Hyperimmune sets and degrees **October 14, 2014**

This is a handout for the mastermath course in computability theory.

1 Hyperimmune sets

Definition 1.1. An *array* is a c.e. set of (indices of) finite sets. If these are canonical indices it is called a *strong array*. If the finite sets in the array are disjoint it is called a *disjoint array*. So an infinite disjoint strong array is of the form $\{D_{f(n)} \mid n \in \omega\}$, where f is a computable function such that $D_{f(n)} \cap D_{f(m)} = \emptyset$ whenever $n \neq m$.

Definition 1.2. A set A is *hyperimmune*, or *h-immune*, if there is no disjoint strong array $\{D_{f(n)} \mid n \in \omega\}$ such that $A \cap D_{f(n)} \neq \emptyset$ for all n.

Note that, as the terminology suggests, hyperimmune is indeed stronger than immune, because if $W_e \subseteq A$ is infinite then the elements of W_e form a disjoint strong array intersecting A. So hyperimmune sets are immune.

The original motivation for this notion, introduced by Post, comes from Post's program for building incomplete c.e. sets for the various notions of reduction. A c.e. set A is *simple* if $\overline{A}$ is immune, and simple sets solve Post's problem for m-reducibility (cf. [3, Corollary 3.6.5]). Likewise, a c.e. set A is *hypersimple*, or simply *h-simple*, if $\overline{A}$ is h-immune. H-simple sets solve Post's problem for some of the weaker reducibilities, such as tt-reducibility, but not for Turing-reducibility. However, h-immune sets still play an important role in computability theory, mainly because of the connection with domination properties.

Definition 1.3. A function f *majorizes* a function g if $g(x) \leqslant f(x)$ for all x. f *dominates* g if $g(x) \leqslant f(x)$ for almost every x. g is *computably dominated* if there is a computable function that dominates g.

Definition 1.4. For an infinite set $A = \{a_0 < a_1 < a_2 < \dots\}$, the function $p_A(n) = a_n$ is called the *principal function* of A.

Proposition 1.5. *The following are equivalent:*

(i) *A is h-immune,*

(ii) *The principal function p_A is not computably dominated.*

Proof. (i)⇒(ii). Suppose f is computable and $a_n = f(n)$ for all n. Then $a_0 < \dots < a_m \leqslant f(m)$, so $\{m, \dots, f(m)\} \cap A \neq \emptyset$ for every m. Hence the sequence of finite sets $\{m_k + 1, \ldots, f(m_k + 1)\}$, where $m_0 = f(0)$ and $m_{k+1} = f(m_k + 1)$, is a disjoint strong array intersecting A, so A is not h-immune.

(ii)⇒(i). If A is not h-immune, say $\{D_{g(n)} \mid n \in \omega\}$ is a disjoint strong array intersecting A, let $f(n) = \max \bigcup_{m \leqslant n} D_{g(m)}$. Then $p_A(n) \leqslant f(n)$: If $D \subseteq A$ has n elements then $\max D \geqslant p_A(n)$. □

Post's simple set (from Theorem 3.6.4 in [3]) is not h-simple. Namely, for this set A we have $|A \cap \{0, \ldots, 2x\}| \leqslant x$ by construction, hence $|\overline{A} \cap \{0, \ldots, 2x\}| > x$, so $p_{\bar{A}}(n) \leqslant 2n$. So by Proposition 1.5, $\overline{A}$ is not h-immune. In particular we see that h-immune is strictly stronger than immune.

References

[1] P. G. Odifreddi, *Classical recursion theory*, Vol. 1, Studies in logic and the foundations of mathematics Vol. 125, North-Holland, 1989.

[2] P. G. Odifreddi, *Classical recursion theory*, Vol. 2, Studies in logic and the foundations of mathematics Vol. 143, North-Holland, 1999.

[3] S. A. Terwijn, *Syllabus computability theory*, 2004.