Exercises Proof Theory
Master Class Logic, Utrecht 2006–2007

Exercise 1. Show that the following propositional sequents are derivable in *PK*.

1. $\varphi \Rightarrow \varphi$ for arbitrary propositional φ.
2. $\varphi \Rightarrow \psi \to \varphi$.
3. $\neg(\varphi \wedge \psi) \Rightarrow \neg\varphi \vee \neg\psi$.
4. $\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$.
5. $\varphi \to \psi \Rightarrow \neg\varphi \vee \psi$.
6. $\neg\varphi \vee \psi \Rightarrow \varphi \to \psi$.
7. $\neg\neg\varphi \Rightarrow \varphi$.
8. $\Rightarrow \varphi \vee \neg\varphi$.
9. $\varphi \wedge \neg\varphi \Rightarrow$.
10. $\varphi \wedge (\psi \vee \chi) \Rightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.
11. $\varphi \to (\psi \vee \chi) \Rightarrow (\varphi \to \psi) \vee (\varphi \to \chi)$.

Exercise 2. Prove the statement on page 13 [Buss]: For any (tree-like) proof P of $\Gamma \Rightarrow \Delta$ there is a proof of an end-sequent $\Gamma' \Rightarrow \Delta'$ with at most $||P||^2$ sequents and with $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$.

Exercise 3. Prove propositional soundness, Theorem 1.2.6 [Buss].

Exercise 4. Prove the Inversion Theorem, Theorem 1.2.7 [Buss].

Exercise 5. Complete the missing parts of Lemma 1.2.9 [Buss].

Exercise 6. (Cf. 2.1.3 [Buss]) Let EQ be the set of equality axioms defined on page 29 [Buss].

(i) Show that for every model M of EQ the relation $=^M$ is an equivalence relation by showing that it is symmetric and transitive.

(ii) Show that in general $M \models' \mathrm{EQ}, A$ does *not* imply $M \models A$.

(iii) For models M of EQ, show that the interpretation of terms is well-defined on the quotient model $M/=^M$.

(iv) Prove that $M \models' \mathrm{EQ}, A$ if and only if $(M/=^M) \models A$.

(v) Prove that $\Gamma \models A$ if and only if $\Gamma, \mathrm{EQ} \models' A$.

Exercise 7. Show that the following sequents are derivable in *LK*.

1. $\varphi \Rightarrow \varphi$ for arbitrary φ.
2. $\Rightarrow \exists x A(x) \to \exists y A(y)$.
3. $\exists x \varphi(x) \to \psi \Rightarrow \forall x(\varphi(x) \to \psi)$, x not in ψ.
4. $\forall x(\varphi(x) \to \psi) \Rightarrow \exists x \varphi(x) \to \psi$, x not in ψ.
5. $\varphi \to \exists x \psi(x) \Rightarrow \exists x(\varphi \to \psi(x))$, x not in φ.
6. $\exists x(\varphi \to \psi(x)) \Rightarrow \varphi \to \exists x \psi(x)$, x not in φ.
7. $\forall x \neg \varphi(x) \Rightarrow \neg \exists x \varphi(x)$.
8. $\neg \exists x \varphi(x) \Rightarrow \forall x \neg \varphi(x)$.
9. $\neg \forall x \neg \varphi(x) \Rightarrow \exists x \varphi(x)$.

Exercise 8. [Buss] p32: Let S be the sequent

$$P(x, y) \Rightarrow \exists y \exists x P(y, x).$$

Note that S is not an admissible LK-sequent by the conventions of 2.3.1. Show that without the variable conventions, S is derivable but not without the cut-rule.

Exercise 9. Show that the side conditions on eigenvariables in 2.3.2 [Buss] are necessary.

Exercise 10. Prove Theorem 2.3.4 [Buss].

Exercise 11. Prove the statement in 2.3.5 [Buss].

Exercise 12. Prove the Soundness Theorem 2.3.6 [Buss].

Exercise 13. Derive $\exists x\big(R(x) \lor \neg R(x)\big)$ in LK. Note that this squares with the convention that all models are nonempty ([Buss] p27).

