GEOMETRIC INVARIANT THEORY AND SYMPLECTIC QUOTIENTS

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1. Introduction

In this course we study methods for constructing quotients of group actions in algebraic and symplectic geometry and the links between these areas. Often the spaces we want to take a quotient of are a parameter space for some sort of geometric objects and the group orbits correspond to equivalence classes of objects; the desired quotient should give a geometric description of the set of equivalence classes of objects so that we can understand how these objects may vary continuously. These types of spaces are known as moduli spaces and this is one of the motivations for constructing such quotients.

For an action of a group G on a topological space X, we endow the set $X/G := \{G \cdot x : x \in X\}$ of G-orbits with the quotient topology; that is, the weakest topology for which the quotient map $\pi : X \to X/G$ is continuous. Then the orbit space X/G is also a topological space which we call the topological quotient. If X has some property (for example, X is connected or Hausdorff), then we may ask if the orbit space X/G also has this property. Sometimes this is the case: for example, if X is compact or connected, then so is the orbit space X/G. Unfortunately, it is not always the case that the orbit space inherits the geometric properties of X; for example, it is easy to construct actions on a Hausdorff topological space for which the orbit space is non-Hausdorff. However, if G is a topological group, such as a Lie group, and the graph of the action

$$\begin{aligned} \Gamma : G \times X &\to X \times X \\ (g, x) &\to (x, g \cdot x) \end{aligned}$$

is proper, then X/G is Hausdorff: given two distinct orbits $G \cdot x_1$ and $G \cdot x_2$ as (x_1, x_2) is not in the image of Γ (which is closed in $X \times X$) there is an open neighbourhood $U_1 \times U_2 \subset X \times X$ of (x_1, x_2) preserved by G which is disjoint from the image of Γ and $\pi(U_1)$ and $\pi(U_2)$ are disjoint open neighbourhoods of x_1 and x_2 in X/G. More generally, if we have a smooth action of a Lie group on a smooth manifold M for which the action is free and proper (i.e. the graph of the action is proper), then the orbit space M/G is a smooth manifold and $\pi : M \to M/G$ is a smooth submersion. In fact, this is the unique smooth manifold structure on M/G which makes the quotient map a smooth submersion.

As we saw above, the orbit space can have nice geometric properties for certain types of group actions. However one could also ask whether we should relax the idea of having an orbit space, in order to get a quotient with better geometrical properties. The idea of this course is: given a group G acting on some space X in a geometric category (for example, the category of topological spaces, smooth manifolds, algebraic varieties or symplectic manifolds), to find a categorical quotient; that is, a G-invariant morphism $\pi : X \to Y$ in this category which is universal so that every other G-invariant morphism $X \to Z$ factors uniquely through π. With this definition it is not necessary for Y to be an orbit space and so it may be the case that π identifies some orbits. Of course, if the topological quotient $\pi : X \to X/G$ exists in the geometric category we are working in, then it will be a categorical quotient.

In the algebraic setting, given the action of a linear algebraic group G on a algebraic variety X the aim of Geometric Invariant Theory (GIT) is to construct a quotient for this action which is an algebraic variety. The topological quotient X/G in general will not have the structure of an algebraic variety. For example, it may no longer be separated (this is an algebraic notion of Hausdorffness) as in the Zariski topology the orbits of G in X are not always closed; therefore, a lower dimensional orbit might be contained in the closure of another orbit and so we cannot separate these orbits in the quotient X/G. Geometric Invariant theory, as developed by

Mumford in [31], shows for a certain class of groups that one can construct an open subvariety $U \subset X$ and a categorical quotient of the G-action on U which is a quasi-projective variety. In general the quotient will not be an orbit space but it contains an open subvariety V/G which is the orbit space for an open subset $V \subset U$. In the case when X is affine, we shall see that $U = X$ and the categorical quotient is also an affine variety. Whereas in the case when X is a projective variety in $\mathbb{P}^n$, we see that the categorical quotient is a projective variety; however in general this is only a categorical quotient of an open subset of X. We briefly summarise the main results in affine and projective GIT below.

If $X \subset \mathbb{A}^n$ is an affine variety over an algebraically closed field k which is cut out by the polynomials $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, then its coordinate ring

$$A(X) \cong k[x_1, \dots, x_n]/(f_1, \dots, f_s)$$

is a finitely generated k-algebra of regular functions on X. If there is an action of an algebraic group G on X, then there is an induced action of G on the coordinate ring $A(X)$ of regular functions on X. For any G-invariant morphism $f : X \to Z$ of affine varieties, the image of the associated morphism of coordinate rings $f^* : A(Z) \to A(X)$ is contained in the ring $A(X)^G$ of G-invariant regular functions. If $A(X)^G$ is a finitely generated k-algebra, then it corresponds to an affine variety $Y = \operatorname{Spec} A(X)^G$ (the idea is to use a set of generators $f_1, \dots, f_m$ to construct Y as an algebraic subvariety of $\mathbb{A}^m$). In this case, the morphism $\varphi : X \to Y$ corresponding to the inclusion $A(X)^G \hookrightarrow A(X)$ is universal as every other G-invariant morphism $X \to Z$ of affine varieties factors uniquely through φ and so φ is a categorical quotient. The categorical quotient is constant on orbits and also on their closures; hence, the categorical quotient identifies orbits whose closures meet. In particular, it is not necessarily an orbit space; although, we will prove that it contains an open subvariety X^s/G which is an orbit space of a open subvariety $X^s \subset X$ of so-called stable points.

Of course, in the above discussion we assumed that $A(X)^G$ was finitely generated so that we were able to realise Y as an algebraic subset of $\mathbb{A}^m$ for some finite m. We now turn our attention to the question of whether the ring of G-invariant functions is finitely generated. This classical problem in Invariant theory was studied by Hilbert who built much of the foundations for the modern theory of GIT. For $G = \mathrm{GL}_n(\mathbb{C})$ and $SL_n(\mathbb{C})$, Hilbert showed the answer is yes. This problem, known as Hilbert's 14th problem, has now been answered in the negative by Nagata who gave an action of the additive group $\mathbb{C}^+$ for which the ring of invariants is not finitely generated. However there are a large number of groups for which the ring of invariants is always finitely generated. Nagata also showed that if a reductive linear algebraic group G acts on an affine variety X, then the invariant subalgebra is finitely generated [33]. Fortunately, many of the algebraic groups we are interested in are reductive (for example, the groups GL_n and SL_n are reductive). However, more recently there has been work on non-reductive GIT by Doran and Kirwan [12].

The affine GIT quotient serves as a guide for the general approach: as every algebraic variety is constructed by gluing affine algebraic varieties, the general theory is obtained by gluing the affine theory. However, we need to cover X by certain nice G-invariant open affine sets to be able to glue the affine GIT quotients and so in general we can only cover an open subset X^{ss} of X of so-called semistable points. Then GIT provides a categorical quotient $\pi : X^{ss} \to Y$ where Y is a quasi-projective variety. In fact it also provides an open subset $X^s/G \subset Y$ which is an orbit space for an open set $X^s \subset X^{ss}$ of stable points.

Suppose we have an action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ which is linearised; that is, G acts via a representation $\rho : G \to GL(n+1)$ and so the action lifts to the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. As X is a complete variety (i.e. compact), we would like to have a complete quotient or at least a quotient with a natural completion. In this case we use the homogeneous graded coordinate ring of X and the inclusion $R(X)^G \hookrightarrow R(X)$ which induces a rational morphism of projective varieties $X \dashrightarrow Y$; that is, this is only a well defined morphism on an open subset of X which we call the semistable locus and denote by X^{ss}. More explicitly, given homogeneous generators $h_0, \dots, h_m$ of $R(X)^G$ of the same degree, we define a rational map

$$\varphi : X \dashrightarrow \mathbb{P}^m$$
$$x \mapsto [h_o(x) : \cdots : h_m(x)]$$

whose image $Y \subset \mathbb{P}^m$ is a projective variety. Then φ is undefined on points x for which all non-constant G-invariant homogeneous functions vanish at x. The induced morphism $\varphi : X^{ss} \to Y$ is a categorical quotient of the G-action on the semistable locus. There is an open set $X^s \subset X^{ss}$ of stable points and an open subset X^s/G of Y which is an orbit space of the stable locus X^s. The nicest case is when $X^{ss} = X^s$ and so Y is a projective quotient which is also an orbit space.

The techniques of GIT have been used to construct many moduli spaces in algebraic geometry and we briefly mention a few examples here to give some idea of the importance of GIT in algebraic geometry today. There is still no better introduction to the theory of moduli in algebraic geometry than the excellent notes of Peter Newstead [36]. In fact, in the final chapter of this book, there is an overview of Seshadri's construction of the moduli space of (semi)stable (algebraic) vector bundles on a smooth projective curve as a GIT quotient of a subscheme of a Quot scheme by a projective linear group. Since then, this result has been generalised in two different directions. Firstly, one can consider higher dimensional base schemes and this generalisation was made by Simpson [45]. Secondly, one can give the bundle some additional structure and this generalisation was made by Schmitt [41]. Classically, Mumford was motivated in studying moduli spaces of both vector bundles and curves and this motivated his development of GIT. Today moduli spaces of (marked) curves have GIT constructions. An important example, is the moduli space of elliptic curves (i.e. genus 1 curves with 1 marked point) which is constructed as a quotient of an open subset in $\mathbb{P}^9$ by SL_3; this quotient, is shown to be the affine line $\mathbb{A}^1$. More generally, the moduli space of genus g curves is constructed as a quotient of a subscheme of a Hilbert scheme by the action of a projective general linear group.

In the symplectic setting, suppose we have a Lie group K acting smoothly on a symplectic manifold (X, ω) where X is a real manifold and ω is a the symplectic form on X (that is; a closed non-degenerate 2-form on X). The symplectic form gives a duality between vector fields and 1-forms by sending $Z \in \mathrm{Vect}(X)$ to the 1-form $\omega(Z, -) \in \Omega^1(X)$. We say the action is symplectic if the image of the map $K \to \mathrm{Diff}(X)$ given by $k \mapsto (\sigma_k : x \mapsto g \cdot x)$ is contained in the subset of symplectomorphisms i.e. $\sigma_k^* \omega = \omega$. We can also consider the 'infinitesimal action' which is a Lie algebra homomorphism $\mathfrak{K} \to \mathrm{Vect}(X)$ given by

$$A \mapsto A_{X,x} = \frac{d}{dt} \exp(tA) \cdot x|_{t=0} \in T_x X$$

where $\mathfrak{K} = \mathrm{Lie}\ K$. We say the action is Hamiltonian if we can lift the infinitesimal action to a Lie algebra homomorphism

$$\begin{tikzcd} & \mathcal{C}^\infty(X) \arrow[d] \\ \mathfrak{K} \arrow[ur, dashed] \arrow[r] & \mathrm{Vect}(X) \end{tikzcd}$$

where the vertical map is the composition of the exterior derivative $d : \mathcal{C}^\infty(X) \to \Omega^1(X)$ with the isomorphism $\Omega^1(X) \cong \mathrm{Vect}(X)$ given by ω. The lift $\phi : \mathfrak{K} \to \mathcal{C}^\infty(X)$ is called a comoment map, although often one works with the associated moment map $\mu : X \to \mathfrak{K}^*$ which is defined by $\mu(x) \cdot A = \phi(A)(x)$ for $x \in X$ and $A \in \mathfrak{K}$. If $\mu_A : X \to \mathbb{R}$ is given by $x \mapsto \mu(x) \cdot A$, then by construction the 1-form $d\mu_A$ corresponds under the duality defined by ω to the vector field A_X given by the infinitesimal action of A. As ϕ is a Lie algebra homomorphism, the moment map is K-equivariant where K acts on $\mathfrak{K}^*$ by the coadjoint representation.

In general, the topological quotient X/K is not a manifold, let alone a symplectic manifold. In fact, even if it is a manifold it may have odd dimension over $\mathbb{R}$ and so cannot admit a symplectic structure. To have a hope of finding a quotient which will admit a symplectic structure we need to ensure the quotient is even dimensional.

When a Lie group K acts on X with moment map $\mu : X \to \mathfrak{K}^*$, the equivariance of μ implies that the preimage $\mu^{-1}(0)$ is invariant under the action of K. We consider the symplectic reduction

$$\mu^{-1}(0)/K$$

of Marsden and Weinstein [26] and Meyer [28]. If 0 is a regular value of μ, then the preimage $\mu^{-1}(0)$ is a closed submanifold of X of dimension $\dim X - \dim \mathfrak{K}$. If the action of K on $\mu^{-1}(0)$ is free and proper, then the symplectic reduction is a smooth manifold of dimension $\dim X - 2\dim \mathfrak{K}$. Furthermore, there is a unique symplectic form ω^{red} on $\mu^{-1}(0)/K$ such that $i^*\omega = \pi^*\omega^{\mathrm{red}}$ where $i : \mu^{-1}(0) \hookrightarrow X$ is the inclusion and $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/K$ the quotient. The symplectic reduction has a universal property (in a suitable category of symplectic manifolds) and so can be considered as a categorical quotient.

So far it may seem that there is not much similarity between the symplectic reduction and the GIT quotient. However, one of the key results we shall see in this course is the Kempf-Ness Theorem which states that they give the same quotient when a complex reductive group G acts linearly on a smooth complex projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$. A group G is complex reductive if and only if it is the complexification of its maximal compact subgroup K; that is, $\mathfrak{g} = \mathfrak{K} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathfrak{g}$ and $\mathfrak{K}$ denote the Lie algebras of G and K respectively. Complex projective space has a natural Kähler, and hence also symplectic, structure; therefore we can restrict the symplectic form to X to make X a symplectic manifold. One can explicitly write down a moment map $\mu : X \to \mathfrak{K}^*$ which shows that the action is Hamiltonian. Then the Kempf-Ness theorem states that there is an inclusion $\mu^{-1}(0) \subset X^{ss}$ which induces a homeomorphism

$$\mu^{-1}(0)/K \cong X/\!/G.$$

Moreover, 0 is a regular value of μ if and only if $X^s = X^{ss}$. In this case, the GIT quotient is a projective variety which is an orbit space for the action of G on X^s.

There is a further generalisation of this result which gives a correspondence between an algebraic and symplectic stratification of X. The Kempf-Ness Theorem and the agreement of these stratifications can be seen as a 'finite-dimensional version' of many of the classical results in gauge theory (where an infinite-dimensional set up is used); for example, the Narasimhan–Seshadri correspondence [34] can be seen as an analogous type of result. These results show there is a rich interplay between the fields of algebraic and symplectic geometry. Another famous link between these fields is Kontsevich's homological mirror symmetry conjecture, although we will not be covering this in this course!

Notation and conventions. Throughout we fix an algebraically closed field k. On a few occasions, we will assume the characteristic of k is zero to simplify the proofs (for example, we only provide a proof of Nagata's theorem in characteristic zero) and we provide references for those interested in the proofs in positive characteristic. We work with varieties over k rather than schemes. From a technical point of view, there is no difference between the theory for schemes and the theory for varieties for GIT; however, we state everything in the language of varieties as this avoids introducing the full machinery of schemes and, after all, the Kempf–Ness theorem is a result about (smooth complex projective) varieties rather than schemes.

2. Types of algebraic quotients

In this section we consider group actions on algebraic varieties and also describe what type of quotients we would like to have for such group actions. Since the groups we will be interested will also have the structure of an affine variety, we start with a review of affine algebraic geometry.

2.1. Affine algebraic geometry.

In this section we shall briefly review some of the terminology from algebraic geometry. For a good introduction to the basics of algebraic geometry see [21, 14, 16, 42]; for further reading see also [17, 30, 43].

We fix an algebraically closed field k and let $\mathbb{A}^n = \mathbb{A}^n_k$ denote affine n-space over k:

$$\mathbb{A}^n = \{(a_1, \dots, a_n) : a_i \in k\}.$$

Every polynomial $f \in k[x_1, \dots x_n]$ can be viewed as a function $f : \mathbb{A}^n \to k$ which sends $(a_1, \dots, a_n)$ to $f(a_1, \dots, a_n)$. An algebraic subset of the affine space $\mathbb{A}^n \cong k^n$ is a subset

$$V(f_1, \dots f_m) = \{p = (a_1, \dots, a_n) \in \mathbb{A}^n : f_i(p) = 0 \text{ for } i = 1, \dots, m\}$$

defined as the set of zeros of finitely many polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$. As the k-algebra $k[x_1, \dots, x_n]$ is Noetherian, every ideal $I \subset k[x_1, \dots x_n]$ is finitely generated, say $I = (f_1, \dots, f_m)$, and we write $V(I) := V(f_1, \dots, f_m)$. The Zariski topology on $\mathbb{A}^n$ is given by defining the algebraic subsets to be the closed sets.

