Bridgeland Stability Conditions on $\mathbb{P}^1$

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§1 Overview of proof

$K(\mathbb{P}^1) = \mathcal{N}(\mathbb{P}^1) = \mathbb{Z}^2 \Rightarrow \dim_{\mathbb{C}} \mathrm{Stab}\, \mathbb{P}^1 = 2.$

Recall, we have actions:

$$\begin{tikzcd}
\mathrm{Aut}\, D(\mathbb{P}^1) \arrow[r, phantom, "\curvearrowright"] \arrow[d] & \mathrm{Stab}\, \mathbb{P}^1 \arrow[r, phantom, "\curvearrowleft"] & \widetilde{GL_+(2,\mathbb{R})} \arrow[d, phantom, "\cup" rotate=90] & \\
\text{There is a s.e.s of gps} & & \mathbb{C}^+ & \arrow[l] \text{acts holomorphically \& } M = \mathrm{Stab}\, \mathbb{P}^1/\mathbb{C} \text{ is a complex manifold. (see Talk I)}
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \times \underbrace{\mathrm{Pic}\, \mathbb{P}^1}_{\cong \, \mathbb{Z}} \arrow[r] & \mathrm{Aut}\, D(\mathbb{P}^1) \arrow[r] & \mathrm{Aut}\, \mathbb{P}^1 \arrow[r] \arrow[d, equal] & 1 \\
\text{shifts (contained in } \mathbb{C}\text{-action)} \arrow[u] & & & PGL(2,\mathbb{C}) & \\
& & & \text{acts trivially} \arrow[u] &
\end{tikzcd}$$

Main Theorem (Okada)

Stab $\mathbb{P}^1 \cong \mathbb{C}^2$.

We'll prove this result assuming 2 results, which we prove later.

Lemma (*) M is connected.

Theorem (**) There is a fundamental domain $K \subseteq \mathrm{Stab}\, \mathbb{P}^1$ for the action of $\mathrm{Pic}\, \mathbb{P}^1 \times \mathbb{C}$ that is conformally equivalent to $\mathbb{C}^*$.

Proof of the main theorem:

The fibration

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & M := \mathrm{Stab}\, \mathbb{P}^1/\mathbb{C} \arrow[d] \\
& M/\mathbb{Z} \cong K \cong \mathbb{C}^* \text{ (by Thm (**))}
\end{tikzcd}$$

gives a long exact sequence in homotopy:

$$\begin{tikzcd}
1 = \pi_1(\mathbb{Z}) \arrow[r] & \pi_1(M) \arrow[r] & \pi_1(M/\mathbb{Z}) \arrow[r, "\alpha"] \arrow[d, phantom, "\cong" rotate=90] & \pi_0(\mathbb{Z}) \arrow[r] \arrow[d, equal] & \pi_0(M) \arrow[r] \arrow[d, equal] & \pi_0(M/\mathbb{Z}) \arrow[d, equal] \\
& & \pi_1(\mathbb{C}^*) \cong \mathbb{Z} & \mathbb{Z} & 1 & 1 \\
& & & & \text{by Lemma (*)} \arrow[u] &
\end{tikzcd}$$

$\Rightarrow \alpha : \mathbb{Z} \to \mathbb{Z}$ surjective.

As every homo $\mathbb{Z} \to \mathbb{Z}$ is given by multiplication by n, we have α is an isomorphism. Hence, $\pi_1(M) = 0$ i.e. M is the universal covering of $\mathbb{C}^* \Rightarrow M \cong \mathbb{C}$.

Finally as every $\mathbb{C}^+$-bundle on $\mathbb{C}$ is trivial, we have

$$\mathrm{Stab}\, \mathbb{P}^1 \xrightarrow[\mathbb{C}^+\text{-bdle}]{} M = \mathrm{Stab}\, \mathbb{P}^1/\mathbb{C} \cong \mathbb{C} \Rightarrow \mathrm{Stab}\, \mathbb{P}^1 \cong \mathbb{C}^2. \qquad \blacksquare$$

§2 The Kronecker heart of $D(\mathbb{P}^1)$

By tilting the standard heart $\mathcal{A} = \mathrm{Coh}\,\mathbb{P}^1$, we obtain the Kronecker heart $\mathcal{B} \cong \mathrm{Rep}(\cdot \rightrightarrows \cdot)$.

Proposition The pair

$$\mathcal{T} = \langle \mathcal{O}(n),\ n \geq 0;\ \mathcal{O}_x,\ x \in \mathbb{P}^1 \rangle^{\oplus}$$

$$\mathcal{F} = \langle \mathcal{O}(n),\ n < 0 \rangle^{\oplus}$$

give a torsion thy on $\mathcal{A} = \mathrm{Coh}\,\mathbb{P}^1 = \underset{\substack{\text{standard}\\ t\text{-structure}}}{\underbrace{D^{\leq 0} \cap D^{\geq 0}}}$

with tilted t-structure

$${}^tD^{\leq 0} = \{ X \in D(\mathbb{P}^1) : H^i(X) = 0\ \forall i > 0\ \&\ H^0(X) \in \mathcal{T} \} \subseteq D^{\leq 0}$$

$${}^tD^{\geq 0} = \{ X \in D(\mathbb{P}^1) : H^i(X) = 0\ \forall i < -1\ \&\ H^{-1}(X) \in \mathcal{F} \} \subseteq D^{\geq -1}$$

and heart $\mathcal{B} = {}^t\mathcal{A} = \langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle^{ext}$.

Proof: To prove $(\mathcal{T}, \mathcal{F})$ are a torsion theory, we need to check:

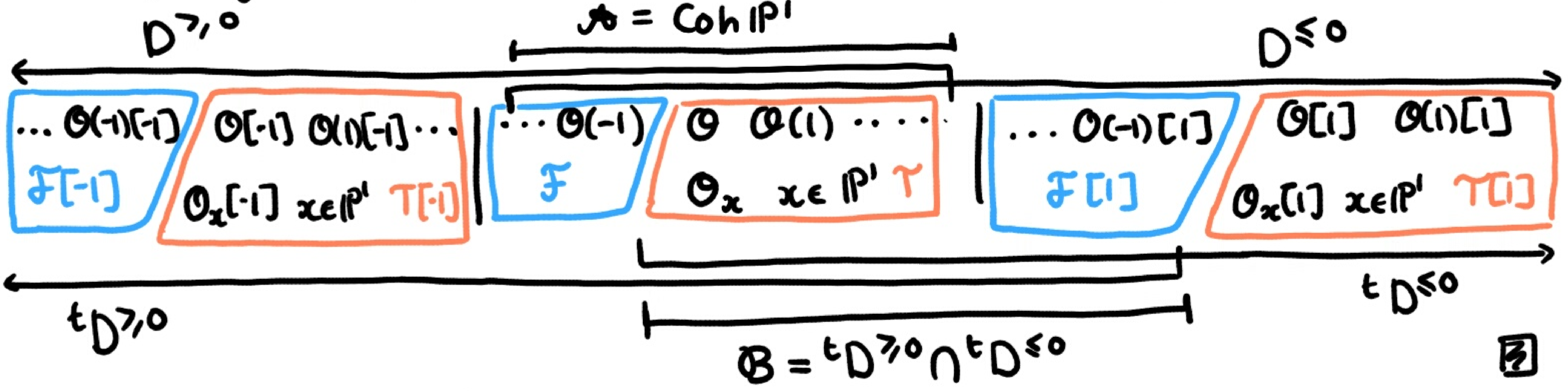
(i) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$, which is clear as every homo $\mathcal{O}_x \to \mathcal{O}(n)$ & $\mathcal{O}(n) \to \mathcal{O}(m)$ for $n > m$ is zero.

