Bridgeland Stability Conditions on curves I

by V. Hoskins

§1 Overview of stability conditions

Def^n: A stability condition on a triangulated category $\mathcal{T}$ is a pair $\sigma = (Z, \mathcal{P})$ where

$Z : K(\mathcal{T}) \to \mathbb{C}$ is a group homomorphism (called the central charge)

$\mathcal{P} = \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ is a slicing

ie. $\mathcal{P}(\phi) \subseteq \mathcal{T}$ are full additive subcategories satisfying:

- $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$
- $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ if $\phi_1 > \phi_2$.
- $\forall\ 0 \neq E \in \mathcal{T}$, $\exists\ \phi_1 > \dots > \phi_n$ and a seq. of distinguished Δs

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] & \end{tikzcd}$$

with $A_i \in \mathcal{P}(\phi_i)$

such that $\forall\ 0 \neq E \in \mathcal{P}(\phi)$, $Z(E) = m_E \cdot e^{i\pi\phi}$ for $m_E \in \mathbb{R}_{>0}$.

Theorem (Bridgeland)

1) $\mathrm{Stab}\,\mathcal{T} = \{\sigma = (Z, \mathcal{P})$ "locally finite" stability conditions$\}$ has a natural structure of a complex manifold.

2) There are commuting group actions

$\Phi \in \mathrm{Aut}\,\mathcal{T} \curvearrowright \mathrm{Stab}\,\mathcal{T} \curvearrowleft \widetilde{GL_+(2,\mathbb{R})} \ni (T, f)$ where $T \in GL_+(2,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is increasing with $f(t+1) = f(t) + 1$ s.t. T and f induce the same map on $S^1 \cong \mathbb{R}^2 - \{0\} / \mathbb{R}_{>0} \cong \mathbb{R}/2\mathbb{Z}$

given by

$$\Phi \cdot (Z, \mathcal{P}) = (Z \circ K(\Phi)^{-1}, \mathcal{P}_\Phi(\phi) = \Phi(\mathcal{P}(\phi)))$$

$$(Z, \mathcal{P}) \cdot (T, f) = (T^{-1} \circ Z, \mathcal{P}_f(\phi) = \mathcal{P}(f(\phi)))$$

Def^n For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}\,\mathcal{T}$

i) The heart of σ is $\mathcal{P}(0,1] := \langle \mathcal{P}(\phi) : \phi \in (0,1] \rangle$ ← ext closed subcategory generated by these cats.

ii) $0 \neq E \in \mathcal{T}$ is semistable if $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$.

Rmk:
- The action of $\widetilde{GL_+(2,\mathbb{R})}$ preserves the semistable objects, but relabels their phases (so the heart can change).
- For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}\,\mathcal{T}$, the stabiliser group $\widetilde{GL_+(2,\mathbb{R})}_\sigma \neq \mathrm{Id}$ $\iff$ Image $Z \subseteq \mathbb{R}$-line through 0 in $\mathbb{C}$.

Proposition (Okada)

The subgroup $H \cong \mathbb{C}^*$ of $\mathbb{C}$-linear maps in $GL_+(2,\mathbb{R})$ lifts via the exponential to an additive subgroup $\tilde{H} \cong \mathbb{C}^+ \subseteq \widetilde{GL_+(2,\mathbb{R})}$ which acts freely and holomorphically on $\operatorname{Stab} \mathcal{T}$. For $n \in \mathbb{Z}$, the action of $-\pi i n \in \mathbb{C} \cong \tilde{H}$ coincides with the action of $[n] \in \operatorname{Aut} \mathcal{T}$. Furthermore $\operatorname{Stab}\mathcal{T}/\mathbb{C}$ is a complex manifold.