For any subset $X \subset \mathbb{A}^n$, we let

$$I(X) := \{f \in k[x_1, \dots, x_n] : f(p) = 0 \text{ for all } p \in X\}$$

denote the ideal in $k[x_1, \dots, x_n]$ of polynomials which vanish on X.

We have defined operators

$$V : \{\text{ideals of } k[x_1, \dots, k_n]\} \longleftrightarrow \{\text{subsets of } \mathbb{A}^n\} : I$$

but these operators are not inverse to each other. Hilbert's Nullstellensatz gives the relationship between V and I:

- $X \subset V(I(X))$ with equality if and only if X is a closed subset.
- $I \subset I(V(I))$ with equality if and only if I is a radical ideal.

Definition 2.1. An affine variety X over k is an algebraic subset of $\mathbb{A}^n (= \mathbb{A}^n_k)$.

Often varieties are assumed to be irreducible; that is, they cannot be written as the union of two proper closed subsets. However we shall refer to reducible (i.e. not irreducible) varieties also as varieties. In particular, an affine variety is a topological space with its topology induced from the Zariski topology on $\mathbb{A}^n$.

Exercise 2.2. (1) Show $V(fg) = V(f) \cup V(g)$.
(2) Let $X = \mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$; then what is $V(I(X))$?
(3) Let $I = (x^2) \in k[x]$; then what is $I(V(I))$?
(4) Let $f(x) \in k[x]$ and $X = V(f)$ be the associated affine variety in the affine line $\mathbb{A}^1$. When is X irreducible?

For an affine variety X, we define the (affine) coordinate ring of X as

$$A(X) = k[x_1, \dots, x_n]/I(X)$$

and we view elements of $A(X)$ as functions $X \to k$. Since we can add and multiply these functions and scale by elements in k, we see that $A(X)$ is a k-algebra and, moreover, that $A(X)$ is a finitely generated k-algebra (the functions x_i provide a finite set of generators). In fact, $A(X)$ is a reduced k-algebra (that is, there are no nilpotents) and, if X is irreducible, then $A(X)$ is an integral domain (that is, there are no zero divisors). The coordinate ring of $\mathbb{A}^n$ is $k[x_1, \dots, x_n]$.

Definition 2.3. Let $X \subset \mathbb{A}^n$ be an affine variety and $U \subset X$ be an open subset. A function $f : U \to k$ is regular at a point $p \in U$ if there is an open neighbourhood $V \subset U$ containing p on which $f = g/h$ where $g, h \in A = k[x_1, \dots, x_n]$ and $h(p) \neq 0$. We say $f : U \to k$ is regular on U if it is regular at every point $p \in U$.

For any open set $U \subset X$, we let $\mathcal{O}_X(U) := \{f : U \to k : f \text{ is regular on } U\}$ denote the k-algebra of regular functions on U. For $V \subset U \subset X$, there is a natural morphism

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

given by restricting a regular function on U to the smaller set V. For those familiar with the terminology from category theory, $\mathcal{O}_X(-)$ is a contravariant functor from the category of open subsets of X to the category of k-algebras and is in fact a sheaf called the structure sheaf (we shall not need this fact, but for those who are interested see [17] II §1 for a precise definition).

The following theorem summarises some of the results about $\mathcal{O}_X$ which we shall use without proof (for details of the proofs, see [17] I Theorem 3.2 and II Proposition 2.2).

Theorem 2.4. *Let $X \subset \mathbb{A}^n$ be an irreducible affine variety. Then:*

i) *The ring $\mathcal{O}(X) := \mathcal{O}_X(X)$ of regular functions on X is isomorphic to the coordinate ring $A(X) = k[x_1, \dots, x_n]/I(X)$ of X.*
ii) *There is a one-to-one correspondence between the points p in X and maximal ideals $\mathfrak{m}_p$ in $A(X)$ where m_p is the ideal of functions which vanish at p.*
iii) *For $f \in A(X)$, we define $X_f := \{x \in X : f(x) \neq 0\} = X - V(f)$ and let $A(X)_f$ denote the ring obtained by localising $A(X)$ by the multiplicative set $\{f^n : n \geq 0\}$. Then*

$$\mathcal{O}_X(X_f) \cong A(X)_f.$$

iv) *The open affine sets X_f for $f \in A(X)$ form a basis for the Zariski topology on X.*

Example 2.5. Let $X = \mathbb{A}^1$ and $f(x) = x \in k[x] = A(X)$. Then $X_f = \mathbb{A}^1 - \{0\}$ is an affine variety with coordinate ring $A(X_f) = A(X)_f = k[x, x^{-1}]$. We refer to $\mathbb{A}^1 - \{0\}$ as the punctured line. For $n \geq 2$, we note that $\mathbb{A}^n - \{0\}$ is no longer an affine variety.

Given an affine variety $X \subset \mathbb{A}^n$, we can associate to X a finitely generated k-algebra, namely its coordinate ring $A(X)$ which is equal to the ring of regular functions on X. If X is irreducible, then its coordinate ring $A(X)$ is an integral domain (i.e. it has no zero divisors). Conversely given a finitely generated k-algebra A, we can associate to A an affine variety $\operatorname{Spec} A$ (called the spectrum of A) as follows. As A is a finitely generated k-algebra, we can take generators $x_1, \dots, x_n$ of A which define a surjection

$$k[x_1, \dots, x_n] \to A$$

with finitely generated kernel $I = (f_1, \dots, f_m)$. Then $\operatorname{Spec} A := V(f_1, \dots, f_m) = V(I)$. One can also define the spectrum $\operatorname{Spec} A$ without making a choice of generating set for A: we define $\operatorname{Spec} A$ to be the set of prime ideals in A and we can also define a Zariski topology on $\operatorname{Spec} A$ (for example, see [17] II §2). We use the term point to mean closed point; that is, a point corresponding to a maximal ideal.

Definition 2.6. A morphism of varieties $\varphi : X \to Y$ is a continuous map of topological spaces such that for every open set $V \subset Y$ and regular function $f : V \to k$ the morphism $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

Exercise 2.7. Let X be an affine variety, then show the morphisms from X to $\mathbb{A}^1$ are precisely the regular functions on X.

Given a morphism of affine varieties $\varphi : X \to Y$, there is an associated k-algebra homomorphism

$$\varphi^* : A(Y) \to A(X)$$
$$(f : Y \to k) \mapsto (f \circ \varphi : X \to k).$$

Conversely given a k-algebra homomorphism $\varphi^* : A \to B$, we can define a morphism of varieties $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ by $\varphi(p) = q$ where $\mathfrak{m}_q = (\varphi^*)^{-1}(\mathfrak{m}_p)$ (recall that by Theorem 2.4 ii), the points p of an affine variety correspond to maximal ideals $\mathfrak{m}_p$ in the coordinate ring).

Formally, the coordinate ring operator $A(-)$ is a contravariant functor from the category of affine varieties to the category of k-algebra homomorphisms and the maximal spectrum operator $\operatorname{Spec}(-)$ is a contravariant functor in the opposite direction. These define an equivalence between the category of (irreducible) affine varieties over k and finitely generated reduced k-algebras (without zero divisors).

Exercise 2.8. (1) Which affine variety X corresponds to the k-algebra $A(X) = k$?
(2) The inclusion $k[x_1, \dots, x_{n-1}] \hookrightarrow k[x_1, \dots, x_n]$ of k-algebras corresponds to which morphism of varieties.
(3) Show $V(xy) \subset \mathbb{A}^2$ is isomorphic to the affine variety $\mathbb{A}^1 - \{0\}$ by showing an isomorphism of their k-algebras. Equivalently, one could explicitly write down morphisms $V(xy) \to \mathbb{A}^1 - \{0\}$ and $\mathbb{A}^1 - \{0\} \to V(xy)$ which are inverse to each other.

2.2. **Algebraic groups.**

Definition 2.9. An affine algebraic group over k is an affine variety G (not necessarily irreducible) over k whose set of points has a group structure such that group multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are morphisms of affine varieties.

It is a classical result that every affine algebraic group over k is isomorphic to a linear algebraic group over k; that is, a Zariski closed subset of a general linear group $\mathrm{GL}_n(k)$ which is also a subgroup of $\mathrm{GL}_n(k)$ (for example, see [4] 1.10). In particular, we will use the terms affine algebraic group over k and linear algebraic group over k interchangeably.

Remark 2.10. Let $A(G)$ denote the k-algebra of regular functions on G. Then the above morphisms of affine varieties correspond to k-algebra homomorphisms $m^* : A(G) \to A(G) \otimes A(G)$ (comultiplication) and $i^* : A(G) \to A(G)$ (coinversion). In fact we can write down the co-group operations to define the group structure on G.

Example 2.11. Many of the groups that we are already familiar with are algebraic groups.

(1) The additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$ over k is the algebraic group whose underlying variety is the affine line $\mathbb{A}^1$ over k and whose group structure is given by addition:
$$m^*(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad i^*(t) = -t.$$
(2) The multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ over k is the algebraic group whose underlying variety is the $\mathbb{A}^1 - \{0\}$ and whose group action is given by multiplication:
$$m^*(t) = t \otimes t \quad \text{and} \quad i^*(t) = t^{-1}.$$
(3) The general linear group $\mathrm{GL}_n(k)$ over k is an open subvariety of $\mathbb{A}^{n^2}$ cut out by the condition that the determinant is nonzero. It is an affine variety with coordinate ring $k[x_{ij} : 1 \leq i, j \leq n]_{\det(x_{ij})}$. The co-group operations are defined by:
$$m^*(x_{ij}) = \sum_{s=1}^{n} x_{is} \otimes x_{sj} \quad \text{and} \quad i^*(x_{ij}) = (x_{ij})^{-1}_{ij}$$
where $(x_{ij})^{-1}_{ij}$ is the regular function on $\mathrm{GL}_n(k)$ given by taking the (i, j)th entry of the inverse of a matrix.

Exercise 2.12. Show that any finite group G is an affine algebraic group over any field k. For example, write down the coordinate ring of $G = \{\mathrm{id}\}$ and $\mu_n := \{c \in k : c^n = 1\}$.

Exercise 2.13. Show that an affine algebraic group is smooth.

2.3. **Linear algebraic groups.** In this section, we state a few important results about the structure of linear algebraic groups over an algebraically closed field; for further details and proofs, see [4, 19, 46]. As above, we continue to let G denote a linear algebraic group over the algebraically closed field k. Our starting point is to recall the Jordan decomposition for linear algebraic groups over (the algebraically closed field) k.

Definition 2.14. Let G be a linear algebraic group over k.

(1) An element g is semisimple (resp. unipotent) if there is a faithful linear representation $\rho : G \to \mathrm{GL}_n$ such that $\rho(g)$ is diagonalizable (resp. unipotent).
(2) A unipotent subgroup is a subgroup of unipotent elements.

Theorem 2.15 (Jordan decomposition)**.** *Let G be a linear algebraic group over k. For every $g \in G$, there exists a unique semisimple element g_{ss} and a unique unipotent element g_u such that*
$$g = g_{ss} g_u = g_u g_{ss}.$$
Furthermore, this decomposition is functorial with respect to morphisms of linear algebraic groups.

Definition 2.16. A Borel subgroup of G is a maximal connected solvable linear algebraic subgroup of G.

Theorem 2.17 ([4] 11.2)**.** *All Borel subgroups in G are conjugate.*

Proof. If we fix a Borel B, then we can view the quotient G/B as the set of Borel subgroups. As B is a Borel subgroup, this quotient G/B is a projective variety (cf. [4] 11.1). Then we apply Borel's fixed point theorem: a solvable group action on a projective variety has a fixed point. □

Definition 2.18. An algebraic k-torus is an affine algebraic group over k isomorphic to $\mathbb{G}_m^n \cong (k^*)^n$.

Alternatively, as we are working over an algebraically closed field, a torus is a connected abelian semisimple group (cf. [4] 11.5). Furthermore, if G admits no nontrivial tori then it is unipotent (this is also proved in [4] 11.5).

Often one makes use of the lattices of (co)characters of a torus. More precisely, for a torus T, we define commutative groups

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \quad X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$$

called the character and cocharacter group respectively, where we consider homomorphisms of linear algebraic groups over k between T and the multiplicative group $\mathbb{G}_m$. As the automorphism group of $\mathbb{G}_m$ can be identified with the integers $\mathbb{Z}$ by $t \mapsto t^n$, we see that the (co)character groups are finite free $\mathbb{Z}$-modules of rank $\dim T$. There is a perfect pairing between these lattices given by composition

$$<,>: X^*(T) \times X_*(T) \to \mathbb{Z}$$

where $< \chi, \lambda >:= \chi \circ \lambda$.

An important fact about tori is that their linear representations are completely reducible. We will often use this result to diagonalise a torus action. More precisely, we state this result as a weight space decomposition.

Proposition 2.19. *For a finite dimensional linear representation of a torus $\rho : T \to \mathrm{GL}(V)$, there is a weight space decomposition*

$$V \cong \bigoplus_{\chi \in X^*(T)} V^\chi$$

where $V^\chi = \{v \in V : t \cdot v = \chi(t)v \ \forall t \in T\}$.

Alternatively, there is an equivalence between the category of linear representations of T and $X^*(T)$-graded k-vector spaces. We refer to the collection of characters χ for which $V_\chi \neq 0$ as the weights for the T-action.

Any torus in G is contained in a maximal torus by dimension considerations. Moreover, as every torus is contained in a Borel, it follows from Theorem 2.17 that all maximal tori are conjugate.

Proposition 2.20 ([4], 11.3)**.** *All maximal tori in a linear algebraic group G over k are conjugate.*

2.4. **Group actions.**

Definition 2.21. An action of an affine algebraic group G on a variety X is an action of G on X which is given by a morphism of varieties $\sigma : G \times X \to X$.

Remark 2.22. If X is an affine variety over k and $A(X)$ denotes its algebra of regular functions, then an action of G on X gives rise to a coaction homomorphism of k-algebras:

$$\sigma^* : A(X) \to A(G) \otimes A(X)$$
$$f \mapsto \sum h_i \otimes f_i.$$

This gives rise to an action $G \to \mathrm{Aut}(A(X))$ where the automorphism of $A(X)$ corresponding to $g \in G$ is given by

$$f \mapsto \sum h_i(g) f_i \in A(X)$$

where $f \in A(X)$ and $\sigma^*(f) = \sum h_i \otimes f_i$. The actions $G \to \mathrm{Aut}(A(X))$ which arise from actions on affine varieties are called rational actions.

Lemma 2.23. *For any $f \in A(X)$, the linear space spanned by the translates $g \cdot f$ for $g \in G$ is finite dimensional.*

Proof. If we write $\sigma^*(f) = \sum_{i=1}^n h_i \otimes f_i$ for the coaction homomorphism σ^* as above, then $g \cdot f = \sum_i h_i(g) f_i$ and so $f_1, \dots, f_n$ are a basis. □

Definition 2.24. Let G be an affine algebraic group acting on a variety X. We define the orbit $G \cdot x$ of x to be the image of the morphism $\sigma_x = \sigma(-, x) : G \to X$ given by $g \mapsto g \cdot x$. We define the stabiliser G_x of x to be the fibre of σ_x over x.

The stabiliser G_x of x is a closed subvariety of G (as it is the preimage of a closed subvariety of X under the continuous map $\sigma_x : G \to X$). In fact it is also a subgroup of G. The orbit $G \cdot x$ is a locally closed subvariety of X (this follows from a theorem of Chevalley which states that the image of morphisms of varieties is a constructible subset; for example, see [17] II Exercise 3.19).

Lemma 2.25. *Let G be a linear algebraic group acting on a variety X.*

i) *If Y and Z are subvarieties of X such that Z is closed, then*
$$\{g \in G : gY \subset Z\}$$
is closed.

ii) *For any subgroup, $H \subset G$ the fixed point locus*
$$X^H = \{x \in X : H \cdot x = x\}$$
is closed in X.