(ii) For $E \in \mathrm{Coh}\,\mathbb{P}^1$, by Grothendieck's Thm we have

$$E = \underbrace{\bigoplus_x \mathcal{O}_x^{\oplus m_x} \oplus \bigoplus_{n \geq 0} \mathcal{O}(n)^{\oplus r_n}}_{\overset{\shortparallel}{T}\ \in\ \mathcal{T}} \oplus \underbrace{\bigoplus_{n<0} \mathcal{O}(n)^{\oplus r_n}}_{\overset{\shortparallel}{F}\ \in\ \mathcal{F}}$$

and $0 \to T \to E \to F \to 0$ is a s.e.s.

Picture of the standard t-structure & tilted t-structure:



$\square$

Proposition $D^b(\mathcal{B}) = D^b(\mathcal{A}) = D(\mathbb{P}^1)$

Proof 1 : $(\mathcal{T}, \mathcal{F})$ are cotilting : ie $\forall E \in \mathcal{A}\ \exists\ F \in \mathcal{F}$ s.t. $F \twoheadrightarrow E$

Indeed for $n \gg 0$, $\underbrace{H^0(E(n)) \otimes \mathcal{O}(-n)}_{\in\,\mathcal{F}} \to E$ is surjective

Proof 2 (Argument of Bondal)

Let $T = \mathcal{O} \oplus \mathcal{O}(1)$ and $A = \mathrm{End}\,T = \mathbb{C}(\cdot \overset{x}{\underset{y}{\rightrightarrows}} \cdot)$ path algebra of the Kronecker quiver $Q = \cdot \rightrightarrows \cdot$

$$\mathrm{Hom}(\mathcal{O},\mathcal{O}) \to \begin{pmatrix} \mathbb{C} & 0 \\ \langle x,y \rangle & \mathbb{C} \end{pmatrix} \leftarrow \mathrm{Hom}(\mathcal{O}(1),\mathcal{O}(1))$$

$\mathrm{Hom}(\mathcal{O},\mathcal{O}(1)) \to \langle x,y \rangle$

Then we claim the following functors are quasi-inverse:

$$F = R\mathrm{Hom}(T,-) : D(\mathbb{P}^1) \rightleftarrows D^b(A\text{-mod}) = D^b(\mathrm{Rep}\,Q) : \; G = - \overset{\mathbb{L}}{\otimes}_A T.$$

We have

(1) $F \circ G = \mathrm{Id}$ (as $\mathrm{Hom}^{>0}(T,T) = 0 \Rightarrow R\mathrm{Hom}(T, E \overset{\mathbb{L}}{\otimes}_A T) = E$)

(2) $G \circ F \cong \mathrm{Id}$ as $F(E) \neq 0 \; \forall \, E \neq 0 \leftarrow$ (resolve any cx by line bundles & torsion sheaves + use Serre duality)

therefore if we apply F to the distinguished triangle

$$\begin{tikzcd} G \circ F(E) \arrow[r, "\eta_E"] & E \arrow[d] \\ & \mathrm{Cone}\,\eta_E \arrow[ul, dashed] \end{tikzcd} \quad \longrightarrow \quad \begin{tikzcd} F \circ G \circ F(E) \arrow[r, "\cong \text{ by } (1)"] & F(E) \arrow[d] \\ & F(\mathrm{Cone}\,\eta_E) \cong 0 \arrow[ul, dashed] \end{tikzcd} \Rightarrow \mathrm{Cone}\,\eta_E = 0$$

i.e. η_E is an isomorphism. $\square$

where η_E is the counit of the adjunction

Remark: $F(\mathcal{O}) = (\overset{\mathbb{C}}{\bullet} \rightrightarrows \overset{0}{\bullet}) = S_0$
$F(\mathcal{O}(-1)[1]) = (\underset{0}{\bullet} \rightrightarrows \underset{\mathbb{C}}{\bullet}) = S_1$ } simple reps of Q

$0 = \mathrm{Hom}(\mathcal{O}(1), \mathcal{O})^* \underset{S.D}{=} \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-1)[1])$ $\quad \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(-1)[1]) \overset{S.D}{=} \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)) = \mathbb{C}$

Hence $\mathbb{B} = \langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle^{ext} \cong \langle S_0, S_1 \rangle^{ext} = \mathrm{Rep}\,Q = A\text{-mod}$

is called the <u>Kronecker heart</u>.

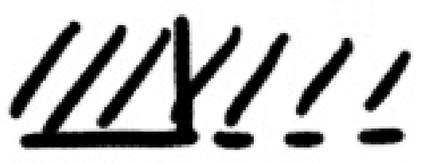
We'll soon see there are more stability conditions with heart $\mathbb{B}$ than $\heartsuit = \mathcal{A}$.

<u>Proposition A</u> Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}\,\mathbb{P}^1$ with $\mathcal{P}(0,1] = \mathcal{A} = \mathrm{Coh}\,\mathbb{P}^1$. Then Z is determined by $Z(\mathcal{O}_x) \in \mathbb{R}_{<0}$ and $Z(\mathcal{O}) \in \mathbb{H} = \{ z \in \mathbb{C} : \mathrm{Im}\, z > 0 \}$.

Furthermore
- all line bundles & torsion sheaves are σ-semistable
- $\exists \, \lambda \in \mathbb{C}$ s.t $\sigma' = \sigma . \lambda$ has heart $\mathcal{P}'(0,1] = \mathbb{B}$.

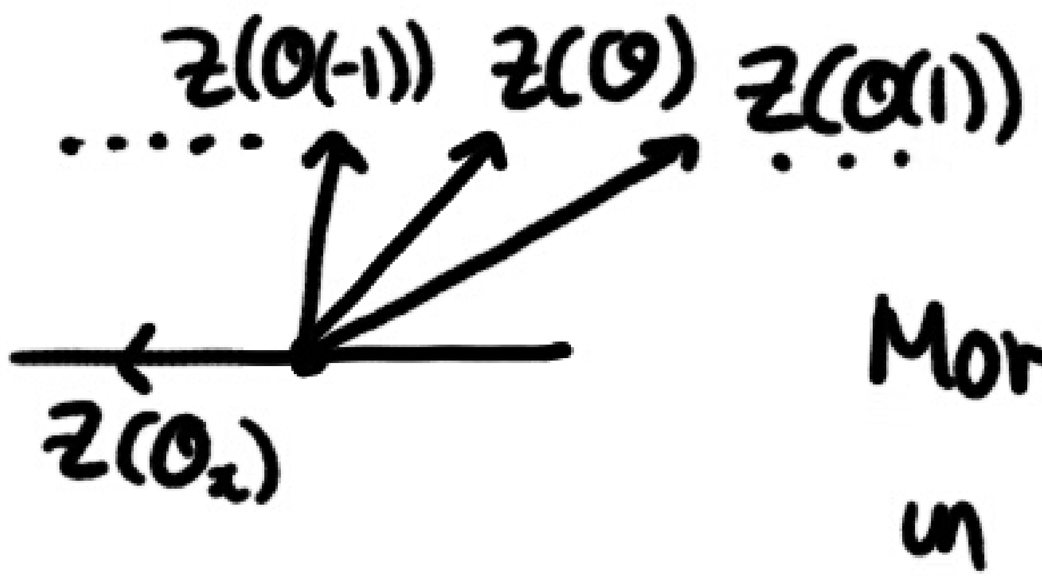
<u>Proof</u>: As $Z : K(\mathbb{P}^1) \longrightarrow \mathbb{C}$ is a gp homomorphism, the s.e.s

$0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O}_x \to 0$ gives: $Z(\mathcal{O}(n)) = Z(\mathcal{O}) + n Z(\mathcal{O}_x)$. (*)

By assumption $\mathcal{P}(0,1] = \mathcal{A} \Rightarrow Z(\mathcal{A}) \subseteq H = \mathbb{H} \cup \mathbb{R}_{\leq 0} =$ .