<u>Proof</u>: The isomorphism $\mathbb{C} \cong \tilde{H} \subseteq \widetilde{GL_+(2,\mathbb{R})}$ is given by

$$\lambda \in \mathbb{C} \mapsto \left(\begin{array}{cc} T_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, & f_\lambda : \mathbb{R} \to \mathbb{R} \\ (a+ib) \mapsto (a+ib)\cdot e^{\lambda} & t \mapsto t - \frac{\operatorname{Im}\lambda}{\pi} \end{array} \right) \in \widetilde{GL_+(2,\mathbb{R})}$$

Let us check that T_λ^{-1} and f_λ induce the same map on S^1

$$\begin{tikzcd} 1 \arrow[r] & 2\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{R} \arrow[r] \arrow[d, "f_\lambda"'] & S^1 \arrow[r] \arrow[d, "\bar{f}_\lambda"] & 1 \\ 1 \arrow[r] & 2\mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1 \arrow[r] & 1 \end{tikzcd}$$

$$\begin{array}{ccc} t & \longmapsto & e^{i\pi t} \\ \mathrm{I} & & \mathrm{I} \\ t - \frac{\operatorname{Im}\lambda}{\pi} & \longmapsto & e^{i\pi t} e^{-i\operatorname{Im}\lambda} \end{array}$$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{R}_{>0} \arrow[r] \arrow[d, equal] & \mathbb{R}^2 - \{0\} \arrow[r] \arrow[d, "T_\lambda^{-1}"'] & S^1 \arrow[r] \arrow[d, "\bar{T}_\lambda^{-1}"] & 1 \\ 1 \arrow[r] & \mathbb{R}_{>0} \arrow[r] & \mathbb{R}^2 - \{0\} \arrow[r] & S^1 \arrow[r] & 1 \end{tikzcd}$$

use polar coords

$$\begin{array}{ccc} re^{i\theta} & \longmapsto & e^{i\theta} \\ \mathrm{I} & & \mathrm{I} \\ re^{i\theta}e^{-\lambda} & \longmapsto & e^{i(\theta - \operatorname{Im}\lambda)} \end{array}$$

$$re^{-\operatorname{Re}\lambda} e^{i(\theta - \operatorname{Im}\lambda)} \longmapsto e^{i\theta} e^{-i\operatorname{Im}\lambda}$$

$\Rightarrow \bar{T}_\lambda = \bar{f}_\lambda =$ multiplication by $e^{i\operatorname{Im}\lambda}$.

The $\mathbb{C}^+$-action is free:

$$(Z,\mathcal{P}) \cdot \lambda = (Z,\mathcal{P}) \Rightarrow f_\lambda = \mathrm{Id} \ \& \ e^{-\lambda} = 1$$
$$\Downarrow$$
$$\operatorname{Im}\lambda = 0 \ \& \ \operatorname{Re}\lambda = 0.$$

If $\lambda = -\pi i n$, then $e^{\lambda} = (-1)^n$, $f_\lambda(t) = t + n$

$$(Z,\mathcal{P}) \cdot \lambda = ((-1)^n Z, \mathcal{P}[n]) = [n] \cdot (Z,\mathcal{P}).$$

Let $\Sigma \subseteq \operatorname{Stab} \mathcal{D}$ be a connected component & consider the local homeo $\Sigma \xrightarrow{\text{forget slicing}} V(\Sigma) \subseteq \operatorname{Hom}(K(\mathcal{T}),\mathbb{C})$ (see [B])
linear subspace

This is equivariant:

$$\begin{tikzcd} \Sigma \arrow[r, "\text{forget slicing}"] & V(\Sigma) \\ \mathbb{C}^+ \arrow[u, bend left] \arrow[r, "\exp"'] & \mathbb{C}^* \arrow[u, bend left] \end{tikzcd}$$

scaling action (holomorphic)

$\Rightarrow \mathbb{C}^+ \curvearrowright \operatorname{Stab}\mathcal{D}$ is holomorphic & $\operatorname{Stab}\Sigma/\mathbb{C}$ is locally modelled on $\mathbb{P}(V(\Sigma))$. $\boxtimes$

<u>Def^n</u>: $\mathcal{T} = D^b(X)$ for X smooth proj variety

$K(X) = K(\mathcal{T}) \twoheadrightarrow \mathcal{N}(X) := K(X)/\ker\langle\, , \rangle$ where $\langle \mathcal{E},\mathcal{F}\rangle = \sum_i (-1)^i \operatorname{ext}^i(\mathcal{E},\mathcal{F})$

"numerical grothendieck gp"   Euler pairing

Theorem (Bridgeland)

$\mathrm{Stab}_N(X) \subseteq \mathrm{Stab}(D^b(X))$ is a finite dim^e complex mfld.