Proof. For i) we write
$$\{g \in G : gY \subset Z\} = \bigcap_{y \in Y} \sigma_y^{-1}(Z)$$
where $\sigma_y : G \to X$ is given by $g \mapsto g \cdot y$. As Z is closed, its preimage under the morphism $\sigma_y : G \to X$ is closed which proves i). For ii) and any $h \in H$ we may consider the graph $\Gamma_h : X \to X \times X$ of the action given by $x \mapsto (x, \sigma(h, x))$, then
$$X^H = \bigcap_{h \in H} \Gamma_h^{-1}(\Delta_X)$$
where Δ_X is the diagonal in $X \times X$. As $\Delta_X \subset X \times X$ is closed we see that X^H is also closed. □

Proposition 2.26. *The boundary of an orbit $\overline{G \cdot x} - G \cdot x$ is a union of orbits of strictly smaller dimension. In particular, each orbit closure contains a closed orbit (of minimal dimension).*

Proof. The boundary of an orbit $G \cdot x$ is invariant under the action of G and so is a union of G-orbits. As the orbit is a locally closed subvariety of G, it contains a subset U which is open and dense in its closure $\overline{G \cdot x}$. The orbit $G \cdot x$ is the union over g of the translates gU of U and so is open and dense in its closure $\overline{G \cdot x}$. Thus the boundary $\overline{G \cdot x} - G \cdot x$ is closed and of strictly lower dimension than $G \cdot x$. It is clear that orbits of minimum dimension are closed and so each orbit closure contains a closed orbit. □

Proposition 2.27. *Let G be an affine algebraic group acting on a variety X. Then the dimension of the stabiliser subgroup viewed as a function $\dim G_{-} : X \to \mathbb{N}$ is upper semi-continuous; that is, for every n, the set*
$$\{x \in X : \dim G_x \geq n\}$$
is closed in X. Equivalently,
$$\{x \in X : \dim G \cdot x \leq n\}$$
is closed in X for all n.

Proof. Consider the graph of the action

$$\Gamma : G \times X \to X \times X$$

$$(g, x) \mapsto (x, \sigma(g, x))$$

and the fibre product P

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d] & X \arrow[d, "\Delta"] \\ G \times X \arrow[r, "\Gamma"] & X \times X, \end{tikzcd}$$

where $\Delta : X \to X \times X$ is the diagonal morphism; then the fibre product P consists of pairs (g, x) such that $g \in G_x$. The function on P given by sending $p = (g, x) \in P$ to the dimension of $P_{\varphi(p)} := \varphi^{-1}(\varphi(p)) = (G_x, x)$ is upper semi-continuous; that is, for all n

$$\{p \in P : \dim P_{\varphi(p)} \geq n\}$$

is closed in P. By restricting to the closed set $X \cong \{(\mathrm{id}, x) : x \in X\} \subset P$, we see that the dimension of the stabiliser of x is upper semi-continuous; that is,

$$\{x \in X : \dim G_x \geq n\}$$

is closed in X for all n. Then by the orbit stabiliser theorem

$$\dim G = \dim G_x + \dim G \cdot x,$$

which gives the second statement. □

Example 2.28. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The orbits of this action are

- Conics $\{xy = \alpha\}$ for $\alpha \in k^*$,
- The punctured x-axis,
- The punctured y-axis.
- The origin.

The origin and the conic orbits are closed whereas the punctured axes both contain the origin in their orbit closures. The dimension of the orbit of the origin is strictly smaller than the dimension of $\mathbb{G}_m$, indicating that its stabiliser has positive dimension.

Example 2.29. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scalar multiplication: $t \cdot (a_1, \ldots, a_n) = (ta_1, \ldots, ta_n)$. In this case there are two types of orbits:

- punctured lines through the origin.
- the origin.

In this case the origin is the only closed orbit and there are no closed orbits of dimension equal to that of $\mathbb{G}_m$. In fact, every orbit contains the origin in its closure.

Exercise 2.30. For Examples 2.28 and 2.29, write down the coaction homomorphism.

2.5. **First notions of quotients.** Let G be an affine algebraic group acting on a variety X over k. In this section and the following section (§2.6) we discuss types of quotients for the action of G on X; the main references for these sections are [10], [31] and [36].

The orbit space $X/G = \{G \cdot x : x \in X\}$ for the action of G on X unfortunately does not always admit the structure of a variety. For example, often the orbit space is not separated (this is an algebraic notion of Hausdorff topological space) as we saw in Examples 2.28-2.29. Instead, we can look for a universal quotient in the category of varieties:

Definition 2.31. A categorical quotient for the action of G on X is a G-invariant morphism $\varphi : X \to Y$ of varieties which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through φ so that there exists a unique morphism $h : Y \to Z$ such that $f = \varphi \circ h$.

As φ is continuous and constant on orbits, it is also constant on orbit closures. Hence, a categorical quotient is an orbit space only if the action of G on X is closed; that is, all the orbits $G \cdot x$ are closed.

Remark 2.32. The categorical quotient has nice functorial properties in the following sense: if $\varphi : X \to Y$ is G-invariant and we have an open cover U_i of Y such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ is a categorical quotient for each i, then φ is a categorical quotient.

Exercise 2.33. Let $\varphi : X \to Y$ be a categorical quotient of the G action on X. If X is either connected, reduced or irreducible, then show that Y is too.

Example 2.34. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ as in Example 2.28. As the origin is in the closure of the punctured axes $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$, all three orbits will be identified by the categorical quotient. The conic orbits $\{xy = \alpha\}$ for $\alpha \in k^*$ are closed and clearly parametrised by the parameter $\alpha \in k^*$. Hence the categorical quotient $\varphi : \mathbb{A}^2 \to \mathbb{A}^1$ is given by $(x, y) \mapsto xy$.

Example 2.35. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ as in Example 2.29. As the origin is in the closure of every single orbit, all orbits will be identified by the categorical quotient and so the categorical quotient is the structure map $\varphi : \mathbb{A}^2 \to \operatorname{Spec} k$ to the point $\operatorname{Spec} k$.

We now see the sort of problems that may occur when we have non-closed orbits. In Example 2.29 our geometric intuition tells us that we would ideally like to remove the origin and then take the quotient of $\mathbb{G}_m$ acting on $\mathbb{A}^n - \{0\}$. In fact, we already know what we want this quotient to be: the projective space $\mathbb{P}^{n-1} = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$ which is an orbit space for this action. However, in general just removing lower dimensional orbits does not suffice in creating an orbit space. In fact in Example 2.28, if we remove the origin the orbit space is still not separated and so is a scheme rather than a variety (it is the affine line with a double origin obtained from gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 - \{0\}$).

2.6. **Second notions of quotient.** Let G be an affine algebraic group acting on a variety X over k. The group G acts on the k-algebra $\mathcal{O}_X(X)$ of regular functions on X by

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

and we denote the subalgebra of invariant functions by

$$\mathcal{O}_X(X)^G := \{f \in \mathcal{O}_X(X) : g \cdot f = f \text{ for all } g \in G\}.$$

Similarly if $U \subset X$ is a subset which is invariant under the action of G (that is, $g \cdot u \in U$ for all $u \in U$ and $g \in G$), then G acts on $\mathcal{O}_X(U)$ and we write $\mathcal{O}_X(U)^G$ for the subalgebra of invariant functions.

Ideally we want our quotient to have nice geometric properties and so we give a new definition of a good quotient:

Definition 2.36. A morphism $\varphi : X \to Y$ is a good quotient for the action of G on X if

i) φ is constant on orbits.
ii) φ is surjective.
iii) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.
iv) If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W)$ is closed in Y.
v) If W_1 and W_2 are disjoint G-invariant closed subsets, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
vi) φ is affine (i.e. the preimage of every affine open is affine).

If moreover, the preimage of each point is a single orbit then we say φ is a geometric quotient.

Remark 2.37. We note that the two conditions iv) and v) together are equivalent to:

v)′ If W_1 and W_2 are disjoint G-invariant closed subsets, then the closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

Proposition 2.38. *If $\varphi : X \to Y$ is a good quotient for the action of G on X, then it is a categorical quotient.*

Proof. Part i) of the definition of a good quotient shows that φ is G-invariant and so we need only prove that it is universal. Let $f : X \to Z$ be a G-invariant morphism; then we may pick a finite affine open cover U_i of Z. As $W_i := X - f^{-1}(U_i)$ is closed and G-invariant, its image $\varphi(W_i) \subset Y$ is closed by iv). Let $V_i := Y - \varphi(W_i)$; then we have an inclusion $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$. As U_i are a cover of Z, the intersection $\cap W_i$ is empty and so by v) we have $\cap\varphi(W_i) = \phi$; that is, V_i are also an open cover of Y. As f is G-invariant the homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ has image in $\mathcal{O}_X(f^{-1}(U_i))^G$. We consider the composition

$$\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))^G \to \mathcal{O}_X(\varphi^{-1}(V_i))^G \cong \mathcal{O}_Y(V_i)$$

where the second homomorphism is the restriction map associated to the inclusion $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$ and the final isomorphism is given by property iii) of good quotients. As U_i is affine, the algebra homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ defines a morphism $h_i : V_i \to U_i$ (see [17] I Proposition 3.5). Moreover, we have that

$$f| = h_i \circ \varphi| : \varphi^{-1}(V_i) \to U_i.$$

Therefore we can glue the morphisms h_i to obtain a morphism $h : Y \to Z$ such that $f = h \circ \varphi$. One can check that this morphism is independent of the choice of affine open cover of Z. $\square$

Corollary 2.39. *Let $\varphi : X \to Y$ be a good quotient; then:*

i) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$ *if and only if* $\varphi(x_1) = \varphi(x_2)$.
ii) *For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed (i.e. all orbits are closed), then φ is a geometric quotient.*

Proof. For i) we know that φ is continuous and constant on orbit closures which shows that $\varphi(x_1) = \varphi(x_2)$ if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$ and by property v) of φ being a good quotient we get the converse. For ii), suppose we have two distinct closed orbits W_1 and W_2 in $\varphi^{-1}(y)$, then the fact that their images under φ are both equal to y contradicts property v) of φ being a good quotient. $\square$

Corollary 2.40. *If $\varphi : X \to Y$ is a good (resp. geometric) quotient, then for every open $U \subset Y$ the restriction $\varphi| : \varphi^{-1}(U) \to U$ is also a good (resp. geometric) quotient of G acting on $\varphi^{-1}(U)$.*

Proof. It is clear that conditions i)-iii) and vi) in the definition of the good quotient hold also for the restriction. For iv), if $W \subset \varphi^{-1}(U)$ is closed and G-invariant and we suppose y belongs in the closure $\overline{\varphi(W)}$ of $\varphi(W)$ in U. Then if $\overline{W}$ denotes the closure of W in X, we have that $y \in \overline{\varphi(W)} \subset \varphi(\overline{W})$. As $\varphi : X \to Y$ is a good quotient, we see $\varphi^{-1}(y) \cap \overline{W} \neq \phi$ (for example, apply v) to the unique closed orbit in $\varphi^{-1}(y)$ and $\overline{W}$). But as $\varphi^{-1}(y) \subset \varphi^{-1}(U)$, this implies $\varphi^{-1}(y) \cap W \neq \phi$ and so $y \in \varphi(W)$ which proves this set is closed.

For v), let W_1 and W_2 be disjoint G-invariant closed subset of $\varphi^{-1}(U)$; we denote their closures in X by $\overline{W_1}$ and $\overline{W_2}$. If there is a point $y \in \varphi(W_1) \cap \varphi(W_2) \subset U$, then as φ is a good quotient,

$$\varphi^{-1}(y) \cap \overline{W_1} \cap \overline{W_2} \neq \phi.$$

However $\varphi^{-1}(y) \subset \varphi^{-1}(U)$ and as W_1 and W_2 are both closed in $\varphi^{-1}(U)$, we have

$$\varphi^{-1}(U) \cap \overline{W_1} \cap \overline{W_2} = W_1 \cap W_2 = \phi.$$

Therefore, the restriction $\varphi| : \varphi^{-1}(U) \to U$ is also a good quotient. If φ is a geometric quotient then it is clear that $\varphi| : \varphi^{-1}(U) \to U$ is too. $\square$

Remark 2.41. The definition of good and geometric quotients are local; thus if $\varphi : X \to Y$ is G-invariant and we have a cover of Y by open sets U_i such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ are all good (respectively geometric) quotients, then so is $\varphi : X \to Y$.

3. Affine Geometric Invariant Theory

In this section we consider an action of an affine algebraic group G on an affine variety X over k. The main references for this section are [31] and [36] (for further reading, see also [5], [10] and [37]).

Let $A(X)$ denote the coordinate ring of an affine variety X; then $A(X)$ is a finitely generated k-algebra. In the opposite direction, the maximal spectrum operator associates to a finitely generated k-algebra A an affine variety $\operatorname{Spec} A$.

The action of G on X induces an action of G on $A(X)$ by $g \cdot f(x) = f(g^{-1} \cdot x)$. Of course any G-invariant morphism $f : X \to Z$ of affine varieties induces a morphism $f^* : A(Z) \to A(X)$ given by $h \mapsto h \circ f$ whose image is contained in

$$A(X)^G := \{f \in A(X) : g \cdot f = f \text{ for all } g \in G\}$$

the subalgebra of G-invariant regular functions on X. Therefore, if $A(X)^G$ is also a finitely generated k-algebra, the categorical quotient is the morphism $\varphi : X \to Y := \operatorname{Spec} A(X)^G$ induced by the inclusion $A(X)^G \hookrightarrow A(X)$ of finitely generated k-algebras. This leads us to an interesting problem in invariant theory which was first considered by Hilbert:

3.1. **Hilbert's 14th problem.** Given a rational action of an affine algebraic group G on a finitely generated k-algebra A, is the algebra of G-invariants A^G finitely generated?

Unfortunately the answer to Hilbert's 14th problem is not always yes, but for a very large class of groups we can answer in the affirmative (see [33] and also Nagata's Theorem below). In the 19th century, Hilbert showed that for $G = \mathrm{GL}_n(\mathbb{C})$, the answer is always yes. The techniques of Hilbert were then used by many others to prove that for a very large class of groups (known as reductive groups), the answer is yes.

3.2. **Reductive groups.** As mentioned above, Hilbert's 14th problem has a positive answer for a large class of groups known as reductive groups. In order to define reductive groups we need to introduce the unipotent radical of a linear algebraic group G over k. A unipotent group is isomorphic to an algebraic subgroup of the unipotent $U_n \subset \mathrm{GL}_n$ consisting of upper triangular matrices with diagonal entries equal to 1. We also see that subgroups, quotients and extensions of unipotents groups are also unipotent. Given two connected unipotent normal linear algebraic subgroups $U, U' \subset G$, the normal closed subgroup $U \cdot U'$ that they generate is also unipotent. Therefore, due to dimension reasons, there is a unique maximal connected unipotent normal linear algebraic subgroup, called the unipotent radical. In particular, we need to impose the term normal to get a unique unipotent radical.

Definition 3.1. Let G be a linear algebraic group over k.

(1) The unipotent radical of G, denoted $\mathcal{R}_u(G)$ is the maximal connected unipotent normal linear algebraic subgroup of G.
(2) The radical of G, denoted $\mathcal{R}(G)$ is the maximal connected solvable normal linear algebraic subgroup of G.
(3) G is semisimple if it has trivial radical $\mathcal{R}(G) = \{1\}$.
(4) G is reductive if it has trivial unipotent radical $\mathcal{R}_u(G) = \{1\}$.

Every unipotent linear algebraic subgroup is solvable; this is a consequence of the Lie–Kolchin Theorem (for example, see [4] 10.5): for a connected solvable linear algebraic group G over an algebraically closed field k, any linear representation $\rho : G \to \mathrm{GL}_n$ can be conjugated to have image in the upper triangular matrices. Furthermore, any unipotent linear algebraic group can be realised as a closed subgroup of the strictly upper triangular matrices in GL_n. In particular, we have that $\mathcal{R}_u(G) \subset \mathcal{R}(G)$ and every semisimple group is reductive.

Example 3.2. The general linear group GL_n has radical consisting of the scalar matrices $\mathcal{R}(\mathrm{GL}_n) \cong \mathbb{G}_m$ and trivial unipotent radical. In particular, this is reductive but not semisimple. The special linear group and projective linear group are both semisimple (and thus reductive).

Remark 3.3. If G is connected and reductive, then $\mathcal{R}(G)$ is solvable and connected; hence, as its reductive, the radical $\mathcal{R}(G)$ must be a torus $\mathbb{G}_m^n$. As this is a normal torus in a connected linear algebraic group, it must in fact be central. In particular, connected reductive groups are central extensions of connected semisimple groups by a torus.

We should mention a few further notions that are closely related to reductivity.

