \sim_{R_\gamma} g, \tag{4}$$
$$f \sim_\gamma g \Longrightarrow f \sim_{L_\gamma} g. \tag{5}$$

This holds because in Definition 4.5, we can either take the right sides equal, obtaining (4), or take the left sides equal, obtaining (5). Also note that (5) actually implies (4) by Proposition 7.2. We could call (4) *right-algebraic* and (5) *left-algebraic*.

Proposition 8.1. *γ-algebraic is equivalent with* (5).

Proof. That (5) follows from γ-algebraicity was noted above. Conversely, assume (5) and suppose that $a \sim_\gamma a'$ and $b \sim_\gamma b'$. We have to prove that $ab \sim_\gamma a'b'$. Indeed we have

$$\begin{aligned} ab &\sim_\gamma a'b && \text{by (4)} \\ &\sim_\gamma a'b' && \text{by (5).} \end{aligned}$$
□

On the other hand, if $\mathcal{A}$ is strongly γ-extensional, we have

$$f \sim_{R_\gamma} g \Longrightarrow f \sim_\gamma g \tag{6}$$

which is the converse of (4). So we see that in a sense, the notions of algebraicity and extensionality are complementary. We now show that neither of them implies the other.

Proposition 8.2. *γ-extensional does not imply γ-algebraic.*

Proof. Consider Kleene's first model $\mathcal{K}_1$, and let $\gamma = \gamma_e$ be the numbering from Proposition 4.2. Every pca is trivially γ_e-extensional, as $f \sim_e g \Rightarrow f \sim_e g$. However, $\mathcal{K}_1$ is not γ_e-algebraic. Namely, (5) above does not hold: There are $n, m \in \mathcal{K}_1$ such that $n \sim_e m$, i.e. n and m are codes of the same partial computable function, but $n \neq m$, so that $\forall z\; zn \sim_e zm$ does not hold: There is a p.c. function φ such that $\varphi(n) \not\sim_e \varphi(m)$. □

We can strengthen Proposition 8.2 to the following.

Theorem 8.3. *Strong γ-extensional does not imply γ-algebraic.*

Proof. We show that (6) does not imply (5). As a pca we take Kleene's first model $\mathcal{K}_1$, and we define a generalized numbering γ on it as follows. We start with the equivalence $\sim_e$ on $\mathcal{K}_1$, and we let $\sim_\gamma$ be the smallest extension of $\sim_e$ such that

$$f \sim_\gamma g \Longleftrightarrow \forall x\; fx \sim_\gamma gx. \tag{7}$$

(Here we read $fx \sim_\gamma gx$ as in Definition 4.4.) The equivalence relation $\sim_\gamma$ is the smallest fixed point of the monotone operator that, given an equivalence $\sim$ on $\mathcal{K}_1$ that extends $\sim_e$, defines a new equivalence $\approx$ by

$$f \approx g \Longleftrightarrow \forall x\; fx \sim gx. \tag{8}$$

The existence of $\sim_\gamma$ is then guaranteed by the Knaster-Tarski theorem on fixed points of monotone operators [14]. Note that by Remark 4.3, $\mathcal{K}_1/{\sim_e}$ is not a pca, and neither are the extensions $\mathcal{K}_1/{\approx}$, but this is not a problem for the construction (8), since the application fx keeps taking place in the pca $\mathcal{K}_1$. Note that by (7) we have that $\mathcal{K}_1$ is strongly γ-extensional.

We claim that (5) fails for γ, and hence that $\mathcal{K}_1$ is not γ-algebraic. First we observe that γ is not trivial, i.e. does not consist of only one equivalence class. Namely, let $ax\uparrow$ for every x, and let $b \in \mathcal{K}_1$ be total. Then obviously $\forall x\; ax \sim_\gamma bx$ does not hold, hence by (7) we have $a \not\sim_\gamma b$.