By (*), we must have $Z(\mathcal{O}_x) \in \mathbb{R}_{<0} \; \forall \, x \in \mathbb{P}^1$ and $Z(\mathcal{O}) \in \mathbb{H}$.

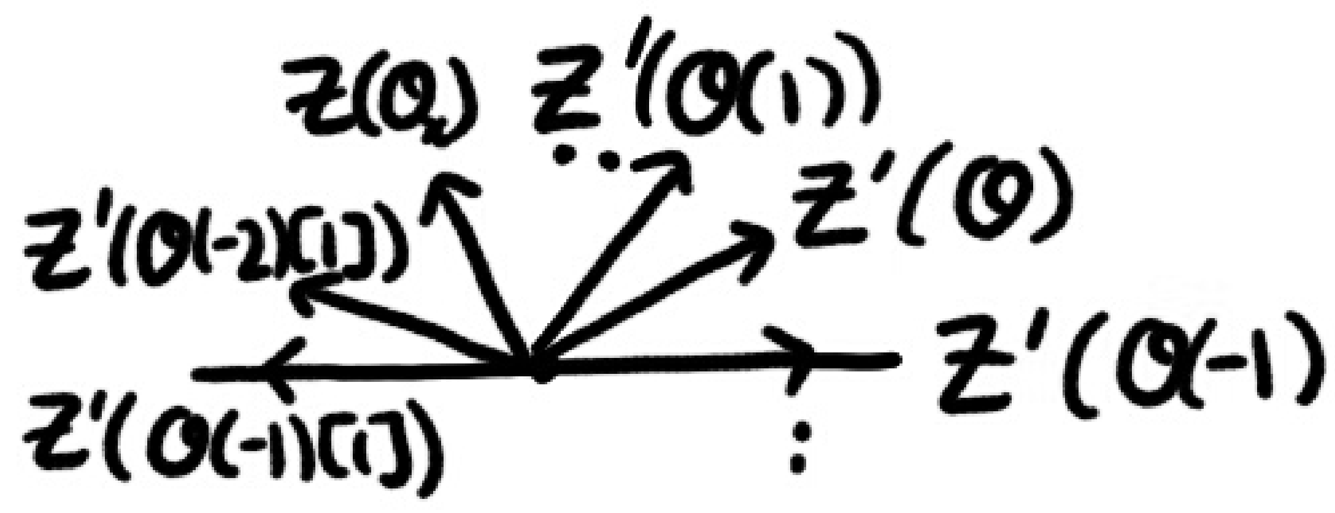
<u>Picture</u>:



Since $\mathcal{O}_x$ has no subobjects in $\mathcal{A}$, it is σ-st.
Moreover $\mathcal{O}(n)$ is σ-stable, as its subobjects $\{\mathcal{O}(k)\}_{k<n}$ in $\mathcal{A}$ have smaller phases (see picture).

Finally, let $\lambda = i\pi \phi_\sigma(\mathcal{O}(-1))$, then the action of λ on Z is $Z' = e^{-\lambda} Z$ which rotates $\mathcal{O}(-1)$ onto the +ve real axis.

Picture :



We have $\phi_{\sigma'}(\mathcal{O}(-1)) = 0 \Rightarrow \phi_{\sigma'}(\mathcal{O}(-1)[1]) = 1$

and $\phi_{\sigma'}(\mathcal{O}) \in (0,1)$.

Since the action of $\lambda \in \mathbb{C}$ does not alter semistability, but reorders the phases, we have $\mathcal{B} = \langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle^{ext} \subseteq \mathcal{P}'(0,1]$, which must be an equality as both $\mathcal{B}$ and $\mathcal{P}'(0,1]$ are hearts of t-structures on $D(\mathbb{P}^1)$. $\square$

$\rightsquigarrow$ i.e. there at least as many stability conditions with $\heartsuit = \mathcal{B}$ as there are with $\heartsuit = \mathcal{A}$.

§3 Stability analysis

Proposition 1 [GKR]

a) In $D(\mathbb{P}^1)$ there are the following families of distinguished triangles

$$\begin{tikzcd} \Delta_1: & \mathcal{O}(k+1)^{\oplus n-k} \arrow[rr] & & \mathcal{O}(n) \arrow[dl] \\ \text{for } n>k+1 & & \mathcal{O}(k)^{\oplus n-k-1} \arrow[ul, dashed] & \end{tikzcd}$$

,

$$\begin{tikzcd} \Delta_2: & \mathcal{O}(k+1)^{\oplus(k-n)} \arrow[rr] & & \mathcal{O}(n) \arrow[dl] \\ \text{for } n<k & & \mathcal{O}(k)^{\oplus(k-n+1)} \arrow[ul, dashed] & \end{tikzcd}$$

and

$$\begin{tikzcd} \Delta_3: & \mathcal{O}(k+1) \arrow[rr] & & \mathcal{O}_x \arrow[dl] \\ \text{for } k\in\mathbb{Z} \ \& \ x \in \mathbb{P}^1 & & \mathcal{O}(k)[1] \arrow[ul, dashed] & \end{tikzcd}$$

(b) If $E = \mathcal{O}_x$ or $\mathcal{O}(n)$ and we have a distinguished Δ:
$$\begin{tikzcd} E' \arrow[rr] & & E \arrow[dl] \\ & F \arrow[ul] & \end{tikzcd}$$
with $\mathrm{Hom}^{\leq 0}(E', F)$, then $\Delta = \Delta_i$ for some $i \in \{1,2,3\}$.

(c) If $E = \mathcal{O}_x$ or $\mathcal{O}(n)$ is σ-unstable for some $\sigma \in \mathrm{Stab}\,\mathbb{P}^1$, then its Harder-Narasimhan "filtration" is given by one of the Δ_i's above.

(d) If $\mathcal{O}(k)$ and $\mathcal{O}(k+1)$ are σ-semistable and $\phi_\sigma(\mathcal{O}(k+1)) > \phi_\sigma(\mathcal{O}(k)[1])$ then $\mathcal{O}_x$ (for $x \in \mathbb{P}^1$) and $\mathcal{O}(n)$ (for $n \notin \{k, k+1\}$) are σ-unstable.

Proof a) The distinguished Δs appear as rotations of s.e.s in $\mathrm{Coh}\,\mathbb{P}^1$

Δ_1: for $n>k+1$, we have a s.e.s

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(k)^{\oplus(n-k+1)} \arrow[r] & \mathcal{O}(k+1)^{\oplus(n-k)} \arrow[r, "ev"] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{O}(n) \arrow[r] & 0 \\ & & \mathrm{Hom}(\mathcal{O}(k+1), \mathcal{O}(n)) \otimes \mathcal{O}(k+1) & & \end{tikzcd}$$

Δ_2: for $n<k$, we have s.e.s

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(n) \arrow[r] & \mathcal{O}(k)^{\oplus(k-n+1)} \arrow[r] & \mathcal{O}(k+1)^{\oplus(k-n)} \arrow[r] & 0 \\ & \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(k)) = H^0(\mathcal{O}(k-n)) \cong \mathbb{C}^{k-n+1} \arrow[u, shift left=12] & & & \end{tikzcd}$$

Δ_3: $0 \to \mathcal{O}(k) \to \mathcal{O}(k+1) \to \mathcal{O}_x \to 0 \quad \forall x \in \mathbb{P}^1, k \in \mathbb{Z}$.