Definition 3.4. A linear algebraic group G is

(1) reductive if it has trivial unipotent radical $\mathcal{R}_u(G) = \{1\}$.
(2) linearly reductive if for every finite dimensional linear representation $\rho : G \to \mathrm{GL}_n(k)$ decomposes as a direct sum of irreducibles.
(3) geometrically reductive if for every representation $\rho : G \to \mathrm{GL}_n(k)$ and every non-zero G-invariant point $v \in \mathbb{A}^n$, there is a G-invariant non-constant homogeneous polynomial $f \in k[x_1, \dots, x_n]$ such that $f(v) \neq 0$.

Remark 3.5. An alternative and often used definition of linear reductivity is that for every finite dimensional linear representation $\rho : G \to \mathrm{GL}_n(k)$ and every non-zero G-invariant point $v \in \mathbb{A}^n$, there is a G-invariant homogeneous polynomial $f \in k[x_1, \dots, x_n]$ of degree 1 such that $f(v) \neq 0$. The equivalence is seen as follows. If we have a representation $\rho : G \to \mathrm{GL}_n(k)$ decomposes as a direct sum of irreducibles, then for $v \neq 0$ fixed by G, there is a projection $f : \mathbb{A}^n \to \mathbb{A}^1$ onto the line spanned by v. This is clearly G-invariant and linear; i.e. corresponds to a degree 1 homogeneous G-invariant polynomial and $f(v) = v \neq 0$. Conversely, if for every $v \neq 0$ fixed by G, there is a projection $f : \mathbb{A}^n \to \mathbb{A}^1$ such that $f(v) \neq 0$, then we have a direct sum decomposition $\mathbb{A}^n_k \cong \mathrm{span}(v) \oplus \ker(v)$. This allows us to decompose the representation into a direct sum of irreducibles.

It is clear from the above remark that a linearly reductive group is also geometrically reductive. Nagata showed that every geometrically reductive group is reductive [33]. In characteristic zero we have that all three notions coincide as a Theorem of Weyl shows that every reductive group is linearly reductive. In positive characteristic, the different notions of reductivity are related as follows:

$$\text{linearly reductive} \implies \text{geometrically reductive} \iff \text{reductive}$$

where the implication that every reductive group is geometrically reductive is the most recent result (this was conjectured by Mumford and then proved by Haboush [15]).

Example 3.6. Every torus $(\mathbb{G}_m)^r$ and finite group is reductive. Also the groups $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$ and $\mathrm{PGL}(n, k)$ are all reductive. The additive group $\mathbb{G}_a$ of k under addition is not reductive. In positive characteristic, the groups $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$ and $\mathrm{PGL}_n(k)$ are not linearly reductive for $n > 1$.

Exercise 3.7. Show that the additive group $\mathbb{G}_a$ is not geometrically reductive by giving a representation $\rho : \mathbb{G}_a \to \mathrm{GL}_2(k)$ and a G-invariant point $v \in \mathbb{A}^2$ such that every non-constant G-invariant homogeneous polynomial in two variables vanishes at v.

Lemma 3.8. *Suppose G is geometrically reductive and acts on an affine variety X. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in A(X)^G$ which separates these sets i.e.*

$$f(W_1) = 0 \quad \text{and} \quad f(W_2) = 1.$$

Proof. As W_i are disjoint and closed

$$(1) = I(\phi) = I(W_1 \cap W_2) = I(W_1) + I(W_2)$$

and so we can write $1 = f_1 + f_2$. Then $f_1(W_1) = 0$ and $f_1(W_2) = 1$. The linear subspace V of $A(X)$ spanned by $g \cdot f_1$ is finite dimensional by Lemma 2.23 and so we can choose a basis $h_1, \dots, h_n$. This basis defines a morphism $h : X \to k^n$ by

$$h(x) = (h_1(x), \dots, h_n(x)).$$

For each i, we have that $h_i = \sum a_i(g) g \cdot f_i$ and so $h_i(x) = \sum_g a_i(g) f_1(g^{-1} \cdot x)$. As W_i are G-invariant, we have $h(W_1) = 0$ and $h(W_2) = v \neq 0$. The functions $g \cdot h_i$ also belong to V and so we can write them in terms of our given basis as

$$g \cdot h_i = \sum_j a_{ij}(g) h_j.$$

This defines a representation $G \to \mathrm{GL}_n(k)$ given by $g \mapsto (a_{ij}(g))$. We note that $h : X \to \mathbb{A}^n$ is then G-equivariant with respect to the action of G on X and $\mathrm{GL}_n(k)$ on $\mathbb{A}^n$; therefore $v = h(W_2)$ is a G-invariant point. As G is geometrically reductive, there is a non-constant homogeneous polynomial $f_0 \in k[x_1, \dots, x_n]^G$ such that $f_0(v) \neq 0$ and $f_0(0) = 0$. Then $f = c f_0 \circ h$ is the desired invariant function where $c = 1/f_0(v)$. □

3.3. **Nagata's theorem.** We recall that an action of G on a k-algebra A is rational if $A = A(X)$ for some affine variety X and this action comes from an (algebraic) action of G on X.

Theorem 3.9 (Nagata). *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. Then the G-invariant subalgebra A^G is finitely generated.*

As every reductive group is geometrically reductive, we can use Nagata's theorem for reductive groups. In the following section, we will prove this result for linearly reductive groups using Reynolds operators (so in characteristic zero this also proves Nagata's theorem). Nagata also gave a counterexample of a non-reductive group action for which the ring of invariants is not finitely generated (see [32] and [33]). The problem of constructing quotients for non-reductive groups is very interesting and already there is progress in this direction; for example, see the foundational paper on non-reductive GIT by Doran and Kirwan [12].

3.4. **Reynolds operators.** Given a linearly reductive group G, for any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we can write $V = V^G \oplus W$ where W is the direct sum of all non-trivial irreducible sub-representations. In particular, there is a projection $p : V \to V^G$. This motivates the following definition.

Definition 3.10. For a group G acting rationally on a k-algebra A. A linear map $R_A : A \to A^G$ is called a Reynolds operator if if it a projection onto A^G and for $a \in A^G$ and $b \in B$ we have $R_A(ab) = a R_A(b)$.

Proposition 3.11. *Let G be a linearly reductive group acting rationally on a k-algebra A; then there exists a Reynolds operator and A^G is finitely generated.*

Proof. As the action is rational, there is an affine variety X such that $A(X) = A$ and the action comes from an algebraic action on X. There is a natural homogeneous grading of $A(X)$ and so we write $A = \oplus_{r \geq 0} A_r$ with $A_0 = k$. As each graded piece A_n is a finite dimensional vector space and G is linearly reductive, there is a Reynold's operator $R_r : A_n \to A_n^G$. This allows us to define a Reynold's operator R_A on A. In particular, $A^G = \oplus_{n \geq 0} A_n^G$ is a graded k-algebra: if we take $f \in A^G$ and write $f = f_0 + \cdots + f_d$ with f_i homogeneous of degree m, then

$$f = R_A(f) = R_A(f_0 + \cdots + f_d) = R_0(f_0) + \cdots + R_d(f_d)$$

with $R_i(f_i) \in A_i^G$.

By Hilbert's basis theorem, A is a Noetherian ring and so the ideal $I = \oplus_{r>0} A_r^G$ is finitely generated; that is, we have $I =< f_1, \dots, f_m >$ for f_i of degree d_i. We claim that the elements $f_1, \dots, f_m$ generate A^G as a k-algebra. The proof is by induction on degree. The degree zero case is trivial as we have $A_0^G = 0$. We fix $d > 0$ and assume that all elements of degree strictly less than d can be written as a polynomial in the f_i with coefficients in k. Now take $f \in A_d^G$; then we can write

$$f = a_0 f_0 + \cdots + a_m f_m$$

for $a_i \in A$. If we replace a_i by its homogeneous piece of degree $d - d_i$ then the same equation still holds and so we can assume each a_i is homogeneous of degree $d - d_i$. We apply the Reynolds operator R_A to the above expression for f to obtain

$$f = R_A(f) = R_A(a_0) f_0 + \cdots + R_A(a_m) f_m$$

as a_i is homogeneous of degree $d-d_i$, we have that $R_A(a_i) \in A^G$ is also homogeneous of degree $d-d_i$. Hence each $R_A(a_i)$ is homogeneous of degree strictly less than d and so we deduce by induction that they are polynomials in the f_j with coefficients in k. In particular, f is also a polynomial in the f_i with coefficients in k. □

3.5. **Construction of the affine GIT quotient.** Let G be a reductive group acting on an affine variety X. We have seen that this induces an action of G on the affine coordinate ring $A(X)$ which is a finitely generated k-algebra without zero-divisors. By Nagata's Theorem the subalgebra of invariants $A(X)^G$ is finitely generated.

Definition 3.12. The affine GIT quotient is the morphism $\varphi : X \to X//G := \operatorname{Spec} A(X)^G$ of varieties associated to the inclusion $\varphi^* : A(X)^G \hookrightarrow A(X)$.

Remark 3.13. The double slash notation $X//G$ used for the GIT quotient is a reminder that this quotient is not necessarily an orbit space and so it may identify some orbits. In nice cases, the GIT quotient is also a geometric quotient and in this case we shall often write X/G instead to emphasise the fact that it is an orbit space.

Theorem 3.14. *Let G be a reductive group acting on an affine variety X. Then the affine GIT quotient $\varphi : X \to Y := X//G$ is a good quotient and in particular Y is an affine variety. Moreover if the action of G is closed on X, then it is a geometric quotient.*

Proof. As G is reductive and so also geometrically reductive, it follows from Nagata's Theorem that the algebra of G-invariant regular functions on X is a finitely generated reduced k-algebra. Hence $Y := \operatorname{Spec} A(X)^G$ is an affine variety. The affine GIT quotient is defined by the inclusion $A(X)^G \hookrightarrow A(X)$ and so is G-invariant and affine: this gives part i) and vi) in the definition of good quotient.

To prove ii), we take $y \in Y$ and want to construct $x \in X$ whose image under $\varphi : X \to Y$ is y. Let $\mathfrak{m}_y$ be the maximal idea in $A(Y) = A(X)^G$ corresponding to the point y. We choose generators $f_1, \dots, f_m$ of $\mathfrak{m}_y$ and, as G is reductive, we claim that

$$\sum_{i=1}^{m} f_i A(X) \neq A(X).$$

In general, this claim can be deduced by using [36] Lemma 3.4.2. Here we provide a quick and simple proof of this statement for linearly reductive groups (and so, at least in characteristic zero, this proves the result). It suffices to prove that

$$\left(\sum_{i=1}^{m} f_i A(X) \right) \cap A(X)^G = \sum f_i A(X)^G;$$

as the ideal $\mathfrak{m}_y$ in $A(X)^G$ generated by the f_i is a proper maximal ideal. The right hand side of this expression is contained in the left hand side and to prove the opposite containment we use the Reynold's operator $R_A : A(X) \to A(X)^G$ (which exists as G is linearly reductive). We write $f = \sum f_i a_i \in A(X)^G$ with $a_i \in A(X)$ and apply the Reynold's operator; then

$$f = R_A(f) = \sum R_A(a_i) f_i$$

with $R_A(a_i) \in A(X)^G$ which proves the result. Then, as $\sum_{i=1}^m f_i A(X)$ is not equal to $A(X)$, it is contained in some maximal idea $\mathfrak{m}_x \subset A(X)$ corresponding to a closed point $x \in X$. In particular, we have that $f_i(x) = 0$ for $i = 1, \dots, m$ and so $\varphi(x) = y$ as required.

As the open sets of the form $U = Y_f := \{y \in Y : f(y) \neq 0\}$ for non-zero $f \in A(X)^G$ form a basis for the open sets of Y it suffices to verify iii) for these open sets. If $U = Y_f$, then $\mathcal{O}_Y(U) = (A(X)^G)_f$ is the localisation of $A(X)^G$ with respect to f and

$$\mathcal{O}_X(\varphi^{-1}(U))^G = \mathcal{O}_X(X_f)^G = (A(X)_f)^G = (A(X)^G)_f = \mathcal{O}_Y(U)$$

as localisation commutes with taking G-invariants. Hence the image of the inclusion homomorphism $\mathcal{O}_Y(U) = (A(X)^G)_f \to \mathcal{O}_X(\varphi^{-1}(U)) = A(X)_f$ is $\mathcal{O}_X(\varphi^{-1}(U))^G = (A(X)_f)^G$ and this homomorphism is an isomorphism onto its image.

By Remark 2.37 iv) and v) are equivalent to v)$'$ and so it suffices to prove v)$'$. For this, we use the fact that G is geometrically reductive: by Lemma 3.8 for any two disjoint closed subsets W_1 and W_2 in X there is a function $f \in A(X)^G$ such that f is zero on W_1 and equal to 1 on W_2. We may view f as a regular function on Y and as $f(\varphi(W_1)) = 0$ and $f(\varphi(W_2)) = 1$, we must have

$$\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi.$$

The final statement follows immediately from Corollary 2.39. □

Example 3.15. Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$ as in Example 2.28. In this case $A(X) = k[x, y]$ and $A(X)^G = k[xy] \cong k[z]$ so that $Y = \mathbb{A}^1$ and the GIT quotient $\varphi : X \to Y$ is given by $(x, y) \mapsto xy$. The three orbits consisting of the punctured axes and the origin are all identified and so the quotient is not a geometric quotient.

Example 3.16. Consider the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ by $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ as in Example 2.29. Then $A(X) = k[x_1, \dots, x_n]$ and $A(X)^G = k$ so that $Y = \operatorname{Spec} k$ is a point and the GIT quotient $\varphi : X \to Y = \operatorname{Spec} k$ is given by the structure morphism. In this case all orbits are identified and so this good quotient is not a geometric quotient.

Remark 3.17. We note that the fact that G is reductive was used several times in the proof, not just to show the ring of invariants is finitely generated. In particular, there are non-reductive group actions which have finitely generated invariant rings but for which other properties listed in the definition of good quotient fail. For example, consider the additive group $\mathbb{G}_a$ acting on $X = \mathbb{A}^4$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}_4$

$$s \mapsto \begin{pmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{pmatrix}.$$

Even though $\mathbb{G}_a$ is non-reductive, the invariant ring is finitely generated:

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{G}_a} = \mathbb{C}[x_2, x_4, x_1x_4 - x_2x_3].$$

However the GIT 'quotient' map $X \to X//\mathbb{G}_a = \mathbb{A}^3$ is not surjective; its image misses the punctured line $\{(0, 0, \lambda) : \lambda \in k^*\} \subset \mathbb{A}^3$. For further differences, see [12].

3.6. **Geometric quotients on open subsets.** As we saw above, when a reductive group G acts on an affine variety X in general a geometric quotient (i.e. orbit space) does not exist because in general the action is not closed. For finite groups G, every good quotient is a geometric quotient as the action of a finite group is always closed (every orbit is a finite number of points which is a closed subset). For general G, we ask if there is an open subset of X for which there is a geometric quotient.

Definition 3.18. We say $x \in X$ is stable if its orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We let X^s denote the set of stable points.

Proposition 3.19. *Suppose a reductive group G acts on an affine variety X and let $\varphi : X \to Y$ be the associated good quotient. Then $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$ is also open. Moreover, $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. We first show that X^s is open by showing for every $x \in X^s$ there is an open neighbourhood of x in X^s. By Lemma 2.27, the set $X_+ := \{x \in X : \dim G_x > 0\}$ of points with positive dimensional stabilisers is a closed subset of X. If $x \in X^s$, then by Lemma 3.8 there is a function $f \in A(X)^G$ such that

$$f(X_+) = 0, \quad f(G \cdot x) = 1.$$

It is clear that x belongs to the open subset X_f, but in fact we claim that $X_f \subset X^s$ so it is an open neighbourhood of x. It is clear that all points in X_f must have stabilisers of dimension zero but we must also check that their orbits are closed. Suppose $z \in X_f$ has a non closed orbit so $w \notin G \cdot z$ belongs to the orbit closure of z; then $w \in X_f$ too as f is G-invariant and so w must have stabiliser of dimension zero. However, by Proposition 2.26 the boundary of the orbit

$G \cdot z$ is a union of orbits of strictly lower dimension and so the orbit of w must be of dimension strictly less than that of z which contradicts the orbit stabiliser theorem. Hence X^s is open and is covered by sets of the form X_f.

