Hyperimmune sets and degrees

October 14, 2014

This is a handout for the mastermath course in computability theory.

1 Hyperimmune sets

Definition 1.1. An array is a c.e. set of (indices of) finite sets. If these are canonical indices it is called a strong array. If the finite sets in the array are disjoint it is called a disjoint array. So an infinite disjoint strong array is of the form \( \{ D_{f(n)} \mid n \in \omega \} \), where \( f \) is a computable function such that \( D_{f(n)} \cap D_{f(m)} = \emptyset \) whenever \( n \neq m \).

Definition 1.2. A set \( A \) is hyperimmune, or h-immune, if there is no disjoint strong array \( \{ D_{f(n)} \mid n \in \omega \} \) such that \( A \cap D_{f(n)} \neq \emptyset \) for all \( n \).

Note that, as the terminology suggests, hyperimmune is indeed stronger than immune, because if \( W_e \subseteq A \) is infinite then the elements of \( W_e \) form a disjoint strong array intersecting \( A \). So hyperimmune sets are immune.

The original motivation for this notion, introduced by Post, comes from Post’s program for building incomplete c.e. sets for the various notions of reduction. A c.e. set \( A \) is simple if \( \overline{A} \) is immune, and simple sets solve Post’s problem for m-reducibility (cf. [3, Corollary 3.6.5]). Likewise, a c.e. set \( A \) is hypersimple, or simply h-simple, if \( \overline{A} \) is h-immune. H-simple sets solve Post’s problem for some of the weaker reducibilities, such as tt-reducibility, but not for Turing-reducibility. However, h-immune sets still play an important role in computability theory, mainly because of the connection with domination properties.

Definition 1.3. A function \( f \) majorizes a function \( g \) if \( g(x) \leq f(x) \) for all \( x \). \( f \) dominates \( g \) if \( g(x) \leq f(x) \) for almost every \( x \). \( g \) is computably dominated if there is a computable function that dominates \( g \).

Definition 1.4. For an infinite set \( A = \{ a_0 < a_1 < a_2 < \ldots \} \), the function \( p_A(n) = a_n \) is called the principal function of \( A \).

Proposition 1.5. The following are equivalent:

(i) \( A \) is h-immune,

(ii) The principal function \( p_A \) is not computably dominated.

Proof. (i)⇒(ii). Suppose \( f \) is computable and \( a_n = f(n) \) for all \( n \). Then \( a_0 < \ldots < a_m \leq f(m) \), so \( \{ m, \ldots, f(m) \} \cap A \neq \emptyset \) for every \( m \). Hence
the sequence of finite sets \( \{m_k + 1, \ldots, f(m_k + 1)\} \), where \( m_0 = f(0) \) and \( m_{k+1} = f(m_k + 1) \), is a disjoint strong array intersecting \( A \), so \( A \) is not h-immune.

(ii) \( \Rightarrow \) (i). If \( A \) is not h-immune, say \( \{D_{g(n)} \mid n \in \omega\} \) is a disjoint strong array intersecting \( A \), let \( f(n) = \max \bigcup_{m \leq n} D_{g(m)} \). Then \( p_A(n) \leq f(n) \): If \( D \subseteq A \) has \( n \) elements then \( \max D \geq p_A(n) \). \( \square \)

Post’s simple set (from Theorem 3.6.4 in [3]) is not h-simple. Namely, for this set \( A \) we have \( |A \cap \{0, \ldots, 2x\}| \leq x \) by construction, hence \( |\overline{A} \cap \{0, \ldots, 2x\}| > x \), so \( p_{\overline{A}}(n) \leq 2n \). So by Proposition 1.5, \( \overline{A} \) is not h-immune. In particular we see that h-immune is strictly stronger than immune.

References