(b) Mostly homological algebra - see [GKR] Lemma 6.3.

(c) If $E=\mathcal{O}(n)$ or $\mathcal{O}_x$ is σ-unstable, let Δ: $$\begin{tikzcd} E' \arrow[r] & E \arrow[d] \\ & F \arrow[ul] \end{tikzcd}$$ be the final Δ in its HN filtration ie F is σ-ss

$$\phi_\sigma(E') > \phi_\sigma(E) > \phi_\sigma(F) \Rightarrow \operatorname{Hom}^{\leq 0}(E', F) = 0$$

By (b), $\Delta = \Delta_i$ for some $i \in \{1,2,3\}$ and $F = \mathcal{O}(k)^{\oplus \ell}[j]$

$E' = \mathcal{O}(k+1)^{\oplus m}[j-1]$ for $j \in \{0,1\}$, $k \in \mathbb{Z}$, $\ell, m \in \mathbb{N}$

It remains to show E' is σ-ss.

Equivalently, we need to show $\mathcal{O}(k+1)$ is σ-semistable.

If not, let Δ': $$\begin{tikzcd} E'' \arrow[r] & \mathcal{O}(k+1) \arrow[dl] \\ F' \arrow[u] & \end{tikzcd}$$ be the final Δ in its HN filtration

σ-ss $\longrightarrow F'$

Again $\phi_\sigma(E'') > \phi_\sigma(F') \Rightarrow \operatorname{Hom}^{\leq 0}(E'', F') = 0$

So by b), $\Delta' = \Delta_{i'}$ with $i' \in \{1,2\}$

$\Rightarrow F' = \mathcal{O}(k')^{\oplus \ell'}[j']$ for $j' \in \{0,1\}$

If $\Delta' = \Delta_1$, then $k+1 > k'+1$ and $j' = 1$

Then $\operatorname{Hom}(\mathcal{O}(k'), \mathcal{O}(k)) \neq 0 \Rightarrow \operatorname{Hom}(F'[j-1], F) \neq 0$ (which is a contradiction as $\phi_\sigma(F'[j-1]) > \phi_\sigma(F)$)

If $\Delta' = \Delta_2$, then $k+1 < k'$ and $j' = 0$

As $0 \neq \operatorname{Ext}^1(\mathcal{O}(k'), \mathcal{O}(k)) \overset{\text{Serre duality}}{=} \operatorname{Hom}(\mathcal{O}(k')[j-1], \mathcal{O}(k)[j])$

$\Rightarrow \operatorname{Hom}(F'[j-1], F) \neq 0$, which also gives a contradiction.

(d) $\Delta_3 \Rightarrow \mathcal{O}_x$ is σ-unstable & $\phi_\sigma(\mathcal{O}(k+1)) > \phi_\sigma(\mathcal{O}_x) > \phi_\sigma(\mathcal{O}(k)[1])$

$\Delta_1 \Rightarrow \mathcal{O}(n)$ is σ-unstable for $n > k+1$

$\Delta_2 \Rightarrow \mathcal{O}(n)$ is σ-unstable for $n < k$. ∎

Theorem 2 [O.]

Let $\sigma \in \operatorname{Stab} \mathbb{P}^1$; then up to the $\operatorname{Pic} \mathbb{P}^1$-action, $\mathcal{O}$ and $\mathcal{O}(-1)$ are σ-ss.

Moreover, we have:

① $\phi(\mathcal{O}(-1)[1]) < \phi(\mathcal{O}) \iff$ the only semistable sheaves are $\mathcal{O}(-1)^{\oplus n}$ and $\mathcal{O}^{\oplus n}$,

② $\phi(\mathcal{O}(-1)[1]) \geq \phi(\mathcal{O}) \iff \mathcal{O}_x$ for $x \in \mathbb{P}^1$ and $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ are all semistable.

In both cases $\boxed{\phi(\mathcal{O}) > \phi(\mathcal{O}(-1))}$ $(+)$

and $\exists\, r \in \mathbb{R}, p \in \mathbb{Z}_{>0}$ s.t $\mathcal{O}, \mathcal{O}(-1)[p] \in \mathcal{P}(r, r+1]$.

Proof: By Prop 1 (c), $\exists\, k \in \mathbb{Z}$ s.t $\mathcal{O}(k)$ and $\mathcal{O}(k+1)$ are σ-ss.

By tensoring with $\mathcal{O}(-k-1) \in \operatorname{Pic} \mathbb{P}^1 \subset \operatorname{Aut} D(\mathbb{P}^1)$, we can assume that $\mathcal{O}$ and $\mathcal{O}(-1)$ are σ-semistable.

① "$\Rightarrow$": $\mathcal{O}_x$ for $x \in \mathbb{P}^1$ and $\mathcal{O}(n)$ for $n \in \mathbb{Z} - \{0,-1\}$ are σ-unstable by Proposition 1 (d).

"$\Leftarrow$" If $\mathcal{O}$ and $\mathcal{O}(-1)$ are σ-semistable, but $\mathcal{O}_x$ & $\mathcal{O}(n)$ for $n \neq 0, -1$ are not, then $\mathcal{O}_x$ has HN filtration

$$\begin{tikzcd}
\mathcal{O} \arrow[rr] & & \mathcal{O}_x \arrow[dl] \\
& \mathcal{O}(-1)[1] \arrow[ul, dashed] &
\end{tikzcd}$$

by Proposition 1 $\Rightarrow \phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1])$.

Pf of (+) in case ①: $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1]) = \phi(\mathcal{O}(-1)) + 1 > \phi(\mathcal{O}(-1))$

Let $r = \phi(\mathcal{O}) - 1$; then $\exists\ p > 0$ s.t $r < \phi(\mathcal{O}(-1)[p]) \leq r+1$.

② "$\Leftarrow$" follows from ① $\Rightarrow$.

"$\Rightarrow$" Let us first show (+):

If $\phi(\mathcal{O}) \leq \phi(\mathcal{O}(-1))$, then as $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ is a non-zero morphism of σ-ss sheaves,

we must have $\phi(\mathcal{O}) = \phi(\mathcal{O}(-1)) \Rightarrow \mathcal{O}_n$ and $\mathcal{O}_x$ are all σ-ss of phase $\phi = \phi(\mathcal{O})$.

$\Rightarrow$ Coh $\mathbb{P}^1[s] \subseteq \mathcal{P}(0,1]$, and as both are $\heartsuit$s, this is an equality.

but then $Z(\text{Coh}\,\mathbb{P}^1[s]) \not\subset H$ (see the formula (*) in Proposition A) which contradicts the definition of σ.

Therefore $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1))$ ie (+) holds.

In this case, $0 \leq \phi(\mathcal{O}(-1)[1]) - \phi(\mathcal{O}) < 1 \Rightarrow \mathcal{O}, \mathcal{O}(-1)[1] \in \mathcal{P}(r, r+1]$ for some r

Then

$$\begin{tikzcd}
\mathcal{O}(-1) \arrow[rr] & & \mathcal{O} \arrow[dl] \\
& \mathcal{O}_x \arrow[ul, dashed] &
\end{tikzcd}$$

$\Rightarrow \phi(\mathcal{O}(-1)) < \phi(\mathcal{O}) < \phi(\mathcal{O}_x)$

Using similar triangles inductively, one shows

$$\phi(\mathcal{O}(-n)) < \cdots < \phi(\mathcal{O}(-1)) < \phi(\mathcal{O}) < \phi(\mathcal{O}(1)) < \cdots < \phi(\mathcal{O}(n)) < \cdots < \phi(\mathcal{O}_x)$$

Then one deduces $\mathcal{O}_x$ and $\mathcal{O}(n)$ are all σ-ss using Proposition 1 (c). ☒

corresponds to (+) in Thm 2

Proposition 3: For any $\boxed{\alpha > \beta - 1}$ and $m_\alpha, m_\beta \in \mathbb{R}_{>0}$, $\exists !\ \sigma \in \text{Stab}\,\mathbb{P}^1$ such that $Z(\mathcal{O}) = m_\alpha e^{i\pi\alpha}$, $Z(\mathcal{O}(-1)[1]) = m_\beta e^{i\pi\beta}$ and $\mathcal{O}, \mathcal{O}(-1)$ are σ-semistable.