Since $\varphi(X_f) = Y_f$ is also open in Y and $X_f = \varphi^{-1}(Y_f)$, we see that Y^s is open and also $X^s = \varphi^{-1}(\varphi(X^s))$. Then $\varphi : X^s \to Y^s$ is a good quotient and the action of G on X^s is closed; thus $\varphi : X^s \to Y^s$ is a geometric quotient by Corollary 2.39. □

Example 3.20. We can now calculate the stable set for the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ as in Examples 2.28 and 3.15. The closed orbits are the conics $\{xy = a\}$ for $a \in k^*$ and the origin, however the origin has a positive dimensional stabiliser and so

$$X^s = \{(x, y) \in \mathbb{A}^2 : xy \neq 0\} = X_{xy}.$$

In this example, it is clear why we need to insist that $\dim G_x = 0$ in the definition of stability: so that the stable set is open. In fact this requirement should also be clear from the proof of Proposition 3.19.

Example 3.21. We may also consider which points are stable for the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ as in Examples 2.29 and 3.16. In this case the only closed orbit is the origin whose stabiliser is positive dimensional and so $X^s = \phi$. In particular, this example shows that the stable set may be empty.

Example 3.22. Consider $G = \mathrm{GL}_2(k)$ acting on the space of 2×2 matrices $M_{2\times 2}(k)$ by conjugation. The characteristic polynomial of a matrix A is given by

$$\mathrm{char}_A(t) = \det(xI - A) = x^2 + c_1(A)x + c_2(A)$$

where $c_1(A) = -\mathrm{Tr}(A)$ and $c_2(A) = \det(A)$ and is well defined on the conjugacy class of a matrix. The Jordan canonical form of a matrix is obtained by conjugation and so lies in the same orbit of the matrix. The theory of Jordan canonical forms says there are three types of orbits:

- matrices with characteristic polynomial with distinct roots α, β. These matrices are diagonalisable with Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$
These orbits are closed and have dimension 2 - the stabiliser of the above matrix is the subgroup of diagonal matrices which is 2 dimensional.
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is equal to the characteristic polynomial. These matrices are not diagonalisable - their Jordan canonical form is
$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$
These orbits are also 2 dimensional but are not closed - for example
$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is $x - \alpha$. These matrices have Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$
and as scalar multiples of the identity commute with everything, their stabilisers are full dimensional. Hence these orbits are closed and have dimension zero.

We note that every orbit of the second type contains an orbit of the third type and so will be identified in the quotient. There are only two G-invariant functions: the trace and determinant which define the GIT quotient

$$\varphi = (\mathrm{Tr}, \det) : M_{2\times 2}(k) \to \mathbb{A}^2.$$

The subgroup $\mathbb{G}_m I_2$ fixes every point and so there are no stable points for this action.

Example 3.23. More generally, we can consider $G = \mathrm{GL}_n(k)$ acting on $M_{n\times n}(k)$ by conjugation. If A is an $n \times n$ matrix, then the coefficients of its characteristic polynomial

$$\mathrm{char}_A(t) = \det(tI - A) = t^n + c_1(A)t^{n-1} + \cdots + c_n(A)$$

are all G-invariant functions. In fact, these functions generate the invariant ring

$$A(M_{n\times n})^G \cong k[c_1, \dots, c_n]$$

(for example, see [10] §1) and the affine GIT quotient is given by

$$\begin{aligned} \varphi : M_{n\times n} &\to \mathbb{A}^n \\ A &\mapsto (c_1(A), \dots, c_n(A)). \end{aligned}$$

As in Example 3.22 above, we can use the theory of Jordan canonical forms as above to describe the different orbits. The closed orbits correspond to diagonalisable matrices and as every orbit contains $k^* I_n$ in its stabiliser, there are no stable points.

Remark 3.24. In situations where there is a non-finite subgroup $H \subset G$ which is contained in the stabiliser subgroup of every point for a given action of G on X, the stable set is automatically empty. Hence, for the purposes of GIT, it is better to work with the induced action of the group G/H. In the above example, this would be equivalent to considering the action of the special linear group on the space of $n \times n$ matrices by conjugation.

Example 3.25 (Kleinian surface singularities)**.** The Kleinian surface singularities (also know as the du Val singularities) are isolated complex surface singularities that are classified by A-D-E Dynkin diagrams:

$$\begin{array}{rl} A_n: & 0 = x^2 + y^2 + z^{n+1} \\ D_n: & 0 = x^2 + y^2 z + z^{n-1} \\ E_6: & 0 = x^2 + y^3 + z^4 \\ E_7: & 0 = x^2 + y^3 + yz^3 \\ E_8: & 0 = x^2 + y^3 + z^5. \end{array}$$

They can all be constructed as GIT quotients $\mathbb{C}^2//\Gamma$ for $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ a finite subgroup; thus they are all geometric quotients. In fact, the finite subgroups Γ of $\mathrm{SL}_2(\mathbb{C})$ have a corresponding A-D-E classification. The classification of finite subgroups in $\mathrm{SO}_3(\mathbb{R})$ can be used to provide a classification of finite groups in $\mathrm{SL}_2(\mathbb{C})$ up to conjugation. This classification is as follows: a finite subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ is, up to conjugation:

$$\begin{array}{rl} A_n: & \Gamma = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} : \epsilon^{n+1} = 1 \right\} \\ D_n: & \Gamma = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} : \epsilon^{2(n-2)} = 1 \right\} \\ & \\ E_6: & \Gamma \text{ a double cover of } A_4 \subset \mathrm{SO}_3(\mathbb{R}) \\ & \\ E_7: & \Gamma \text{ a double cover of } S_4 \subset \mathrm{SO}_3(\mathbb{R}) \\ & \\ E_8: & \Gamma \text{ a double cover of } A_5 \subset \mathrm{SO}_3(\mathbb{R}). \end{array}$$

The invariant ring for each Γ were calculated by Klein and in particular he showed that the singularity of a given type occurs as the quotient $\mathbb{C}//\Gamma$ for the corresponding group.

4. Projective GIT quotients

In this section we extend the theory of affine GIT developed in the previous section to construct GIT quotients for reductive group actions on projective varieties. The approach we will take is to try and cover as much of X as possible by G-invariant affine open subvarieties and then use the theory of affine GIT to construct a good quotient $\varphi : X^{ss} \to Y$ of an open subvariety X^{ss} of X (known as the GIT semistable set). As X is projective we would also like our quotient Y to be projective (and in fact it turns out this is the case). As in the affine case, we can restrict our attention to an open subvariety X^s of stable points for which there is a geometric quotient $X^s \to Y^s$ where Y^s is an open subvariety of Y. The main reference for the construction of the projective GIT quotient is Mumford's book [31] and other excellent references are [10], [29], [36], [37] and [48]. We start this section by recalling some definitions and results about projective varieties from algebraic geometry.

4.1. **Projective algebraic geometry.** Let k be an algebraically closed field and let $\mathbb{P}^n = \mathbb{P}^n_k$ denote projective n-space over k:

$$\mathbb{P}^n = (\mathbb{A}^{n+1} - \{0\})/\sim$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if there exists $\lambda \in k^*$ such that $(a_0, \cdots, a_n) = (\lambda b_0, \dots, \lambda b_n)$. Thus two points in $\mathbb{A}^{n+1} - \{0\}$ are equivalent if and only if they lie on the same line through the origin. One way to think of $\mathbb{P}^n$ is as the space of punctured lines through the origin in $\mathbb{A}^{n+1}$. Or in terms of group actions, $\mathbb{P}^n$ is the orbit space for the action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{A}^{n+1} - \{0\}$ by scalar multiplication. There is a natural projection map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ sending a point to its equivalence and we refer to $\mathbb{A}^{n+1}$ as the affine cone over $\mathbb{P}^n$. For any point p of projective space we can choose a point $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} - \{0\}$ which is a representative of the equivalence class p (we say $(a_0, \dots, a_n)$ lies over p). We shall often write $p = [a_0 : \cdots : a_n]$ and say the tuple $(a_0, \dots, a_n)$ are homogeneous coordinates for p.

Projective n-space $\mathbb{P}^n$ is a variety as it can be covered by open affine varieties $U_i \cong \mathbb{A}^n$ for $i = 0, \dots, n$ where

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i \neq 0\}$$

and the isomorphism $\mathbb{A}^n \to U_0$ is given by $(a_1, \dots, a_n) \mapsto [1 : a_1 : \cdots : a_n]$.

A polynomial $f(x_0, \dots, x_n)$ in the (affine) coordinate ring $A(\mathbb{A}^{n+1}) = k[x_0, \dots, x_n]$ is homogeneous of degree r if for every $\lambda \in k^*$ we have

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^r f(x_0, \dots, x_n);$$

that is, f is a linear combination of monomials $x_0^{r_0} x_1^{r_1} \dots x_n^{r_n}$ of degree r (i.e. $\sum r_i = r$). Moreover, as any polynomial can be written as a sum of homogeneous polynomials we have a decomposition

$$k[x_0, \dots, x_n] = \oplus_{r \geq 0} k[x_0, \dots, x_n]_r$$

into a graded k-algebra (a k-algebra R is graded if we can write $R = \oplus_r R_r$ where each R_r is a k-vector space and $R_r R_s \subset R_{rs}$). If $(a_0, \dots, a_n) \in \mathbb{A}^{n+1}$ lies over a point $p \in \mathbb{P}^n$, then for a homogeneous polynomial f we have

$$f(a_0, \dots, a_n) = 0 \iff f(\lambda a_0, \dots, \lambda a_n) = 0$$

for all nonzero $\lambda \in k$. Hence whether f is zero or not at a point $p \in \mathbb{P}^n$ is a well defined notion (it is independent of the choice of representative of the equivalence class of p). We can see a homogeneous polynomial f as a two valued function on $\mathbb{P}^n$: it is either zero or non-zero. Given homogeneous polynomials $f_1, \dots, f_m \in k[x_0, \dots, x_n]$, we define the associated algebraic subset

$$V(f_1, \dots, f_m) = \{p \in \mathbb{P}^n : f_i(p) = 0 \text{ for } i = 1, \dots m\} \subset \mathbb{P}^n.$$

An ideal $I \subset k[x_0, \dots, x_n]$ is a homogeneous ideal if $I = \oplus_{r \geq 0} I \cap k[x_0, \dots, x_n]_r$. In this case (as $k[x_0, \dots, x_n]$ is Noetherian), the ideal is finitely generated $I = (f_1, \dots, f_m)$ by homogeneous polynomials f_i. Then we write $V(I) = V(f_1, \dots, f_m)$. The Zariski topology on $\mathbb{P}^n$ is given by letting the algebraic subsets $V(f_1, \dots, f_m)$ be the closed sets.

Given $X \subset \mathbb{P}^n$, we let $I(X)$ denote the ideal in $k[x_0, \dots, x_n]$ of homogeneous polynomials which vanish on X. The projective Nullstellensatz describes the relationship between I and V:

- For a subset $X \subset \mathbb{P}^n$, we have $X \subset V(I(X))$ with equality if and only if X is a closed subset.
- For a homogeneous ideal I, we have $I \subset I(V(I))$ with equality if and only if I is a radical ideal.

Definition 4.1. A projective variety $X \subset \mathbb{P}^n$ is an algebraic subset of $\mathbb{P}^n$ with the induced topology.

If X is a projective subvariety of $\mathbb{P}^n$; then we may consider the affine cone $\tilde{X}$ over X which is an affine subvariety of the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$:

$$\tilde{X} = \{0\} \cup \{(x_0, \dots, x_n) \in \mathbb{A}^{n+1} - \{0\} : [x_0 : \dots : x_n] \in X\}.$$

Thus $X = (\tilde{X} - \{0\})/\sim$. If X is the projective variety cut out as the zero locus of homogeneous polynomials $f_1, \dots, f_m \in k[x_0, \dots, x_n]$, then $\tilde{X} = V(f_1, \dots, f_m) \subset \mathbb{A}^n$ is the affine variety cut out as the zero locus of the regular functions $f_1, \dots, f_m$ on $\mathbb{A}^{n+1}$.

For a projective variety $X \subset \mathbb{P}^n$, we define the homogeneous coordinate ring of X by

$$R(X) = k[x_0, \dots, x_n]/I(X).$$

We may also write $R(X) = \oplus_{l \geq 0} R_l$ as a graded k-algebra and we call $R(X)_+ := \oplus_{l \geq 0} R_l$ the irrelevant ideal (the name comes from the fact that in $\mathbb{P}^n$ the irrelevant ideal is $(x_0, \dots, x_n)$ which corresponds to $0 \in \mathbb{A}^{n+1}$ and so does not project to a point in $\mathbb{P}^n$). The homogeneous coordinate ring $R(\mathbb{P}^n)$ of $\mathbb{P}^n$ is equal to the affine coordinate ring $A(\mathbb{A}^{n+1})$ on its affine cone $\mathbb{A}^{n+1}$

$$R(\mathbb{P}^n) = \oplus_{l \geq 0} k[x_0, \dots, x_n]_l = k[x_0, \dots, x_n] = A(\mathbb{A}^{n+1})$$

and similarly $R := R(X) = R(\mathbb{P}^n)/I(X) = A(\mathbb{A}^{n+1})/I(\tilde{X}) = A(\tilde{X})$.

Remark 4.2. We note that the definition of $R(X)$ depends on how we realise X as a subset of $\mathbb{P}^n$ (as this choice defines the affine cone $\tilde{X}$) and so a different embedding $X \subset \mathbb{P}^m$ will lead to a different homogeneous graded k-algebra. Thus strictly speaking we should write $R(X \subset \mathbb{P}^n)$ rather than just $R(X)$ to emphasise this dependence.

A non-constant polynomial $f \in k[x_0, \dots, x_n]$ does not define a well defined function on $\mathbb{P}^n$, however the quotient f/g of two homogeneous polynomials of degree d is a rational morphism on PP^n (that is, it is only a well defined on the open subset $\mathbb{P}^n - V(g)$ where g is non-zero) as for $\lambda \neq 0$ and $[a_0 : \dots : a_n] \in \mathbb{P}^n - V(g)$ we check that

$$\frac{f}{g}(\lambda a_0, \dots, \lambda a_n) = \frac{f(\lambda a_0, \dots \lambda, a_n)}{g(\lambda a_0, \dots, \lambda a_n)} = \frac{\lambda^d f(a_0, \dots, a_n)}{\lambda^d g(a_0, \dots, a_n)} = \frac{f}{g}(a_0, \dots, a_n).$$

Definition 4.3. Let $X \subset \mathbb{P}^n$ be a projective variety and U an open subset of X; then a function $f : U \to k$ is regular at $p \in U$, if there is an open neighbourhood V of p such that $f = g/h$ on U and $g(p) \neq 0$ where f, g are homogeneous polynomials of the same degree $k[x_0, \dots, x_n]$. We say f is regular if it is regular at all point in U and denote the k-algebra of regular functions by $\mathcal{O}_X(U)$.

We summarise some properties of projective varieties in the following theorem (see also [17] I Theorem 3.4 and II Proposition 2.5).

Theorem 4.4. *Let X be an irreducible projective variety over k. Then*

i) *The ring $\mathcal{O}_X(X)$ of regular functions on X is isomorphic to k.*
ii) *There is a one-to-one correspondence between the points p in X and homogeneous maximal ideals $\mathfrak{m}_p$ in $R(X)$ which do not contain $R(X)_+$ where $\mathfrak{m}_p$ is the ideal of homogeneous polynomials which vanish at p.*
iii) *For homogeneous $f \in R(X)_+$, we define $X_f := \{x \in X : f(x) \neq 0\} = X - V(f)$ and let $(R(X)_f)_0$ denote the degree zero piece of the localised graded k-algebra $R(X)_f$. Then*

$$X_f \cong \mathrm{Spec}(R(X)_f)_0$$

and

$$\mathcal{O}_X(X_f) \cong (R(X)_f)_0.$$

iv) *The open sets X_f for homogeneous $f \in R(X)_+$ form a basis for the Zariski topology of X.*

The structure sheaf $\mathcal{O}_X$ on a projective variety $X \subset \mathbb{P}^n$ is an invertible sheaf (i.e. line bundle) over X and there are two other important invertible sheaves on X: the tautological sheaf $\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}^n}(-1)|_X$ and the Serre twisting sheaf $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_X$, which is dual to $\mathcal{O}_X(-1)$. The fibre of the tautological line bundle on $\mathbb{P}^n$ over a point x is the line in $\mathbb{A}^{n+1}$ which defines this point $x \in \mathbb{P}^n$ and so the total space of the tautological line bundle is the blow up of $\mathbb{A}^{n+1}$ at the origin.