Exercise 14. (i) Show that $\exists x \forall y(R(y) \to R(x))$ is a tautology by deriving it in LK.

(ii) Show that without contraction rules LK is not complete.

Exercise 15. (i) Argue informally that it is decidable whether a given derivation is a correct LK-derivation. How would you prove this formally?

(ii) Show that the set $\{\varphi : \mathrm{LK} \vdash \varphi\}$ is c.e.

(iii) Conclude that $\{\varphi : \mathrm{LK} \vdash \varphi\} \subsetneq \{\varphi : \omega \models \varphi\}$.

Exercise 16. Write down the cases (2c), (2d), and (2f) on p35 [Buss].

Exercise 17. Prove the statement on p35 [Buss]: For all A, $A \Rightarrow A$ is derivable without cut.

Exercise 18. Prove Königs lemma: Every infinite finitely branching tree has an infinite branch.

Exercise 19. Supply the details of the proof on p36 [Buss].

Exercise 20. (i) Suppose that we are given an LK-proof P of the sequent $\neg A, \Gamma \Rightarrow \Delta$. Let P' be the result of replacing every sequent of the form $\neg A, \Gamma' \Rightarrow \Delta'$ in P, where $\neg A$ is an ancestor of the $\neg A$ in the endsequent, by $\Gamma' \Rightarrow \Delta', A$. Show that P' is again a valid LK-derivation. Conclude that if the sequent $\neg A \Rightarrow$ is derivable then so is $\Rightarrow A$.

(ii) Given an LK-derivation P of $\neg A \Rightarrow$, show how one can extend P to a derivation of $\Rightarrow A$. Is it possible to do this without cut?

Exercise 21. (i) Show that there is a p.c. function φ such that if A is provable in LK then $\varphi(A)\downarrow$ and A has a proof of size $\leq \varphi(A)$.

(ii) Show that there is no computable function f such that if A is provable in LK then it has a proof of size $\leq f(A)$.

Exercise 22. Derive the formulas from exercises 1 and 7 using natural deduction, cf. [Buss] p48 (interpreting $\Gamma \Rightarrow \Delta$ as $\bigwedge \Gamma \to \bigvee \Delta$).

Exercise 23. The proof of Herbrand's theorem 2.5.1 [Buss] shows that for a provable sentence $\exists \vec{x} B(\vec{x})$ one can *effectively from the proof* find Herbrand terms $t_{i,j}$ such that $\bigvee_{i=1}^{r} B(t_{i,1} \dots t_{i,k})$ is provable. Show that in general it is not possible to find the Herbrand terms without the proof being given. (First formulate precisely what this would mean.) N.B. Not even r (as a function of the formula) can be computably bounded, cf. [Buss] p55.

Exercise 24. Skolemize the following formulas:

1. $\big(\forall x \exists y P(x,y) \to \big(\forall x B(x) \vee \exists x C(x)\big)\big) \to \exists x\, C(x)$

2. $\forall x \exists y \forall z (P(x,y) \vee Q(y,z)) \vee \neg \forall z (P(z,z) \vee Q(z,z))$

3. $\forall x \Big(P(x) \to \exists y \big(Q(y) \to \forall z \big(R(z) \to \exists u \big(W(u) \to \forall v S(v)\big)\big)\big)\Big)$

Exercise 25. (Cf. [Buss] p50.)

(i) Prove that $A^S \models A$.

(ii) Prove that T^S is conservative over T, i.e. $T^S \vdash A \Rightarrow T \vdash A$.

Exercise 26. Prove that $T \models A \iff T^S \models A$. (Cf. [Buss] p51.)

Exercise 27. Let f be a function symbol, and for every $n \geq 1$ let $f^n x$ abbreviate the term $f(\ldots f(x) \ldots)$ with n occurrences of f. Find Herbrand terms for the sentence $\exists x \exists y \exists z\ \psi(x,y,z)$, in case that

(i) $\psi(x,y,z) = \neg R(x, fx) \vee \big(R(x,y) \wedge R(y,z) \wedge \neg R(x,z)\big) \vee R(x, f^3 x)$.