$\mathrm{Stab}_N(X) = \{\sigma = (Z, P) : Z \text{ factors via } \mathcal{N}(X)\}$

Pf: Its modelled locally on linear subspaces of $\mathrm{Hom}(\mathcal{N}(X), \mathbb{C})$, which is a f-dim^e $\mathbb{C}$-v.space.

§2 Stability conditions on curves

Let C be a smooth connected cx proj. curve of genus g.

Serre duality on $D(C) := D^b(\mathrm{Coh}\, C)$:

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, A \otimes \omega_C[1])$$

Proposition: There is an isomorphism

$$K(C) := K(D(C)) \xrightarrow{(\det, \mathrm{rk})} \mathrm{Pic}(C) \oplus \mathbb{Z}$$

Pf: Define homomorphism on locally free sheaves using det & rk. Then extend to $K(C) = K(\mathrm{Coh}\, C)$ by resolving any coh sheaf by finitely many locally frees (recall C is smooth). This is clearly surjective. For injectivity, study the excision sequence for removing a pt using C-pt is affine. $\blacksquare$

Rmk

$$K(C) \cong \mathrm{Pic}(C) \oplus \mathbb{Z} \cong \begin{cases} \mathbb{Z}^2 & g = 0 \quad \left(\text{Grothendieck's Thm on } \mathbb{P}^1\right) \\ \underbrace{\mathrm{Jac}(C)}_{\text{uncountable.}} \oplus \mathbb{Z}^2 & g > 0 \end{cases}$$

Lemma $\mathcal{N}(C) \cong \mathbb{Z}^2$

Proof For $\mathcal{E}, \mathcal{F}$ locally free $\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = H^i(\mathcal{E}^\vee \otimes \mathcal{F})$

$$\Rightarrow \langle \mathcal{E}, \mathcal{F} \rangle = \chi(\mathcal{E}^\vee \otimes \mathcal{F}) \underset{\text{(Riemann-Roch)}}{=} \deg(\mathcal{E}^\vee \otimes \mathcal{F}) - \mathrm{rk}(\mathcal{E}^\vee \otimes \mathcal{F})(1-g)$$

$$= -d_{\mathcal{E}} r_{\mathcal{F}} + r_{\mathcal{E}} d_{\mathcal{F}} - r_{\mathcal{E}} r_{\mathcal{F}} (1-g)$$

Hence kernel $\langle \, , \rangle = \{\mathcal{E} : \deg \mathcal{E} = \mathrm{rk}\, \mathcal{E} = 0\}$

$$K(C) \twoheadrightarrow \mathcal{N}(C) \cong \mathbb{Z}^2 \cong K(C) / \ker \langle \, , \rangle.$$

$$\mathcal{E} \longmapsto (\mathrm{rk}\, \mathcal{E}, \deg \mathcal{E})$$

$\blacksquare$

§3 The stability manifold for $g>0$

<u>Theorem</u> (Bridgeland $g=1$, Macrì $g \geq 1$)

The group $\widetilde{GL_+(2,\mathbb{R})}$ acts freely and transitively on $\mathrm{Stab}\, C := \mathrm{Stab}_N D^b(C)$ where C is a smooth proj curve genus >0.

In particular, $\mathrm{Stab}\, C \cong \widetilde{GL_+(2,\mathbb{R})} \cong \mathbb{C} \times \mathbb{H}$. ← open upper half plane

We will prove this result after a few lemmas.

Throughout this section, C is a genus $g>0$ smooth proj cx curve.

<u>Lemma 1</u> (GKR) Let
$$\begin{tikzcd} A \arrow[r] & E \arrow[d] \\ & B \arrow[ul] \end{tikzcd}$$
be a distinguished triangle in $D(C)$.

If $E \in \mathrm{Coh}\, C$ and $\mathrm{Hom}(A, B[k]) = 0 \ \forall\, k \leq 0$, then $A, B \in \mathrm{Coh}\, C$.