Example 4.5. Let $X = \mathbb{P}^n$ and $f(x_0, \dots, x_n) = x_0$. Then $X_f = \{[x_0 : \cdots : x_n] : x_0 \neq 0\} = U_0 \cong \mathbb{A}^n$ and the regular functions on X_f are

$$\mathcal{O}_X(X_f) = k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] \cong k[y_1, \dots, y_n].$$

The operator $R(-)$ sends a projective variety X to its homogeneous coordinate ring $R(X)$. Given a reduced finitely generated graded k-algebra $R = \oplus_r R_r$, we can construct an associated projective variety $X = \operatorname{Proj} R$ (called the projective spectrum) which comes with a specified embedding in projective space as follows. A finite set of homogeneous generators of R of the same degree define a surjection of graded k-algebras

$$k[x_0, \dots, x_n] \to R$$

and we let I be the homogeneous ideal equal to the kernel of this surjection. Then $\operatorname{Proj} R$ is the projective variety $V(I) \subset \mathbb{P}^n$. If R cannot be generated by homogeneous elements of the same degree, then instead one realises X as a subvariety of a weighted projective space. Alternatively, if one is willing to work abstractly without taking generators, then $\operatorname{Proj} R$ as a set is the set of prime homogeneous ideals in R which do not contain the irrelevant ideal R_+ and one can also define the Zariski topology in this abstract setting (for example, see [17] II §2).

Given any finitely generated graded subalgebra S of R, the inclusion $S \hookrightarrow R$ of finitely generated graded k-algebras induces a rational morphism (i.e. a morphism that is only well defined on an open subset of X) of projective varieties

$$\varphi : X := \operatorname{Proj} R \dashrightarrow Y := \operatorname{Proj} S$$

which is undefined on the nullcone:

$$N_S(X) = \{x \in X : f(x) = 0 \,\forall f \in S_+ := \oplus_{l>0} S_l\} \subset X$$

which is a closed subvariety of X. We note the following:

- φ is a well defined morphism on $X_S := \cup_{f \in S_+} X_f = X - N_S(X)$.
- $Y = \cup_{f \in S_+} Y_f$ and $\varphi^{-1}(Y_f) = X_f$.
- Moreover, $A(Y_f) \cong (S_f)_0$ where $(S_f)_0$ denotes the degree zero homogeneous piece of the graded homogeneous algebra S_f obtained by localising S at f.
- The morphism $\varphi : X_S \to Y$ is obtained by gluing the morphisms of affine algebraic varieties $\varphi_f : X_f \to Y_f$ for $f \in S_+$ corresponding to the inclusions
$$A(Y_f) \cong (S_f)_0 \subset (R_f)_0 = A(X_f).$$

In the remainder of this section we recall some important properties of abstract algebraic varieties; this can be happily ignored by those who are not interested in constructing GIT quotients for abstract projective varieties. An abstract projective variety does not come with a specified embedding into projective space, but if we choose a very ample line bundle L on X then (by definition of L being very ample) we can pick global sections $s_0, \dots, s_n \in H^0(X, L)$ such that the rational map $X \dashrightarrow \mathbb{P}^n$ given by

$$x \mapsto [s_0(x) : \cdots : s_n(x)]$$

is a closed embedding. In fact we can write this embedding in a coordinate free way: there is an embedding $X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$ given by

$$x \mapsto ev_x : H^0(X, L) \to L_x \cong \mathbb{C}$$

where $ev_x(s) = s(x)$. If $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion of a closed subvariety, then $L = i^*\mathcal{O}_{\mathbb{P}^n}(1)$ is a very ample line bundle on X and the associated closed embedding is equal to i.

We say a line bundle L is ample if some tensor power of itself $L^{\otimes n}$ for $n > 0$ is very ample (this being ample is a slightly weaker notion than being very ample line). Given an ample line bundle L on X, we can consider the associated graded k-algebra

$$R = R(X, L) := \oplus_{l \geq 0} H^0(X, L^{\otimes l}).$$

Then the (maximal) "Proj construction" for graded rings allows us to recover the pair (X, L); we recall that the points of the $\operatorname{Proj} R$ correspond to maximal homogeneous ideals in this graded ring R which do not contain the irrelevant ideal R_+. We can also define a topology and structure sheaf over $\operatorname{Proj} R$ using R (see [17] Chapter II § 2). We note that if we replace L by the associated very ample line bundle L^n for $n > 0$, then $R(X, L^n)$ will be generated in degree 1.

4.2. **Construction of the projective GIT quotient.**

Definition 4.6. An action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ is said to be linear if G acts via a homomorphism $G \to \mathrm{GL}_{n+1}$.

If we have a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, then G acts on the affine cones $\mathbb{A}^{n+1}$ and $\tilde{X}$ over $\mathbb{P}^n$ and X. In particular G acts on $R := A(\tilde{X})$ and preserves the graded pieces so that $R^G = A(\tilde{X})^G$ is a homogeneous graded subalgebra of R. By Nagata's theorem this is also finitely generated and so we can consider the associated projective variety $\operatorname{Proj}(R^G)$.

Definition 4.7. For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we let $X//G$ denote the projective variety $\operatorname{Proj}(R^G)$ associated to the finitely generated graded k-algebra R^G of G-invariant functions where $R = R(X)$ is the homogeneous coordinate ring of X. The inclusion $R^G \hookrightarrow R$ defines a rational map

$$\varphi : X \dashrightarrow X//G$$

which is undefined on the null cone

$$N_{R^G}(X) := \{x \in X : f(x) = 0 \, \forall f \in R^G_+\}.$$

We define the semistable locus $X^{ss} := X - N_{R^G}(X)$ to be the complement to the nullcone. Then the projective GIT quotient for the linear action of G on $X \subset \mathbb{P}^n$ is the morphism $\varphi : X^{ss} \to X//G$.

Proposition 4.8. *The projective GIT quotient for a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ is a good quotient of the action of G on X^{ss}.*

Proof. We let $\varphi : X^{ss} \to Y := X//G$ denote the projective GIT quotient. For $f \in R^G_+$, we have that

$$A(Y_f) \cong ((R^G)_f)_0 = ((R_f)_0)^G = A(X_f)^G$$

and so the corresponding morphism of affine varieties $\varphi_f : X_f \to Y_f$ is a good quotient by Theorem 3.14. The open subsets X_f cover X^{ss} and the open subsets Y_f cover Y. Moreover, the morphism $\varphi : X^{ss} \to Y$ is obtained by gluing the good quotients $\varphi_f : X_f \to Y_f$ and so is also a good quotient. □

We can now ask if there is an open subset X^s of X^{ss} on which this quotient becomes a geometric quotient. For this we want the action to be closed on X^s, or at least the action is closed on some affine open G-invariant subsets which cover X^s. This motivates the definition of stability (see also Definition 3.18):

Definition 4.9. Consider a linear action of a reductive group G on a closed subvariety $X \subset \mathbb{P}^n$. Then a point $x \in X$ is

1. semistable if there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $f(x) \neq 0$.
2. stable if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $x \in X_f$ and the action of G on X_f is closed.

(3) unstable if it is not semistable.

We denote the set of stable points by X^s and the set of semistable points by X^{ss}.

Remark 4.10. The semistable set X^{ss} is the complement of the null cone $N_{R^G}(X)$ and so is open in X. The stable locus X^s is open in X (and also in X^{ss}): let $X_c := \cup X_f$ where the union is taken over $f \in R(X)^G_+$ such that the action of G on X_f is closed; then X_c is open in X and it remains to show X^s is open in X_c. By Proposition 2.27, the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X and so the set of points with zero dimensional stabiliser is open. Therefore, we have open inclusions $X^s \subset X_c \subset X$.

Theorem 4.11. *For a linear action of a reductive group G on a closed subvariety $X \subset \mathbb{P}^n$, we have:*

i) *The GIT quotient $\varphi : X^{ss} \to Y := X//G$ is a good quotient and a categorical quotient. Moreover, Y is a projective variety.*
ii) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq 0$ *if and only if $\varphi(x_1) = \varphi(x_2)$.*
iii) *There is an open subset $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. Part i) is covered by Proposition 4.8 above and part ii) is given by Corollary 2.39 for good quotients. For iii), we let Y_c be the union of Y_f for $f \in R(X)^G_+$ such that the G action on X_f is closed and let X_c be the union of X_f over the same index set so that $X_c = \varphi^{-1}(Y_c)$. Then $\varphi : X_c \to Y_c$ is constructed by gluing $\varphi_f : X_f \to Y_f$ for $f \in R(X)^G_+$ such that the G action on X_f is closed. Each φ_f is a good quotient and as the action on X_f is closed, φ_f is also a geometric quotient (cf. Corollary 2.39). Therefore $\varphi : X_c \to Y_c$ is a geometric quotient. By definition X^s is the open subset of X_c consisting of points with zero dimensional stabilisers and we let $Y^s := \varphi(X^s) \subset Y_c$. As $\varphi : X_c \to Y_c$ is a geometric quotient and X^s is a G-invariant subset of X, $\varphi^{-1}(Y^s) = X^s$ and also $Y_c - Y^s = \varphi(X_c - X^s)$. As $X_c - X^s$ is closed in X_c, property iv) of good quotient states that $\varphi(X_c - X^s) = Y_c - Y^s$ is closed in Y_c and so Y^s is open in Y_c. Since Y_c is open in Y, the subset $Y^s \subset Y$ is open and the geometric quotient $\varphi : X_c \to Y_c$ restricts to a geometric quotient $\varphi : X^s \to Y^s$. □

Remark 4.12. We see from the proof of this theorem that to get a geometric quotient we do not have to impose the condition $\dim G_x = 0$ and in fact in Mumford's original definition of stability this condition was omitted. However, the modern definition of stability, which asks for zero dimensional stabilisers, is now widely accepted. One advantage of the modern definition is that if the semistable set is nonempty, then the dimension of the geometric quotient equals its expected dimension.

Example 4.13. Consider the linear action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n].$$

In this case $R(X) = k[x_0, \ldots, x_n]$ which is graded into homogeneous pieces by degree. It is easy to see that the invariant functions $x_0x_1, \ldots, x_0x_n$ generate the G-invariant subalgebra $R(X)^G$, so

$$R(X)^G = k[x_0x_1, \ldots, x_0x_n] \cong k[y_0, \ldots, y_{n-1}]$$

corresponds to the projective variety $X//G = \mathbb{P}^{n-1}$. The explicit choice of generators for $R(X)^G$ allows us to write down the rational morphism

$$\varphi : X = \mathbb{P}^n \dashrightarrow X//G = \mathbb{P}^{n-1}$$

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0x_1 : \cdots : x_0x_n]$$

and its clear from this description that the nullcone

$$N_{R(X)^G}(X) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0 \text{ or } x_1 \cdots x_n = 0\}$$

is the projective variety defined by the homogeneous ideal $I = (x_0x_1, \cdots, x_0x_n)$. In particular,

$$X^{ss} = \bigcup_{i=1}^n X_{x_0x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \ldots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

where we are identifying the affine chart on which $x_0 \neq 0$ in $\mathbb{P}^n$ with $\mathbb{A}^n$. Therefore

$$\varphi : X^{ss} = \mathbb{A}^n - \{0\} \dashrightarrow X//G = \mathbb{P}^{n-1}$$

is a good quotient of the action on X^{ss}. As the preimage of each point in $X//G$ is a single orbit, this is also a geometric quotient. Moreover, every semistable point is stable as all orbits are closed in $\mathbb{A}^n - \{0\}$ and have zero dimensional stabilisers.

In general it can be difficult to determine which points are semistable and stable as it is necessary to have a good description of the graded k-algebra of invariant functions. The ideal situation is as above where we have an explicit set of generators for the invariant algebra which allows us to write down the quotient map. However, finding generators for the invariant algebra in general can be hard. We will soon see that there are other criteria that we can use to determine (semi)stability of points.

Lemma 4.14. *A point x is stable if and only if x is semistable and its orbit $G \cdot x$ is closed in X^{ss} and has zero dimensional stabiliser.*

Proof. Suppose x is stable and $y \in \overline{G \cdot x} \cap X^{ss}$; then $\varphi(y) = \varphi(x)$ and so $y \in \varphi^{-1}(\varphi(x)) \subset \varphi^{-1}(Y^s) = X^s$. As the action of G on X^s is closed, $y \in G \cdot x$ and so the orbit $G \cdot x$ is closed in X^{ss}.

Conversely, we suppose x is semistable with closed orbit in X^{ss} of top dimension. As x is semistable, there is a homogeneous $f \in R(X)^G_+$ such that $x \in X_f$. As $G \cdot x$ is closed in X^{ss}, it is also closed in the open affine set $X_f \subset X^{ss}$. By Proposition 2.27, the G-invariant set

$$Z := \{z \in X_f : \dim G \cdot z < \dim G\}$$

is closed in X_f. Since Z is disjoint from $G \cdot x$, by Lemma 3.8, there exists $h \in A(X_f)^G$ such that

$$h(Z) = 0 \text{ and } h(G \cdot x) = 1.$$

It is a (non-trivial) consequence of G being geometrically reductive, that there is a homogeneous G-invariant polynomial h' such that $h^s = h'/f^r$ for positive integers r, s (we do not give a proof of this fact but instead reference [36] Lemma 3.4.1). Then $x \in X_{fh'}$ and as $X_{fh'}$ is disjoint from Z, every orbit $G \cdot y$ in $X_{fh'}$ has dimension $\dim G$. It follows from Proposition 2.26 that the action of G on $X_{fh'}$ is closed and so this completes the proof that x is stable. □

Definition 4.15. A semistable point x is said to be polystable if its orbit is closed in X^{ss}. We say two semistable points are S-equivalent if their orbit closures meet in X^{ss}.

By Lemma 4.14 above, every stable point is polystable.

Lemma 4.16. *Let x be a semistable point; then its orbit closure $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.*

Proof. The first statement follows from Corollary 2.39. For the second statement we note that if a semistable orbit $G \cdot x$ is not closed, then the unique closed orbit in $\overline{G \cdot x}$ has dimension strictly less than $G \cdot x$ and so cannot be stable. □

Corollary 4.17. *Let x and x' be semistable points; then $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, there is a bijection of sets*

$$X//G \cong X^{ps}/G$$

where $X^{ps} \subset X^{ss}$ is the set of polystable points.

4.3. **Linearisations.** An abstract projective variety X does not come with a specified embedding in projective space. However, an ample line bundle L on X (or more precisely some power of L) determines an embedding of X into a projective space. In order to construct a GIT quotient of an abstract projective variety X we need the extra data of a lift of the G-action to a line bundle on X; such a choice is called a linearisation of the action.

Definition 4.18. Let $\pi : L \to X$ be a line bundle on X. A linearisation of the action of G with respect to L is an action of G on L such that

i) For all $g \in G$ and $l \in L$, we have $\pi(g \cdot l) = g \cdot \pi(l)$,
ii) For all $x \in X$ and $g \in G$ the map of fibres $L_x \to L_{g \cdot x}$ is a linear map.

The linearisation is often also denoted by L. We say a linearisation L is (very) ample if the invertible sheaf associated to L is (very) ample.

Remark 4.19. Let $\mathcal{L}$ denote the invertible sheaf associated to the line bundle L; then the notion of a linearisation of an action $\sigma : G \times X \to X$ can be stated in terms of sheaves as follows. Let $\pi_X : G \times X \to X$ denote the projection onto the second factor and $\mu : G \times G \to G$ denote the group action. A linearisation of the action with respect to $\mathcal{L}$ is an isomorphism

$$\Phi : \sigma^* \mathcal{L} \to \pi_X^* \mathcal{L}$$

which satisfies the cocycle condition:

$$(\mu \times \mathrm{id}_X)^* \Phi = \pi_{23}^* \Phi \circ (\mathrm{id}_G \times \sigma)^* \Phi$$

where $\pi_{23} : G \times G \times X \to G \times X$ is the projection onto the last two factors.

Example 4.20. Let $L = X \times k$ be the trivial line bundle on a variety X over k; then a linearisation of a G-action on X with respect to L corresponds to a character $\chi : G \to \mathbb{G}_m$. The character χ defines a lift of the action to L by

$$g \cdot (x, z) = (g \cdot x, \chi(g) z)$$

where $(x, z) \in X \times k$. More generally, we can use a character $\chi : G \to \mathbb{G}_m$ to modify any given linearisation L of a G-action on X by defining

$$g \cdot (x, z) = (g \cdot x, \chi(g) g \cdot z)$$

for $x \in X$ and $z \in L_x$.