Furthermore:

1) If $\alpha = \beta$, then $\mathcal{P}(0,1] = \mathcal{B}[j]$ for some $j \in \mathbb{Z}$

2) If $\alpha > \beta$, then $\mathcal{P}(0,1] = \langle \mathcal{O}(-1)[p+1], \mathcal{O}[q] \rangle^{\text{ext}}$ for $p, q \in \mathbb{Z}$ s.t $\alpha - \beta - 1 < p - q < \alpha - \beta + 1$

3) If $\alpha < \beta$ then
   - a) If $\phi(\mathcal{O}_x) = k \in \mathbb{Z}$, then $\mathcal{P}(0,1] = \mathcal{A}[k]$
   - b) else, $\mathcal{P}(0,1] = \langle \mathcal{O}(k-1)[1+j], \mathcal{O}(k)[j] \rangle^{\text{ext}}$ for some $j, k \in \mathbb{Z}$.

Proof: Since the central charge Z is uniquely determined by $Z(\mathcal{O})$ and $Z(\mathcal{O}(-1)[1])$, it remains to show there is a ! compatible slicing.

We need a slicing $\mathcal{P}$ such that
$$\begin{aligned}\mathcal{P}(\alpha+k) &\geq \langle \mathcal{O}[k]\rangle^{ext}\\ \mathcal{P}(\beta+k) &\geq \langle \mathcal{O}(-1)[1+k]\rangle^{ext}\end{aligned} \quad \text{for } k\in\mathbb{Z}.$$

We construct $\mathcal{P}$ case by case:

1) If $\alpha=\beta$, $\quad \mathcal{P}(\alpha+k) \underset{\uparrow}{\geq} \langle \mathcal{O}(-1)[1],\mathcal{O}\rangle^{ext}[k] = \mathcal{B}[k] \quad \forall\ k\in\mathbb{Z}$

must be equality, as RHS is a heart

Let $\mathcal{P}(\phi)=0 \quad \forall\ \phi\notin\mathbb{Z}+\alpha$. Then $\mathcal{P}$ is a slicing:

$\mathrm{Hom}(\mathcal{P}(\alpha+k+1),\mathcal{P}(\alpha+k))=0$ as $\mathrm{Hom}(\mathcal{O}(-1)[2]\oplus\mathcal{O}[1],\mathcal{O}(-1)[1]\oplus\mathcal{O})=0$.

$\exists\ k\in\mathbb{Z}$ s.t $0<\alpha+k\leq 1$. Then $\mathcal{P}(0,1]=\mathcal{P}(\alpha+k)=\mathcal{B}[k] \quad (*)$ and all non-zero objects are semistable.

Since a stability condition $\Leftrightarrow$ stability function on heart w/ HN property, we see that the construction of $\mathcal{P}$ is uniques, as $(*)$ shows we need $\mathcal{P}(\phi)=0 \quad \forall\ \phi\notin\mathbb{Z}+\alpha$.

2) If $\alpha>\beta$, then by Thm 2: if $\sigma\in\mathrm{Stab}\,\mathbb{P}^1$ and $\underset{\substack{\shortparallel\\ \alpha}}{\phi_\sigma(\mathcal{O})} > \underset{\substack{\shortparallel\\ \beta}}{\phi_\sigma(\mathcal{O}(-1)[1])}$, then the only σ-stable sheaves are $\mathcal{O}$ and $\mathcal{O}(-1)$.

Therefore, we let
$$\begin{aligned}\mathcal{P}(\alpha+k) &= \langle\mathcal{O}[k]\rangle^{ext}\\ \mathcal{P}(\beta+k) &= \langle\mathcal{O}(-1)[k+1]\rangle^{ext}\end{aligned} \quad \text{for } k\in\mathbb{Z} \text{ and } \mathcal{P}(\phi)=0 \text{ for } \phi\notin\{\mathbb{Z}+\alpha,\mathbb{Z}+\beta\}$$

$\mathcal{P}$ is a slicing as $\mathrm{Hom}(\mathcal{O},\mathcal{O}(-1)[1]) \overset{\text{serre duality}}{=} \mathrm{Hom}(\mathcal{O}(1),\mathcal{O})^*=0 \Rightarrow \mathrm{Hom}(\mathcal{P}(\alpha),\mathcal{P}(\beta))=0$.

The triangles Δ_i give the HN filtrations of $\mathcal{O}(n)$ for $n\neq 0,1$ and $\mathcal{O}_x$.

$\exists!\ p,q\in\mathbb{Z}$ s.t
$$\begin{aligned}0&<\phi(\mathcal{O}[q])=\alpha+q\leq 1\\ 0&<\phi(\mathcal{O}(-1)[1+p])=\beta+p\leq 1\end{aligned}$$

Then $\mathcal{P}(0,1]=\langle\mathcal{O}[q],\mathcal{O}(-1)[1+p]\rangle^{ext}$ and $\alpha-\beta-1<p-q<\alpha-\beta+1$.

3) If $\alpha<\beta$, then by Thm 2: all $\mathcal{O}(n)$ and $\mathcal{O}_x$ are σ-semistable for $\sigma\in\mathrm{Stab}\,\mathbb{P}^1$ with $\alpha=\phi(\mathcal{O})<\phi(\mathcal{O}(1)[1])=\beta$.

Since $\beta-1<\alpha<\beta \Rightarrow \mathcal{B}=\langle\mathcal{O},\mathcal{O}(-1)[1]\rangle^{ext} \underset{\uparrow}{\subseteq} \mathcal{P}(\beta-1,\beta] \rightsquigarrow$ uniquely determines slicing

must be an equality as both are hearts

As $\mathcal{O}_x,\mathcal{O},\mathcal{O}(-1)$ are σ-ss

Δ_3 and $\alpha<\beta \Rightarrow \alpha=\phi(\mathcal{O})<\psi=\phi(\mathcal{O}_x)<\beta=\phi(\mathcal{O}(-1)[1])$

Let $\phi_0=1+\alpha-\psi \in(0,1)$ as $\alpha<\psi$ and $\phi_0>1-\beta+\alpha>0$.

Moreover $\beta-\psi=\phi_0-\beta-\alpha-1\in(0,1)$ as $\beta-\psi<\phi_0$ and $0<\beta-\psi<\beta-\alpha<1$.