<u>Proof</u> As $D(C)$ has homological dimension 1:
$$A \simeq \bigoplus A_i[-i] \quad \& \quad B \simeq \bigoplus B_i[-i] \quad \text{for } A_i, B_i \in \mathrm{Coh}\, C.$$

In fact, the l.e.s in cohomology sheaves associated to the above distinguished triangle & $\mathrm{Hom}^{\leq 0}(A,B) = 0 \implies$
$$A \simeq A_0 \oplus A_1[-1] \quad \& \quad B \simeq B_0 \oplus B_{-1}[1]$$
and $0 \to B_{-1} \xrightarrow{\varphi} A_0 \to E \to B_0 \xrightarrow{\psi} A_1 \to 0$ is exact.

If $\varphi \neq 0$, then as $g>0$, ie. $\omega_C = \mathcal{O}_C$ $(g=1)$ or is effective $(g>1)$, we can tensor φ by a non-zero section $s \in H^0(\omega_C)$.

Then $B_{-1} \xrightarrow{\varphi \otimes s} A_0 \otimes \omega_C$ is non-zero

$$\implies \quad \mathrm{Hom}(B, A \otimes \omega_C[1]) \neq 0$$
$$\wr\wr \ \text{Serre Duality}$$
$$\mathrm{Hom}(A, B) \ \text{which is zero by assumption} \ \Rightarrow\!\Leftarrow$$

Hence $\varphi = 0$ and $B_{-1} = 0$. Similarly $\psi = 0$ and $A_1 = 0$. ∎

<u>Proposition</u> (Macrì)

For any $\sigma \in \mathrm{Stab}\, C$, every line bundle $\mathcal{L}$ is σ-stable and the structure sheaves $\mathcal{O}_x$ of points $x \in C$ is σ-stable.

__Proof__ Let $E = \mathcal{L}$ or $\mathcal{O}_x$. We first show E is σ-semistable.

If E is not σ-semistable, consider the final Δ in its Harder-Narasimhan "filtration"

$$\begin{tikzcd} A \arrow[r] & E \arrow[d] \\ & B \arrow[ul, dashed] \end{tikzcd}$$

ie B is σ-ss

$\phi(E) > \phi(B)$

$\Rightarrow \phi(A) > \phi(E) > \phi(B)$

$\Rightarrow \mathrm{Hom}(A, B[k]) = 0 \quad \forall\, k \leq 0$

By the lemma $A, B \in \mathrm{Coh}\, C$.

(*) If $E = \mathcal{O}_x$, $\nexists$ s.e.s $0 \to A \to E \to B \to 0$ in $\mathrm{Coh}\, C$.

If $E = \mathcal{L} \Rightarrow A$ is a line bundle & B is torsion, but then $\mathrm{Hom}(A,B) \neq 0$ $\Rightarrow\!\Leftarrow$

Hence, E is σ-semistable.

If E is not σ-stable consider its Jordan-Holder filtration (which is not nec. unique)

$$\begin{tikzcd} 0 = E^{(0)} \arrow[r] & E^{(1)} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{(s-1)} \arrow[r] & E^{(s)} = E \arrow[d] \\ & A_1 \arrow[ul, dashed] & & & A_s \arrow[ul, dashed] \end{tikzcd}$$

where $A_i \in \mathcal{P}(\phi)$ are all stable of phase $\phi = \phi(E)$.

__Claim__: $A_i \cong A \quad \forall\, i$

__Pf__: If $A_i \ncong A_{i+1} \Rightarrow \mathrm{Hom}(A_i, A_{i+1})$ (any non-zero map of stable objects with the same slope is an isomorphism)

Then $\mathrm{Hom}(A_i, A_{i+1}[k]) = 0 \ \forall\, k \leq 0$

By Lemma 1, we conclude A_i & $E^{(i)} \in \mathrm{Coh}\, C$

The argument at (*) gives a contradiction ✓

We claim $A \in \mathrm{Coh}\, C$. Associated to the J-H filtration above, there is a spectral sequence whose entries are the cohomology sheaves of A that converges to the cohomology of E (which is zero except in degree 0). Since $H^i(A) = 0 \ \forall\, i$ suff small & large $\Rightarrow A \in \mathrm{Coh}\, C$.