Example 4.21. There is a natural linearisation of the $\mathrm{SL}(n+1, k)$ action on $\mathbb{P}^n = \mathbb{P}(k^{n+1})$ with respect to $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ such that the induced action of G on

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = (k^{n+1})^*$$

is dual to the natural action of $\mathrm{SL}(n+1, k)$ on k^{n+1}.

Remark 4.22. Suppose that X is a projective variety and L is a very ample linearisation. There is an induced action of G on $H^0(X, L)$ such that the natural evaluation map

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L$$

is G-equivariant. Moreover, the embedding

$$X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$$

given by $x \mapsto \mathrm{ev}_x$ is G-equivariant. Hence in this case, we are in the same situation of §4.2 where we have a linear representation of G on $H^0(X, L)^*$ and the action of G on X is induced by the linear action of G on $\mathbb{P}(H^0(X, L)^*)$.

Exercise 4.23. The set of line bundles on X is a group $\mathrm{Pic}(X)$ with multiplication given by tensor product and inverse given by taking the dual line bundle. Show that the set of G-linearised line bundles $\mathrm{Pic}^G(X)$ is also a group. In particular, what is the identity element and how do we define the product of linearisations or the inverse linearisation?

4.4. **GIT for projective varieties with ample linearisations.** Suppose a reductive group G acts on a projective variety X with respect to an ample linearisation L. Let

$$R := R(X,L) := \bigoplus_{r\geq 0} H^0(X, L^{\otimes^r})$$

denote the associated graded algebra of sections of powers of L. There is an induced action of G on the space of sections $H^0(X, L^{\otimes^r})$ and we denote the graded subalgebra of G-invariant sections by

$$R^G = \bigoplus_{r\geq 0} H^0(X, L^{\otimes^r})^G.$$

As $R_0 = k$ and the constant functions are G-invariant we also have $R_0^G = k$ and so $\operatorname{Proj} R^G$ is a projective variety over k by Nagata's theorem.

Definition 4.24. We define the GIT quotient (for this G-action on X with respect to L) to be the projective variety

$$X//_L G := \operatorname{Proj} R^G.$$

The inclusion of the subalgebra of invariant sections induces a rational map $X \dashrightarrow X//_L G$ which is a morphism on the semistable locus $X^{ss}(L) := X - N_{R^G}(X)$. The morphism $\varphi : X^{ss} \to X//_L G$ is the GIT quotient with respect to L. We define notions of (semi)stability with respect to L as follows:

1) A point $x \in X$ is semistable (with respect to L) if there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$.
2) A point $x \in X$ is stable (with respect to L) if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and the action of G on $X_\sigma := \{x \in X : \sigma(x) \neq 0\}$ is closed.
3) The points which are not semistable are called unstable.

The open subsets of stable and semistable points with respect to L will be denoted by $X^s(L)$ and $X^{ss}(L)$ respectively.

Exercise 4.25. We have already defined notions of semistability and stability when we have a linear action of G on $X \subset \mathbb{P}^n$. In this case the line bundle which is used for the linearisation is $\mathcal{O}_{\mathbb{P}^n}(1)$ with the natural lift of the GL_{n+1} action corresponding to the natural action of GL_{n+1} on k^{n+1}. In this case, show that the two notions of semistability agree; that is,

$$X^{(s)s} = X^{(s)s}(\mathcal{O}_{\mathbb{P}^n}(1)|_X).$$

Theorem 4.26. *Let G be a reductive group acting on a projective variety X and L be an ample linearisation of this action. Then the GIT quotient*

$$\varphi : X^{ss}(L) \to X//_L G = \operatorname{Proj} \bigoplus_{r\geq 0} H^0(X, L^{\otimes^r})^G$$

is a good quotient and hence also a categorical quotient. The GIT quotient $X//_L G$ is also a projective variety. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

Proof. As L is ample, the open sets X_σ are affine and so we can use the affine theory to construct the quotient (we omit the proof as it is very similar to that of Theorem 4.11). $\square$

Remark 4.27 (Variation of geometric invariant theory quotient)**.** It is important to remember that in the case of projective GIT we have a choice to make (that of a linearisation) and so the quotient will depend on this choice. Once can study how the semistable locus $X^{ss}(L)$ and the GIT quotient $X//_L G$ vary with the linearisation L; this area is known as variation of GIT. A key result in this area is that there are only finitely many distinct GIT quotients produced by varying the ample linearisation of a fixed G-action on a projective normal variety X (for example, see [11] and [47]).

Remark 4.28. For an ample linearisation L, we know that some power of L is very ample. Since $X^{ss}(L) = X^{ss}(L^{\otimes n})$ and $X^s(L) = X^s(L^{\otimes n})$, we will often assume without loss of generality that L is very ample and so $X \subset \mathbb{P}^n$ and G acts linearly.

Definition 4.29. We say two semistable points x and x' are S-equivalent if the orbit closures of x and x' meet in the semistable subset $X^{ss}(L)$. We say a semistable point is polystable if its orbit is closed in the semistable locus $X^{ss}(L)$.

Corollary 4.30. *Let x and x' be points in $X^{ss}(L)$; then $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, we have a bijection of sets between the points of the GIT quotient*

$$X/\!/_L G \cong X^{ps}(L)/G$$

and the G-orbits in the polystable locus $X^{ps}(L)$.

4.5. **GIT for general varieties with linearisations.** In this section we state the most general theorem of Mumford for constructing GIT quotients of G-actions on varieties with respect to a (not necessarily ample) linearisation. First we give notions of (semi)stability:

Definition 4.31. Let X be a variety with an action by a reductive group G and L be a linearisation of this action.

1) A point $x \in X$ is semistable (with respect to L) if there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r \geq 0$ such that $\sigma(x) \neq 0$ and $X_\sigma = \{x \in X : \sigma(x) \neq 0\}$ is affine.
2) A point $x \in X$ is stable (with respect to L) if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r \geq 0$ such that $\sigma(x) \neq 0$ and X_σ is affine and the action of G on X_σ is closed.

The open subsets of stable and semistable points with respect to L are denoted $X^s(L)$ and $X^{ss}(L)$ respectively.

Remark 4.32. If X is projective and L is ample, then this agrees with Definition 4.24 as X_σ is affine for any non-constant section σ.

In this case the GIT quotient is constructed by covering X^{ss} by affine G-invariant open subvarieties X_σ and gluing the GIT quotients of these affine varieties.

Theorem 4.33. *(Mumford) Let G be a reductive group acting on a variety X and L be a linearisation of this action. Then there is a quasi-projective variety $X/\!/_L G$ and a good quotient $\varphi : X^{ss}(L) \to X/\!/_L G$ of the G-action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X/\!/_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.*

4.6. **Linearisations for affine varieties.** Often when a reductive group G acts on an affine variety X, the affine GIT quotient $X/\!/G$ collapses too many orbits as no unstable points are removed. As we saw in Example 2.28, if the origin is contained in the closure of every orbit then the affine GIT quotient collapses to a point. We can instead consider the trivial line bundle $L = X \times k$ with linearisation L_χ given by a character $\chi : G \to \mathbb{G}_m$ so that

$$g \cdot (x, c) = (g \cdot x, \chi(g)c)$$

for $(x, c) \in L = X \times k$ and $g \in G$; then the associated GIT quotient $X/\!/_{L_\chi} G$ often provides a better quotient of an open subset of X of 'χ-semistable points'. An application of this construction is King's construction of moduli spaces of χ-semistable quiver representations [24].

More generally, toric varieties are constructed as a linearised GIT quotient of an affine space by a torus action where the linearisation is given by a character of a torus.

5. Criteria for (semi)stability

If a reductive group G acts on a projective variety X with respect to an ample linearisation L, then the definitions of (semi)stability with respect to L require us to calculate the G-invariant sections of all powers of L. In fact since the notions of (semi)stability with respect to L and $L^{\otimes n}$ agree, we can assume without loss of generality that L is very ample and so defines an embedding $X \hookrightarrow \mathbb{P}^n$. For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we know by Nagata's theorem that the invariant graded k-subalgebra $R(X)^G \subset R(X)$ is finitely generated. However, this result does not give a method for finding a set of generators. The original definitions of (semi)stability require us to know this subalgebra of G-invariant homogeneous polynomials. In this section we give some alternative criteria for (semi)stability which do not require us to calculate $R(X)^G$. The main references for the material covered in this section are [10], [31], [36] and [48].

5.1. **A topological criterion.** If G is a reductive group which acts linearly on a projective variety $X \subset \mathbb{P}^n$, then there is an action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$. By definition $X^{ss} = X - N_{R(X)^G}(X)$ where $N_{R(X)^G}(X)$ is the nullcone consisting of points $x \in X$ such that every non-constant invariant function $f \in R(X)^G = A(\tilde{X})^G$ vanishes on x.

Proposition 5.1. *Let $\tilde{x} \in \tilde{X}$ be a point lying over x. Then:*

i) *x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$.*
ii) *x is stable if and if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.*

Proof. i) If x is semistable, then there is a G-invariant homogeneous polynomial $f \in R(X)^G$ which is nonzero on x. We can view f as a G-invariant function on $\tilde{X}$ such that $f(\tilde{x}) \neq 0$. As invariant functions are constant on orbits and also their closures we see that $f(\overline{G \cdot \tilde{x}}) \neq 0$ and so there is a function which separates the closed subvarieties $\overline{G \cdot \tilde{x}}$ and 0; that is, these closed subvarieties are disjoint.

For the converse, if $\overline{G \cdot \tilde{x}}$ and 0 are disjoint G-invariant closed subsets, then there exists a G-invariant polynomial $f \in R(\tilde{X})^G$ such that

$$f(\overline{G \cdot \tilde{x}}) = 1, \quad f(0) = 0$$

by Lemma 3.8. The polynomial f is a sum of G-invariant homogeneous polynomials f_r and so there must be a homogeneous piece f_r of f which does not vanish on $\overline{G \cdot \tilde{x}}$; therefore $f_r(x) = f_r(\tilde{x}) \neq 0$ for f_r a homogeneous G-invariant polynomial which proves that x is semistable.

ii) If x is stable, then $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G$ such that $x \in X_f$ and $G \cdot x$ is closed in X_f. As $G_{\tilde{x}} \subset G_x$, the stabiliser of $\tilde{x}$ is also zero dimensional. We can view f as a function on $\tilde{X}$ and consider the closed subvariety

$$Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

of $\tilde{X}$. Then it suffices to show that $G \cdot \tilde{x}$ is a closed subset of Z. The projection map $\tilde{X} - \{0\} \to X$ restricts to a surjective finite morphism $\pi : Z \to X_f$. The preimage of the closed orbit $G \cdot x$ in X_f under this morphism $\pi : Z \to X_f$ is closed and G-invariant; since π is also finite, the preimage $\pi^{-1}(G \cdot x)$ is a finite number of G-orbits. The finite number of G-orbits in the preimage all lie over $G \cdot x$ and so all have dimension equal to $\dim G$. In particular, these orbits must all be closed as otherwise they would contain lower dimensional orbits in their closure and so $G \cdot \tilde{x} \subset \pi^{-1}(G \cdot x)$ is closed in Z.

Conversely suppose that $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$; then $0 \notin \overline{G \cdot \tilde{x}} = G \cdot \tilde{x}$ and so x is semistable by i). As x is semistable there is a non-constant G-invariant homogeneous polynomial f such that $f(x) \neq 0$. As above, we consider

$$Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

and the finite surjective morphism $\pi : Z \to X_f$. As $\pi(G \cdot \tilde{x}) = G \cdot x$, we see that x must have finite dimensional stabiliser and $G \cdot x$ must be closed in X_f too. This is true for all f such that $f(x) \neq 0$ and so $G \cdot x$ is closed in $X^{ss} = \cup_f X_f$. Hence x is stable by Lemma 4.14. □

Exercise 5.2. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $t \cdot [x : y] = [tx : t^{-1}y]$. By studying the orbits and their closures in the affine cone $\mathbb{A}^2$ over $\mathbb{P}^1$ determine which points are (semi)stable (see also Examples 2.28 and 4.13).

5.2. **The Hilbert–Mumford criterion.** Suppose we have a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$. In this section we describe a numerical criterion which can be used to determine (semi)stability of a point x. Following the topological criterion above, we see that it is important to understand the orbit closure of a point lying over x. The test objects for studying the orbit closure are 1-parameter subgroups:

Definition 5.3. A 1-parameter subgroup (1-PS) of G is a nontrivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

We have an embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ given by sending $a \in \mathbb{G}_m = k^*$ to $[1 : a] \in \mathbb{P}^1$ and we refer to the points $[1 : 0]$ and $[0 : 1]$ in $\mathbb{P}^1$ as zero and infinity respectively. For any 1-PS λ of G and $x \in X$ we can define a morphism

$$\lambda(-) \cdot x : \mathbb{G}_m \to X$$

induced by the action of λ. As X is a complete variety, this morphism extends to a unique morphism $\mathbb{P}^1 \to X$ and we let $\lim_{t\to 0} \lambda(t) \cdot x$ and $\lim_{t\to\infty} \lambda(t) \cdot x$ denote the images of zero and infinity under this morphism. We may also lift x to a point $\tilde{x}$ lying over x in the affine cone $\tilde{X}$ and consider the morphism

$$\sigma_{\tilde{x}} := \lambda(-) \cdot \tilde{x} : \mathbb{G}_m \to \tilde{X}$$

which may no longer extend to $\mathbb{P}^1$. We can study the closure $\overline{\sigma_{\tilde{x}}(\mathbb{G}_m)}$ of the image of $\sigma_{\tilde{x}}$ and its boundary $\overline{\sigma_{\tilde{x}}(\mathbb{G}_m)} - \sigma_{\tilde{x}}(\mathbb{G}_m)$, or equivalently, the closure of the orbit $\lambda(\mathbb{G}_m)\cdot\tilde{x}$ and its boundary. We note that if the boundary is nonempty then any point in the boundary is equal to either $\lim_{t\to 0} \lambda(t) \cdot x$ or $\lim_{t\to\infty} \lambda(t) \cdot x$. In particular, if $\sigma_{\tilde{x}}$ is non-constant (i.e. $\lambda(\mathbb{G}_m) \nsubseteq G_{\tilde{x}}$), then the image is closed if and only if neither limit exists.

The 1-PS $\lambda : \mathbb{G}_m \to G$ induces an action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ which is diagonalisable; that is, there is a basis $e_0, ..., e_n$ of $\mathbb{A}^{n+1}$ such that

$$\lambda(t) \cdot e_i = t^{r_i} e_i \quad \text{for } r_i \in \mathbb{Z}.$$

We call the r_i the λ-weights of this action on $\mathbb{A}^{n+1}$. For $x \in X$ we can pick $\tilde{x} \in \tilde{X}$ lying above this point and write $\tilde{x} = \sum_{i=0}^n a_i e_i$ with respect to this basis; then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} a_i e_i \, .$$

Definition 5.4. We define the Hilbert-Mumford function μ of x at λ by

$$\mu(x, \lambda) := -\min\{r_i : a_i \neq 0\}.$$

Remark 5.5. It is easy to check that the Hilbert–Mumford function is independent of the choices we made (such as the lift $\tilde{x}$ and the basis e_i).

Exercise 5.6. Check the Hilbert–Mumford function has the following properties:

(1) $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t\to 0} t^{\mu}\lambda(t) \cdot \tilde{x}$ exists and is nonzero.
(2) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive n.
(3) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G$.
(4) $\mu(x, \lambda) = \mu(x_0, \lambda)$ where $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$.

.

We note that:

- $\mu(x, \lambda) < 0$ if and only if
$$\tilde{x} = \sum_{r_i > 0} a_i r_i$$
if and only if the limit $\lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ exists and is equal to zero.
- $\mu(x, \lambda) = 0$ if and only if $r_{i_0} = 0$ for some i_0 and
$$\tilde{x} = a_{i_0} e_{i_0} + \sum_{r_i > 0} a_i r_i$$
where $a_{i_0} \neq 0$. This is if and only if the limit $\lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ exists and is equal to $a_{i_0} e_{i_0} \neq 0$.

- $\mu(x,\lambda) > 0$ if and only if $\lim_{t\to 0}\lambda(t)\cdot\tilde{x}$ does not exist.