Let $Z_0=Z\cdot e^{i\pi(\phi_0-\alpha)}$; then
$$\begin{aligned}Z_0(\mathcal{O}) &= m_\alpha e^{i\pi\alpha}\cdot e^{i\pi(\phi_0-\alpha)} = m_\alpha e^{i\pi\phi_0}\in\mathbb{H},\\ Z_0(\mathcal{O}_x) &= m_z(\mathcal{O}_x)e^{i\pi\psi}e^{i\pi(\phi_0-\alpha)} = m_z(\mathcal{O}_x)e^{i\pi}\in\mathbb{R}_{<0},\\ Z_0(\mathcal{O}(-1)[1]) &= m_\beta e^{i\pi\beta}\cdot e^{i\pi(\phi_0-\alpha)} = m_\beta e^{i\pi(\beta-\psi)}\in\mathbb{H}.\end{aligned}$$

Consequently $Z_0(\mathrm{Coh}\,\mathbb{P}^1) \subseteq H := \mathbb{H} \cup \mathbb{R}_{\leq 0} \Rightarrow Z_0$ is a stability function on $\mathcal{A} = \mathrm{Coh}\,\mathbb{P}^1$

Moreover $\mathcal{E} \in \mathrm{Coh}\,\mathbb{P}^1$ is Z_0-semistable $\iff$ $\mathcal{E}$ is torsion or slope semistable.

Since Z_0 has the HN property it is equivalent to a stability condition $\sigma_0 \in \mathrm{Stab}\,\mathbb{P}^1$. Then $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}\,\mathbb{P}^1$ lies in the same $\mathbb{C}^+$-orbit.

More precisely $\sigma = \sigma_0 \cdot e^{i\pi(\phi_0 - \alpha)}$ ↪ This gives the existence.

$$(Z, \mathcal{P}) = \left(Z_0 e^{i\pi(\alpha - \phi_0)},\ \mathcal{P}(\phi) = \mathcal{P}_0(\phi - \phi_0 + \alpha)\right)$$

$$\Rightarrow \mathcal{P}(0,1] = \mathcal{P}_0(\alpha - \phi_0, \underbrace{1 + \alpha - \phi_0}_{\psi}]$$

a) If $\psi = \phi(\mathcal{O}_x) = k \in \mathbb{Z}$, then $\mathcal{P}(0,1] = \mathrm{Coh}\,\mathbb{P}^1[k]$.

b) otherwise we have

$$\underset{\beta}{\underset{\shortparallel}{\phi(\mathcal{O}(-1)[1])}} > \cdots > \underset{\psi}{\underset{\shortparallel}{\phi(\mathcal{O}_x)}} > \cdots > \phi(\mathcal{O}(1)) > \underset{\alpha}{\underset{\shortparallel}{\phi(\mathcal{O})}} > \cdots$$

so $\exists\, k, j \in \mathbb{Z}$ s.t $\phi(\mathcal{O}(k)[j]) > 0 \geq \phi(\mathcal{O}(k-1)[j])$

$\Rightarrow$ $\mathcal{P}(0,1] \geq \langle \mathcal{O}(k)[j], \mathcal{O}(k-1)[j+1] \rangle^{\mathrm{ext}}$ which agree as both are hearts. ☒

§4 A fundametal domain for the $\mathrm{Pic}\,\mathbb{P}^1 \times \mathbb{C}$-action on $\mathrm{Stab}\,\mathbb{P}^1$

Let $G = \mathrm{Pic}\,\mathbb{P}^1 \times \mathbb{C} \curvearrowright \mathrm{Stab}\,\mathbb{P}^1$ and

$$X := \left\{ \sigma \in \mathrm{Stab}\,\mathbb{P}^1 : \begin{array}{l} \text{a) } \mathcal{O} \text{ and } \mathcal{O}(-1) \text{ are } \sigma\text{-ss} \\ \text{b) } \phi(\mathcal{O}(-1)[1]) = m(\mathcal{O}(-1)[1]) = 1 \\ \text{c) } \phi(\mathcal{O}) > 0 \end{array} \right\}$$

Lemma 4

i) $G \cdot X = \mathrm{Stab}\,\mathbb{P}^1$

ii) There is a bijection $\Phi : X \xrightarrow{\simeq} \mathbb{H}$, $\sigma \mapsto \log(m(\mathcal{O})) + i\pi\,\phi(\mathcal{O})$.

Proof i) Let $\sigma \in \mathrm{Stab}\,\mathbb{P}^1$; then up to the $\mathrm{Pic}\,\mathbb{P}^1$-action, we can suppose $\mathcal{O}$ and $\mathcal{O}(-1)$ are σ-semistable by Theorem 2. Moreover $\exists\, r \in \mathbb{R}, p \in \mathbb{Z}_{>0}$ s.t. $\mathcal{O}, \mathcal{O}(-1)[p] \in \mathcal{P}(r, r+1]$.

We can use the $\mathbb{C}$-action to assume $m(\mathcal{O}(-1)[1]) = 1$ and $\phi(\mathcal{O}(-1)[1]) = 1$.

Then $1 \leq \phi(\mathcal{O}(-1)[p]) = p \in (r, r+1] \Rightarrow r > 0 \Rightarrow \phi(\mathcal{O}) > r > 0$.

Hence, up to the G-action, $\sigma \in X$.

ii) Let $a + ib \in \mathbb{H}$ and $\alpha = b/\pi$, $\beta = 1$, $m_\alpha = e^a > 0$ and $m_\beta = 1$.

Then by Proposition 3, $\exists!\, \sigma$ s.t $\mathcal{O}, \mathcal{O}(-1)$ are σ-semistable and

$Z(\mathcal{O}) = m_\alpha e^{i\pi\alpha}$, $Z(\mathcal{O}(-1)[1]) = m_\beta e^{i\pi\beta}$ $\Rightarrow \Phi(\sigma) = a + ib$. ☒

Lemma (*) $M = \mathrm{Stab}\,\mathbb{P}^1 / \mathbb{C}$ is connected.

Proof Since $X \cong \mathbb{H}$ is connected, so is $X \cdot \mathbb{C}$.

Claim: $\mathrm{Pic}\,\mathbb{P}^1$ preserves $S = \{\sigma \in \mathrm{Stab}\,\mathbb{P}^1 : \mathcal{P}(0,1] = \mathrm{Coh}\,\mathbb{P}^1\} \subseteq X \cdot \mathbb{C}$.

If $\sigma \in S$, then $Z(\mathcal{O}_x) \in \mathbb{R}_{<0}$ and $Z(\mathcal{O}) \in \mathbb{H}$ by

$\Rightarrow Z(\mathcal{O}(k)) \in \mathbb{H} \ \forall k$. Then $\mathcal{O}(n) \cdot Z(\mathcal{O}_x) = Z(\mathcal{O}_x)$

$\mathcal{O}(n) \cdot Z(\mathcal{O}(k)) = Z(\mathcal{O}(n+k))$ $\Rightarrow \mathrm{Coh}\,\mathbb{P}^1 \underset{\nearrow}{\subseteq} \mathcal{P}_{\mathcal{O}(n)\cdot\sigma}(0,1]$

equality as both are hearts.

[Hence $\mathcal{O}(n) \cdot \sigma \in S$.

Therefore, $\mathrm{Pic}\,\mathbb{P}^1$ preserves the connected components Σ of $\mathrm{Stab}\,\mathbb{P}^1$ containing $X \cdot \mathbb{C}$ $\xRightarrow[\text{Lemma 4}]{}$ $\mathrm{Stab}\,\mathbb{P}^1$ is connected $\Rightarrow M = \mathrm{Stab}\,\mathbb{P}^1/\mathbb{C}$ is connected. $\boxtimes$

Our goal is to find a fundamental domain for $G \curvearrowright \mathrm{Stab}\,\mathbb{P}^1$.
Since $G \cdot X = \mathrm{Stab}\,\mathbb{P}^1$, we can take a fundamental domain $X' \subseteq X$.
However X itself is too large as we see in part 2) of the following remark.

Remark 1) If $\sigma \in X$ and $\phi_\sigma(\mathcal{O}) > 1$, then $G \cdot \sigma \cap X = \sigma$.