Then we get a contradiction as at (*). Hence E is σ-stable. ∎

__Proof of Theorem__:

For any $\sigma \in \mathrm{Stab}\, C$, $\mathcal{O}_x$ and $\mathcal{L}$ are σ-stable, but not isomorphic. Therefore the non-zero maps $\mathcal{L} \to \mathcal{O}_x$ (and by S.D. $\mathcal{O}_x \to \mathcal{L}[1]$) give inequalities:

$$(\dagger) \quad \phi_\sigma(\mathcal{L}) < \phi_\sigma(\mathcal{O}_x) < \phi_\sigma(\mathcal{L}[1]) = \phi_\sigma(\mathcal{L}) + 1.$$

Hence Image $Z \not\subset \mathbb{R}$-line $\Rightarrow GL_+(2,\mathbb{R})$ acts freely by the Remark on page 1.

To show the action is transitive, we show $\forall\, \sigma \;\; \exists\, (T, f) \in \widetilde{GL_+(2,\mathbb{R})}$ s.t $\sigma \cdot (T, f) = \sigma_\mu$ where

$\sigma_\mu = (Z_\mu, \mathcal{P}_\mu)$ is the <u>slope stability condition</u>:

$$Z_\mu(E) = -\deg E + i \operatorname{rk} E \quad \& \quad \mathcal{P}_\mu(0,1] = \operatorname{Coh} C.$$

$$0 < \phi_\mu(\mathcal{L}) < \phi_\mu(\mathcal{O}_x) = 1 < \phi_\mu(\mathcal{L}) + 1$$

By comparing these inequalities with $(+)$, we can find $T: \mathbb{R}^2 \to \mathbb{R}^2$ orientation preserving and $f: \mathbb{R} \to \mathbb{R}$ s.t T & f induce the same map on S^1

$$Z'(\mathcal{O}_x) = -1$$
$$Z'(\mathcal{O}_C) = i \qquad \text{where } \sigma' = \sigma \cdot (T, f)$$
$$\phi_{\sigma'}(\mathcal{O}_x) = 1$$

Since Z & Z' factor via $\mathcal{N}(C) \cong \mathbb{Z}^2$ (with generators $\mathcal{O}_x, \mathcal{O}_C$)

$\Rightarrow \; Z'(E) = -\deg E + i \operatorname{rk} E = Z_\mu(E).$

By construction $\operatorname{Coh} C \subseteq \mathcal{P}'(0,1]$ as torsion sheaves have phase 1 & line bundles have phase $\in (0,1)$ by $(+)$

$\Rightarrow$ every coh sheaf has HN filtr with factors of slope $\in (0,1]$.

Since $\operatorname{Coh} C$ & $\mathcal{P}'(0,1]$ are both hearts of bdd t-structures on $D(C)$

$\Rightarrow \operatorname{Coh} C = \mathcal{P}'(0,1]$ (If $\exists\, A \in \mathcal{P}'(0,1] - \operatorname{Coh} C$, then for $k > 0$ either
1) $\exists\; B[k] \xrightarrow{\neq 0} A$
2) $\exists\; B \xrightarrow[\neq 0]{} A[-k]$

for $B \in \operatorname{Coh} C \rightsquigarrow$ contradicts $\mathcal{P}'$ being a slicing $\operatorname{Hom}(\mathcal{P}'(k,k+1], \mathcal{P}'(0,1]) \neq 0$)

Hence $\sigma' = \sigma_\mu$ as required.

Finally $\widetilde{GL_+(2,\mathbb{R})} \cong \mathbb{C} \times \mathbb{H}$ due to the following diagram:

$$\begin{tikzcd}
 & & & & 1 \arrow[d] & \\
1 \arrow[r] & \mathbb{Z} = \pi_1(SL(2,\mathbb{R})) \arrow[r] \arrow[dd, equals] & \widetilde{SL(2,\mathbb{R})} \arrow[r] \arrow[dd] \arrow[dr, "\mathbb{R}"] & & SL(2,\mathbb{R}) \arrow[r] \arrow[dd] \arrow[dl, "S^1"'] & 1 \\
 & & & \mathbb{H} & & \\
1 \arrow[r] & \mathbb{Z} = \pi_1(GL_+(2,\mathbb{R})) \arrow[r] & \widetilde{GL_+(2,\mathbb{R})} \arrow[rr] \arrow[ur, "\mathbb{C}"] & & GL_+(2,\mathbb{R}) \arrow[r] \arrow[ul, "\mathbb{C}^*"'] \arrow[d] & 1 \\
 & & & & \mathbb{R}_{>0} \arrow[d] & \\
 & & & & 1 &
\end{tikzcd}$$