For the failure of (5) we further need the existence of $f \sim_\gamma g$ such that $f \neq g$. Such f and g exist, since they already exist for $\sim_e$, and $\sim_\gamma$ extends $\sim_e$. Now let $a \not\sim_\gamma b$ (the existence of which we noted above), and let z be a code of a partial computable function such that $zf = a$

and $zg = b$. Then $zf \not\sim_\gamma zg$, hence $\forall z\ zf \sim_\gamma zg$ does not hold, and thus (5) fails. □

Corollary 8.4. (5) *implies* (4)*, but not conversely.*

Proof. (5) implies (4) by Proposition 7.2. In the counterexample of Theorem 8.3 the equivalence (7) holds, so both (4) and (6) hold, but (5) does not. □

Proposition 8.5. *γ-algebraic does not imply γ-extensional.*

Proof. Let $\mathcal{A}$ be a pca that is not extensional (such as Kleene's $\mathcal{K}_1$), and let γ be the identity on $\mathcal{A}$. Every pca is always γ-algebraic, so this provides a counterexample to the implication. □

The proof of Theorem 8.3 shows that (6) does not imply (5). Since (5) strictly implies (4) by Corollary 8.4, a stronger statement would be to show that (6) does not imply (4). We can obtain this with a variation of the earlier proof (though this construction is less natural, and only serves a technical purpose).

Theorem 8.6. *Strong γ-extensional* (6) *does not imply right-algebraic* (4).

Proof. We show that (6) does not imply (4). As before, we take Kleene's first model $\mathcal{K}_1$, and we define a generalized numbering γ on it. Now we do not want the equivalence (7) to hold, so we do not start with the equivalence relation $\sim_e$.

Let $f, g \in \mathcal{K}_1$ and $x \in \omega$ be such that $fx\!\downarrow$ is total, and $gx\!\uparrow$. Start with $f \sim_\gamma g$, so that γ equates f and g and nothing else. Let $\sim_\gamma$ be the smallest extension of this equivalence relation such that

$$\forall x(fx \sim_\gamma gx) \Longrightarrow f \sim_\gamma g,$$

where again, we read $fx \sim_\gamma gx$ as in Definition 4.4. As before, the equivalence relation $\sim_\gamma$ exists by the Knaster-Tarski theorem [14]. This ensures that γ satisfies (6).

We claim that $\forall x\ fx \sim_\gamma gx$ does not hold, and hence that (4) fails. Note that we would only have $\forall x\ fx \sim_\gamma gx$ if at some stage $\forall x, y\ fxy \sim_\gamma gxy$ would hold. However, since fx is total, fxy is always defined, whereas gxy is never defined by choice of x. Hence, by Definition 4.4, $fxy \not\sim_\gamma gxy$, whatever γ may be. □

Consider the following property, which is the converse of the implication from Proposition 7.2:

$$f \sim_{R_\gamma} g \Longrightarrow f \sim_{L_\gamma} g. \tag{9}$$

This property expresses that whenever f and g denote the same function, they are *inseparable* in the pca (compare Barendregt [4, p48]). Combining algebraicity and extensionality, we obtain the following relation.

Proposition 8.7. *γ-algebraic + strongly γ-extensional $\Longrightarrow$ (9).*

Proof. By γ-algebraicity we have (5), hence

$$\begin{aligned} f \sim_{R_\gamma} g &\Longrightarrow f \sim_\gamma g && \text{by strong } \gamma\text{-extensionality} \\ &\Longrightarrow f \sim_{L_\gamma} g && \text{by (5).} \end{aligned}$$

□

Note that the identity $\gamma_\mathcal{A}$ is algebraic for any pca $\mathcal{A}$. So any extensional $\mathcal{A}$ (meaning $\gamma_\mathcal{A}$-extensional, which in this case coincides with strongly $\gamma_\mathcal{A}$-extensional) is an example where the conditions of Proposition 8.7 hold.

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