To prove this, we note that by Thm 2: $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1]) = 1 \Rightarrow \mathcal{O}$ and $\mathcal{O}(-1)$ are the only σ-stable sheaves

Let $\sigma' = \mathcal{O}(j) \cdot \sigma \cdot (x+iy)$; then

$$\mathcal{P}'(\phi(\mathcal{O}) + y/\pi + k) = \mathcal{O}(j) \cdot \mathcal{P}(\phi(\mathcal{O}) + k) = \langle \mathcal{O}(j)[k] \rangle$$

$$\mathcal{P}'(\phi(\mathcal{O}(-1)[1] + \tfrac{y}{\pi} + k) = \mathcal{O}(j) \cdot \mathcal{P}(\phi(\mathcal{O}(-1)[1]) + k\rangle = \langle \mathcal{O}(j-1)[k+1] \rangle$$

and

$$\mathcal{P}'(\phi) = 0 \quad \forall \ \phi \notin \{\phi(\mathcal{O}) + \tfrac{y}{\pi} + \mathbb{Z},\ \phi(\mathcal{O}(-1)[1]) + y/\pi + \mathbb{Z}\}.$$

If $\sigma' \in X$, then a) $\Rightarrow$ $\mathcal{O}$ and $\mathcal{O}(-1)$ are σ-semistable $\Rightarrow j = 0$

b) $\Rightarrow 1 = \phi'(\mathcal{O}(-1)[1]) = \underbrace{\phi(\mathcal{O}(-1)[1])}_{\overset{\shortparallel}{1}\ (\text{as } \sigma \in X)} + \frac{y}{\pi} \Rightarrow y = 0$

$1 = m'(\mathcal{O}(-1)[1]) = e^{-x} \underbrace{m(\mathcal{O}(-1)[1])}_{\overset{\shortparallel}{1}\ (\sigma \in X)} \Rightarrow x = 0$ $\boxtimes$

2) For $\sigma \in X$ with $0 < \phi(\mathcal{O}) \leq 1$, $\exists\ \mathbb{Z} \hookrightarrow G$ s.t $\mathbb{Z} \cdot \sigma \subseteq X$.

Proof By Thm 2, $\mathcal{O}(n)$, for $n \in \mathbb{Z}$, and $\mathcal{O}_x$, for $x \in \mathbb{P}^1$, are σ-semistable.
Consider the subgroup

$$\begin{aligned} \mathbb{Z} &\hookrightarrow G = \mathrm{Pic}\,\mathbb{P}^1 \times \mathbb{C} \\ j &\longmapsto (\mathcal{O}(j),\ z_j = x_j + i y_j) \end{aligned}$$

and let $\sigma_j = \mathcal{O}(j) \cdot \sigma \cdot z_j$. (we'll determine z_j below)

As $\mathcal{O}(n)$ are σ-semistable $\forall n \Rightarrow \mathcal{O}(n)$ are σ_j-semistable $\forall n \Rightarrow$ a) holds

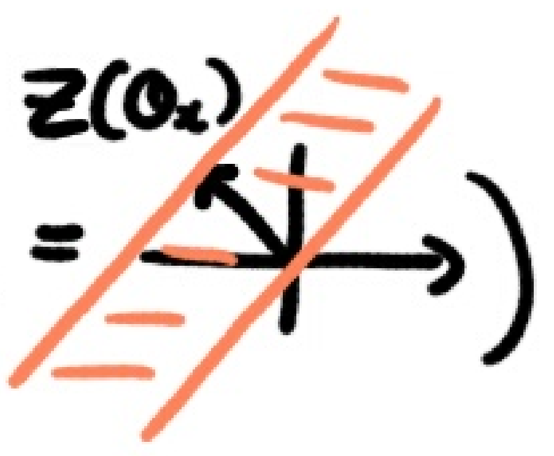
For b), we need to solve $1 = \phi_j(\mathcal{O}(-1)[1]) = \phi(\mathcal{O}(j-1)[1]) + y_j/\pi$

$$\Rightarrow y_j = \pi(1 - \phi(\mathcal{O}(j-1)[1])$$

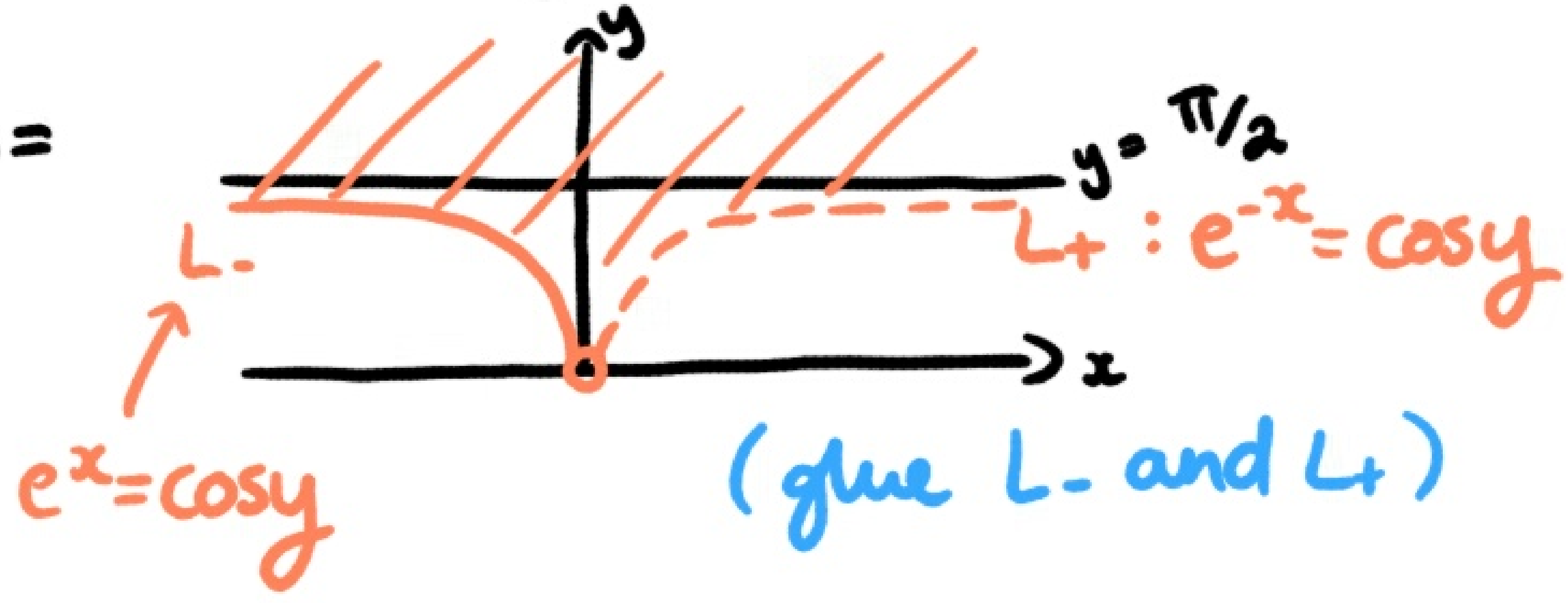
$$1 = m_j(\mathcal{O}(-1)[1]) = e^{x_j} m(\mathcal{O}(j-1)[1]) \Rightarrow x_j = \log \frac{1}{m(\mathcal{O}(j-1)[1])}.$$

c): $\underset{\overset{\shortparallel}{1}}{\phi_j(\mathcal{O}(-1)[1])} - \phi_j(\mathcal{O}) = \phi(\mathcal{O}(j-1)[1]) + \frac{y_j}{\pi} - \phi(\mathcal{O}(j)) + \frac{y_j}{\pi} < 1$ (as $\phi(\mathcal{O}(-1)[1]) - \phi(\mathcal{O}) < 1$ by Thm 2)

$\Rightarrow \phi_j(\mathcal{O}) > 0$ ie. $\sigma_j \in X$ $\boxtimes$

Idea: For $\sigma \in X$ with $0 < \phi(\mathcal{O}) \leqslant 1$, pick single σ_j by minimising $\phi_j(\mathcal{O}(-1)[1]) - \phi_j(\mathcal{O})$ (which happens if $Z_j(\mathcal{O})$ & $Z_j(\mathcal{O}(-1)[1]) \in S_Z =$ )

Then let $X' = \{\sigma \in X : Z(\mathcal{O}) \in S_Z\}$.