(ii) $\psi(x,y,z) = \neg R(x, fx) \vee \big(R(x,y) \wedge R(y,z) \wedge \neg R(x,z)\big) \vee R(x, f^n x)$.

Exercise 28. Prove that the Tait calculus (cf. [Schwicht]) and the Gentzen calculus LK (cf. [Buss]) derive the same formulas.

Exercise 29. Prove the Inversion Lemma 2.5 (i) [Schwicht].

Exercise 30. In the proof of the reduction lemma 2.6 [Schwicht], spell out what happens in Case 1 when the last inference of d_0 is a cut.

Exercise 31. Show that in general $\sup_i(|d_i| + 1) \neq \sup_i(|d_i|) + 1$.

Exercise 32. For every one of the following equations, indicate whether it has a solution in the infinite ordinals, and if so, indicate a least infinite solution α.

(i) $\alpha + \omega = \alpha$

(ii) $\alpha \cdot \omega = \alpha$

(iii) $\alpha^\omega = \alpha$

(iv) $\omega + \alpha = \alpha$

(v) $\omega \cdot \alpha = \alpha$

(vi) $\omega^\alpha = \alpha$

Exercise 33. Prove that ω^α is strictly monotone in α, i.e. $\alpha < \beta$ implies that $\omega^\alpha < \omega^\beta$.

Exercise 34. Suppose that $\alpha < \omega^\beta$. Show that there are $\gamma < \beta$ and $n \in \omega$ such that $\alpha < \omega^\gamma \cdot n$.

Exercise 35. In this exercise we prove that every ordinal has a Cantor normal form: For every ordinal $\alpha > 0$ there are unique $\alpha_1 \geq \ldots \geq \alpha_n$ such that

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}.$$

(i) Prove the uniqueness.

(ii) Show the existence in case that $\forall \beta < \alpha \big(\omega^\beta < \alpha\big)$.

(iii) Prove by induction on β that every ordinal $\alpha < \omega^\beta$ can be written in Cantor normal form.

Exercise 36. (i) Prove by induction on i that for every $i \in \omega$ there is a finite derivation in d_i in PA_∞ such that

$$d_i \vdash \neg\varphi(\bar{0}), \exists n\big(\varphi(n) \wedge \neg\varphi(Sn)\big), \varphi(\bar{\imath}).$$

(ii) Use the ω-rule to prove that there is a derivation of the induction axiom in PA_∞ of length $\omega + 3$.

(iii) Prove the Embedding Lemma 3.4.2 [Schwicht].

Exercise 37. Prove Lemma 3.6.1 [Schwicht] by giving a formal derivation of $\mathrm{Prog}(X)$ in $\mathrm{PA}_\infty + \mathrm{Prog}$.

Exercise 38. Prove Lemma 3.6.2 [Schwicht].

Exercise 39. Draw or describe tree representations of the following ordinals: 0, 3 (give two different trees), ω, $\omega + \omega$, ω^2, ω^ω, ε_0.

Exercise 40. Suppose that tree representations are given for the ordinals α and β. Prove the following statements:

(i) $\alpha + \beta$ can be represented by replacing every leaf in the tree for β by a copy of α (identifying the root of α with the leaf).

(ii) $\alpha \cdot \beta$ can be represented by recursively replacing nodes in the tree for β by copies of α as follows:

- For a successor ordinal $\beta + 1$, $\alpha \cdot (\beta + 1)$ can be represented by the tree $(\alpha \cdot \beta) + \alpha$.
- For a limit ordinal λ, $\alpha \cdot \lambda$ can be represented by taking trees for $\alpha \cdot \xi$ for every $\xi < \lambda$ and connecting all the roots of these trees by a new root.

Exercise 41. Consider the following predicate formulas:

(I) $\big(\forall x\, (A(x) \to B(x))\big) \to \exists y\, (C(y) \wedge \neg B(y))$

(II) $\forall x\, (C(x) \to B(x))$

(III) $\exists x\, (A(x) \wedge \neg B(x))$

Show that LK derives the sequent I, II $\Rightarrow$ III.