$SL(2,\mathbb{R}) \curvearrowright \mathbb{H}$ transitively with stabiliser S^1

Iwasara decomposition:

$$SL(2,\mathbb{R}) = S^1 \cdot D \cdot N$$

$$S^1 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\} \quad D = \left\{ \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$$

∎

§4 The stability manifold of an elliptic curve

Bridgeland & Douglas have conjectured that for a projective Calabi-Yau manifold X, the stability manifold $\mathrm{Stab}\,X$ is related to a stringy Kähler moduli space $\mathcal{M}_K(X)$.

If X & Y are a mirror pair of CYs (Hodge diamonds related by rotⁿ by $\frac{\pi}{2}$)

$$\left[\ \underset{\substack{\text{B-side}\\ \text{(holomorphic)}}}{D(X)} \cong \underset{\substack{\text{A-side}\\ \text{(symplectic)}}}{DFuk(Y,\omega)} \qquad \text{"Homological mirror symmetry" (Kontsevich)}\ \right]$$

then $\mathcal{M}_K(X) \cong \mathcal{M}_{\mathbb{C}}(Y_{top}) :=$ moduli space of complex structures on Y_{top} up to diffeomorphism.

<u>Conjecture</u> (Bridgeland, Douglas)

For a proj CY manifold X

$$\mathcal{M}_K(X) \hookrightarrow {}_{\mathrm{Aut}\,D(X)}\backslash^{\mathrm{Stab}\,X}/\mathbb{C}$$

<u>Theorem</u> (Bridgeland)

The conjecture holds in dimension 1.

For an elliptic curve X, $\mathcal{M}_K(X) = {}_{\mathrm{Aut}\,D(X)}\backslash^{\mathrm{Stab}\,X}/\mathbb{C} = \underbrace{\mathbb{H}/SL(2,\mathbb{Z})}_{\text{modular curve.}}$

<u>Proof</u>: A 1-dimˡ proj CY manifold is an elliptic curve X.

$\mathrm{Aut}\,D(X)$ is generated by

$$\left\{\begin{array}{lll} \check{X} = \mathrm{Pic}^0(X) & \text{twist by deg 0 line bdle} & \multirow{2}{*}{$\Big\}$ act trivially on Stab X} \\ X = \mathrm{Aut}(X) & \text{translation} & \\ \mathbb{Z} & \text{shifts } [n] & \leftarrow \text{contained in } \mathbb{C}\text{-action.} \\ S & = \text{twist by deg 1 line bundle} & \\ T & = \text{Fourier-Mukai transform} & \\ & \quad \text{associated to Poincaré bdle on } X \times \check{X}. & \end{array}\right.$$

Moreover $\langle S, T\rangle = SL(2,\mathbb{Z})$.

We have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \times \underbrace{X \times \check{X}}_{\text{acts trivially}} \longrightarrow \mathrm{Aut}\,D(X) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1.$$

Hence

$$\begin{array}{ccc} {}_{\mathrm{Aut}\,D(X)}\backslash^{\mathrm{Stab}\,X} & = & \dfrac{\widetilde{GL_+(2,\mathbb{R})}}{\widetilde{SL(2,\mathbb{Z})}} \\ \Big\downarrow \ \mathbb{C}\text{-bdle} & & \\ {}_{\mathrm{Aut}\,D(X)}\backslash^{\mathrm{Stab}\,X}/\mathbb{C} & = & \dfrac{GL_+(2,\mathbb{R})/\mathbb{C}}{SL(2,\mathbb{Z})} = \mathbb{H}/SL(2,\mathbb{Z}) = \mathcal{M}_{\mathbb{C}}(X_{top}) \end{array}$$

moduli space of cx structures on a topological 2-torus

$$\mathcal{M}_{\mathbb{C}}(X_{top}) \overset{\text{torus self dual}}{=} \mathcal{M}_{\mathbb{C}}(\check{X}_{top}) \overset{\cdot}{=} \mathcal{M}_K(X) \qquad \blacksquare$$