Proposition 5 $X' \cong K :=$  is a fundamental domain for $G \curvearrowright \mathrm{Stab}\,\mathbb{P}^1$

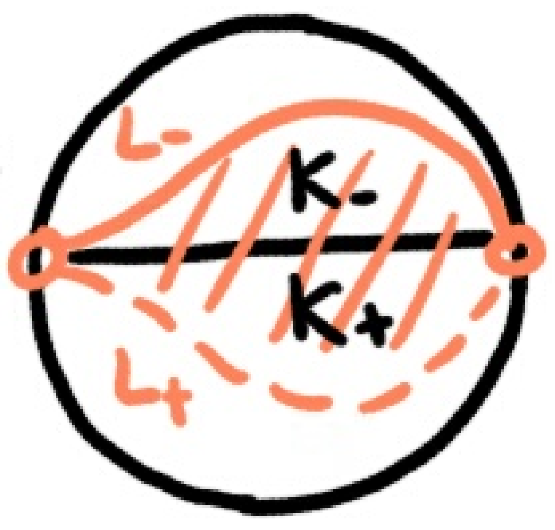
$$\begin{tikzcd} X' \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cap"] & K \arrow[d, phantom, "\cap"] \\ \Phi : X \arrow[r, "\cong"] & \mathbb{H} \end{tikzcd}$$

Theorem (**)

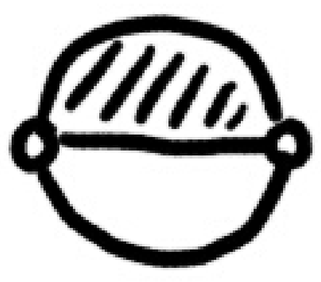
The fundamental domain K is conformally equivalent to $\mathbb{C}^*$

Sketch of proof:

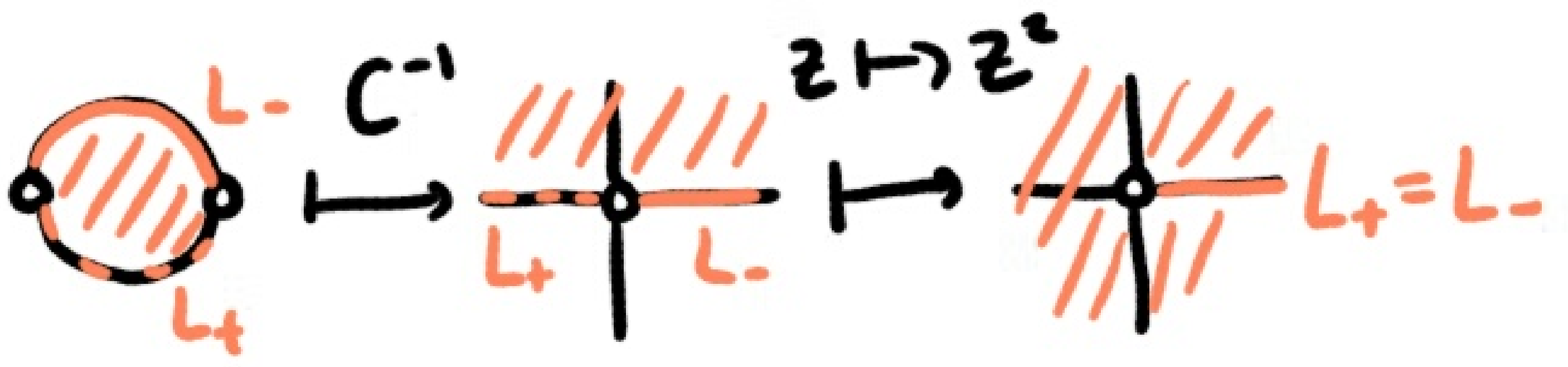
Use the Cayley transformation $C : \mathbb{H} \to \mathbb{U}$

$K \cong C(K) =$ 

By the Riemann mapping theorem and the Schwarz reflection principle

$K_+ \cong$  $\cong$  and $K_- \cong$  $\Rightarrow K \cong$ 

Now consider the following composition:

$K \cong$  $\cong \mathbb{C}^*$ as required. $\boxed{3}$

§5 Wall and chamber structure

Following Proposition 3, the hearts of $D(\mathbb{P}^1)$ on which there is a stability function with the HN property are:

- $\mathcal{A}_k = \mathrm{Coh}\,\mathbb{P}^1[k]$ for $k \in \mathbb{Z}$
- $\mathcal{C}_{p,j,k} = \langle \mathcal{O}(j-1)[p+k], \mathcal{O}(j)[k] \rangle$ for $j, k \in \mathbb{Z}$, $p > 0$.

This determines a decomposition of $\mathrm{Stab}\,\mathbb{P}^1$ into cells:

$$S(\mathcal{C}) = \{\sigma = (Z, P) \in \mathrm{Stab}\,\mathbb{P}^1 : P(0,1] = \mathcal{C}\}$$

where

$$S(\mathcal{A}_k) = \{\sigma = (Z, P) : \phi(\mathcal{O}_x[j]) = 1,\ 0 < \phi(\mathcal{O}[j]) < 1\} \cong \mathbb{R}_{<0} \times \mathbb{H}$$

$$S(\mathcal{C}_{p,j,k}) = \{\sigma = (Z, P) : 0 < \phi(\mathcal{O}(j-1)[p+k]), \phi(\mathcal{O}(j)[k]) \leqslant 1\} \cong \mathbb{H}^2$$

The cell $S(\mathcal{C}_{p,j,k})$ decomposes into two chambers $S^{\pm}(\mathcal{C}_{p,j,k})$ which are separated by a wall $W_{p,j,k} = \{\sigma \in S(\mathcal{C}_{p,j,k}) : \phi(\mathcal{O}(j-1)[p+k]) = \phi(\mathcal{O}(j)[k])\}$

$S^-(\mathcal{C}_{p,j,k}) = \{\sigma \in S(\mathcal{C}_{p,j,k}) : \phi(\mathcal{O}(j-1)[p+k]) > \phi(\mathcal{O}(j)[k])\}$ ← all $\mathcal{O}_x$ and $\mathcal{O}(n)$ are σ-ss

$S^+(\mathcal{C}_{p,j,k}) = \{\sigma \in S(\mathcal{C}_{p,j,k}) : \phi(\mathcal{O}(j-1)[p+k]) < \phi(\mathcal{O}(j)[k])\}$ ← $\mathcal{O}(j)$ & $\mathcal{O}(j-1)$ are the only σ-ss line bundles and all $\mathcal{O}_x$ are σ-unstable

Proposition (0)

The walls $W_{p,j,k}$ are the only walls.

Furthermore, in the fundamental domain K, there is only one wall, namely $W = \{ x+iy \in K : y = \pi \}$.