We can use λ^{-1} to study $\lim_{t\to\infty}\lambda(t)\cdot\tilde{x}$ as

$$\lim_{t\to 0}\lambda^{-1}(t)\cdot\tilde{x} = \lim_{t\to\infty}\lambda(t)\cdot\tilde{x}.$$

Then:

- $\mu(x,\lambda^{-1}) < 0$ if and only if
$$\tilde{x} = \sum_{r_i<0} a_i r_i$$
if and only if the limit $\lim_{t\to\infty}\lambda(t)\cdot\tilde{x} = \lim_{t\to 0}\lambda^{-1}(t)\cdot\tilde{x}$ exists and is equal to zero.
- $\mu(x,\lambda^{-1}) = 0$ if and only if $r_{i_0} = 0$ for some i_0 and
$$\tilde{x} = a_{i_0}e_{i_0} + \sum_{r_i<0} a_i r_i$$
where $a_{i_0} \neq 0$. This is if and only if the limit $\lim_{t\to\infty}\lambda(t)\cdot\tilde{x} = \lim_{t\to 0}\lambda^{-1}(t)\cdot\tilde{x}$ exists and is equal to $a_{i_0}e_{i_0} \neq 0$.
- $\mu(x,\lambda^{-1}) > 0$ if and only if $\lim_{t\to\infty}\lambda(t)\cdot\tilde{x} = \lim_{t\to 0}\lambda^{-1}(t)\cdot\tilde{x}$ does not exist.

Following the discussion above and the topological criterion (see Proposition 5.1), we have the following results for (semi)stability with respect to the action of the subgroup $\lambda(\mathbb{G}_m) \subset G$:

Lemma 5.7. *Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Suppose $x \in X$ and $\tilde{x} \in \tilde{X}$ is a point lying over x; then*

i) *x is semistable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x,\lambda) \geq 0$ and $\mu(x,\lambda^{-1}) \geq 0$.*
ii) *x is stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x,\lambda) > 0$ and $\mu(x,\lambda^{-1}) > 0$.*

Exercise 5.8. Let $\mathbb{C}^*$ act on $\mathbb{P}^2$ by $t\cdot[x:y:z] = [tx:y:t^{-1}z]$. For every point $x \in \mathbb{P}^2$ and the 1-PS $\lambda(t) = t$, calculate $\mu(x,\lambda^{\pm 1})$ and then by using Lemma 5.7 above or otherwise, determine X^s and X^{ss}.

If x is (semi)stable for G, then it is (semi)stable for all subgroups H of G as every G-invariant function is also H-invariant. Hence

$$x \text{ is semistable} \implies \mu(x,\lambda) \geq 0\ \forall \text{ 1-PS } \lambda \text{ of } G,$$
$$x \text{ is stable} \implies \mu(x,\lambda) > 0\ \forall \text{ 1-PS } \lambda \text{ of } G.$$

The Hilbert-Mumford criterion gives the converse to these statements; the idea is that because G is reductive it has enough 1-PSs to detect semistability (see also Theorem 5.10 below).

Theorem 5.9. *(Hilbert–Mumford Criterion) Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then*

$$x \in X^{ss} \iff \mu(x,\lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s} \iff \mu(x,\lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

It follows from the topological criterion given in Proposition 5.1 and also from Lemma 5.7, that the Hilbert–Mumford criterion is equivalent to the following fundamental theorem in GIT.

Theorem 5.10. *Let G be a reductive group acting on an affine space $\mathbb{A}^{n+1}$. If $x \in \mathbb{A}^{n+1}$ and $y \in \overline{G\cdot x}$, then there is a 1-PS λ of G such that $\lim_{t\to 0}\lambda(t)\cdot x = y$.*

Remark 5.11. The proof of the above fundamental theorem relies on a theorem of Iwahori about the abundance of 1-PSs of reductive groups [20] and was given by Mumford in [31] §2.1.

Example 5.12. We consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ as in Example 4.13. As the group is a 1-dimensional torus, we need only calculate $\mu(x,\lambda)$ and $\mu(x,\lambda^{-1})$ for $\lambda(t) = t$ as was the case in Lemma 5.7. Suppose $\tilde{x} = (x_0,\dots,x_n)$ lies over $x = [x_0:\cdots:x_n] \in \mathbb{P}^n$. Then

$$\lim_{t\to 0}\lambda(t)\cdot\tilde{x} = (t^{-1}x_0, tx_1\dots,tx_n)$$

exists if and only if $x_0 = 0$. If $x_0 = 0$, then $\mu(x,\lambda) = -1$ and otherwise $\mu(x,\lambda) > 0$. Similarly

$$\lim_{t\to 0}\lambda^{-1}(t)\cdot\tilde{x} = (tx_0, t^{-1}x_1\dots,t^{-1}x_n)$$

exists if and only if $x_1 = \cdots = x_n = 0$. If $x_1 = \cdots = x_n = 0$, then $\mu(x,\lambda) = -1$ and otherwise $\mu(x,\lambda) > 0$. Therefore, the GIT semistable set and stable coincide:

$$X^{ss} = X^s = \{[x_0 : \cdots : x_n] : x_0 \neq 0 \text{ and } (x_1, \dots, x_n) \neq 0\} \subset \mathbb{P}^n.$$

5.3. **Hilbert–Mumford for ample linearisations.** In this section we consider the following more general set up: suppose X is a projective variety with a G-action and ample linearisation L. If λ is a 1-PS of G and $x \in X$, then the limit

$$x_0 = \lim_{t\to 0} \lambda(t) \cdot x$$

exists in X as X is complete. The limit x_0 is fixed by the $\mathbb{G}_m$-action induced by λ and so $\mathbb{G}_m$ acts on the fibre L_{x_0} by a character $t \mapsto t^r$. We call this the weight of the λ-action on L_{x_0} and define

$$\mu^L(x,\lambda) := r.$$

Remark 5.13. We should check that when $X \subset \mathbb{P}^n$ and the action of G is linear that this definition is consistent with the old definition; that is,

$$\mu^{\mathcal{O}_{\mathbb{P}^n}(1)|_X}(x,\lambda) = \mu(x,\lambda).$$

Let us assume we have chosen a basis $e_0, \dots, e_n$ of $\mathbb{A}^n$ such that $\lambda(t) \cdot e_i = t^{r_i} e_i$. If we let $\tilde{x} = \sum_i a_i e_i$ be a point lying over $x = [a_0 : \cdots : a_n]$, then by definition

$$\mu(x,\lambda) := -\min\{r_i : a_i \neq 0\}.$$

Let $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$; then we may write $x_0 = [b_0 : \cdots : b_n]$ and note that

$$b_i = \begin{cases} a_i & \text{if } r_i = -\mu(x,\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Then if $\tilde{x}_0 = (b_0, \dots, b_n)$ lies over x_0 we have

$$\lambda(t) \cdot \tilde{x}_0 = t^{-\mu(x,\lambda)} \cdot \tilde{x}_0.$$

As $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological line bundle over $\mathbb{P}^n$, the fibre over a given point x_0 is the line consisting of points $\tilde{x}_0 \in \mathbb{A}^{n+1}$ lying over x_0. In particular $\lambda(\mathbb{G}_m)$ acts on the fibre of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over the fixed point x_0 by a character $t \mapsto t^{-\mu(x,\lambda)}$. Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual line bundle, the subgroup $\lambda(\mathbb{G}_m)$ acts on the fibre of $\mathcal{O}_{\mathbb{P}^n}(1)$ over the fixed point x_0 by a character $t \mapsto t^{\mu(x,\lambda)}$ and so

$$\mu^{\mathcal{O}_{\mathbb{P}^n}(1)|_X}(x,\lambda) = \mu(x,\lambda).$$

Exercise 5.14. Fix $x \in X$ and a 1-PS λ of G; then show $\mu^{\bullet}(x,\lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ is a group homomorphism where $\mathrm{Pic}^G(X)$ is the group of G-linearised line bundles on X.

Theorem 5.15. *(Hilbert–Mumford Criterion for ample linearisations) Let G be a reductive group acting on a projective variety X and L be an ample linearisation of this action. Then*

$$x \in X^{ss}(L) \iff \mu^L(x,\lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s}(L) \iff \mu^L(x,\lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

Proof. (Assuming Theorem 5.9) As L is ample, there is $n > 0$ such that $L^{\otimes n}$ is very ample. Then since

$$\mu^{L^{\otimes n}}(x,\lambda) = n\mu^L(x,\lambda)$$

it suffices to prove the statement for L very ample. If L is very ample then it induces a G-equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $L \cong i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then we can just apply the first version of the Hilbert–Mumford criterion (cf. Theorem 5.9 and Remark 5.13). □

5.4. **Weight polytopes for torus actions.** In this section we restate the Hilbert–Mumford criterion in terms of a geometrical condition for weight polytopes for points with respect to a maximal torus T of G. We use some basic notions from representation theory; see [13] for further details.

We start with the case when $G = \mathbb{G}_m$ is a one-dimensional torus which acts linearly on a projective variety $X \subset \mathbb{P}^n$. The action of $G = \mathbb{G}_m$ on $V := \mathbb{A}^{n+1}$ gives us a weight decomposition

$$V = \oplus_{r \in \mathbb{Z}} V_r$$

where

$$V_r = \{v \in V : t \cdot v = t^r v\}.$$

We let $X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ denote the set of characters of G. As G is a one-dimensional torus, $X^*(G) \cong \mathbb{Z}$ where an integer r defines a character $\chi_r : G \to \mathbb{G}_m$ given by $t \mapsto t^r$. Therefore we can see this decomposition as being indexed by a finite number of characters χ_r of G. We refer to the set $\{\chi_r : V_r \neq 0\}$ as the $\mathbb{G}_m$-weights of this action.

Definition 5.16. If $x \in X$ then we can choose $\tilde{x} \in V$ lying over x and write $\tilde{x} = \sum_r v_r$, then we define the $\mathbb{G}_m$-weight set of x to be

$$\mathrm{wt}_{\mathbb{G}_m}(x) := \{\chi_r \in X^*(\mathbb{G}_m) : v_r \neq 0\}.$$

We define the weight polytope of x to be the convex hull in $\mathbb{R} \cong X^*(\mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}$ of its weights and denote this by $\overline{\mathrm{wt}_{\mathbb{G}_m}(x)}$.

We let $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ denote the set of cocharacters of G, which is also isomorphic to $\mathbb{Z}$. Let λ denote the 1-PS corresponding to $1 \in \mathbb{Z}$. For $n > 0$ we have

$$\mu(x, \lambda^n) = n\mu(x, \lambda),$$

and so the Hilbert–Mumford criterion for $G = \mathbb{G}_m$ is simply:

$$x \in X^{ss} \iff \mu(x, \lambda) \geq 0 \text{ and } \mu(x, \lambda^{-1}) \geq 0.$$
$$x \in X^{s} \iff \mu(x, \lambda) > 0 \text{ and } \mu(x, \lambda^{-1}) > 0.$$

By definition

$$\mu(x, \lambda) = -\min\{r : v_r \neq 0\} \quad \text{and} \quad \mu(x, \lambda^{-1}) = -\min\{-r : v_r \neq 0\}.$$

Hence

(1) $\mu(x, \lambda) \geq 0 \iff \tilde{x} \neq \sum_{r>0} x_r$.
(2) $\mu(x, \lambda) > 0 \iff \tilde{x} \neq \sum_{r\geq 0} x_r$.
(3) $\mu(x, \lambda^{-1}) \geq 0 \iff \tilde{x} \neq \sum_{r<0} x_r$.
(4) $\mu(x, \lambda^{-1}) \geq 0 \iff \tilde{x} \neq \sum_{r\leq 0} x_r$.

In conclusion:

$$x \text{ is semistable} \iff \text{there exist } r_1 \leq 0 \leq r_2 \text{ such that } x_{r_i} \neq 0 \text{ for } i = 1, 2.$$

$$x \text{ is stable} \iff \text{there exist } r_1 < 0 < r_2 \text{ such that } x_{r_i} \neq 0 \text{ for } i = 1, 2.$$

We summarise the above by using the weight polytope of x.

Proposition 5.17. *(Hilbert–Mumford criterion for $\mathbb{G}_m$) Let $G = \mathbb{G}_m$ act linearly on a projective variety $X \subset \mathbb{P}^n$ and let $x \in X$ and $\tilde{x}$ lie over x. Then:*

i) $x \in X^{ss} \iff 0 \in \overline{\mathrm{wt}_{\mathbb{G}_m}(x)}$.
ii) $x \in X^{s} \iff 0 \in \mathrm{Int}(\overline{\mathrm{wt}_{\mathbb{G}_m}(x)})$.

Exercise 5.18. Consider the linear action of $G = \mathbb{G}_m$ on $\mathbb{P}^2$ corresponding to the representation $\mathbb{G}_m \to \mathrm{GL}_3(k)$ given by

$$t \mapsto \mathrm{diag}(t, 1, t^{-1}).$$

Write down the weights χ for this action and by drawing the possible weight polytopes or otherwise determine which points are stable, semistable and unstable. What is the GIT quotient for this action?

We now assume $G = T = (\mathbb{G}_m)^r$ is an r-dimensional torus which acts linearly on a projective variety $X \subset \mathbb{P}^n$. In this case the character lattice is $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^r$. As T is commutative, the action of T on $V = \mathbb{A}^{n+1}$ gives a weight decomposition

$$V = \oplus_{\chi \in X^*(T)} V_\chi$$

where

$$V_\chi = \{v \in V : t \cdot v = \chi(t) v \text{ for all } t \in T\}.$$

Definition 5.19. If $x \in X$ and $\tilde{x} \in V$ lies over x, then we may write $\tilde{x} = \sum_\chi v_\chi$. We define the T-weight set of x to be

$$\mathrm{wt}_T(x) := \{\chi \in X^*(T) : v_\chi \neq 0\}$$

and the T-weight polytope of x to be the convex hull of its weights in $\mathbb{R}^r \cong X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ which we denote by $\overline{\mathrm{wt}_T(x)}$.

Since $X^*(T) \cong \mathbb{Z}^r \cong X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$, we can use the dot product on $\mathbb{Z}^r$ as an inner product which allows us to identify characters and cocharaters and define a norm for characters and cocharacters. There are other choices of inner products that we could use, but for (semi)stability we are only interested in the sign of $\mu(x, \lambda)$ and so we will see that the choice of inner product is not important for determining (semi)stability.

For any 1-PS λ of T we have that $\lambda(t)$ acts on $\tilde{x}$ by

$$\lambda(t) \cdot \sum_\chi v_\chi = \sum_\chi \chi \circ \lambda(t) v_\chi$$

and we let $< \lambda, \chi >$ denote the integer r such that $\chi \circ \lambda(t) = t^r$. Then

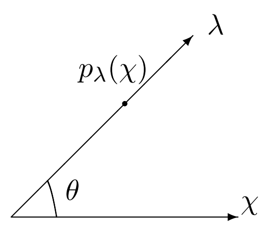
$$\mu(x, \lambda) = -\min\{< \lambda, \chi >: \chi \in \mathrm{wt}_T(x)\}.$$

Since we are interested in only the sign of this quantity we can divide by $||\lambda|| > 0$ and consider the quantity $\mu(x, \lambda)/||\lambda||$. We have that

$$< \lambda, \chi > = ||\chi|| \, ||\lambda|| \cos\theta$$

where $\theta \in [0, \pi]$ is the angle between the two vectors. If we let $p_\lambda(\chi)$ denote the orthogonal projection of χ onto the line spanned by λ, then:

- For $\theta \in [0, \pi/2)$, we have $\mathrm{dist}(0, p_\lambda(\chi)) = ||p_\lambda(\chi)|| = ||\chi|| \cos\theta$:



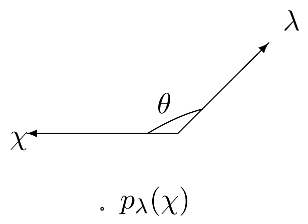
  Therefore,

$$\frac{< \lambda, \chi >}{||\lambda||} = \mathrm{dist}(0, p_\lambda(\chi)).$$

- For $\theta = \pi/2$, we have $p_\lambda(\chi) = 0$ and so

$$0 = \frac{< \lambda, \chi >}{||\lambda||} = \mathrm{dist}(0, p_\lambda(\chi)).$$

- For $\theta \in (\pi/2, \pi]$, we have $\mathrm{dist}(0, p_\lambda(\chi)) = ||p_\lambda(\chi)|| = ||\chi|| \cos(\pi - \theta)$:



Therefore,

$$\frac{<\lambda,\chi>}{||\lambda||} = -\mathrm{dist}(0, p_\lambda(\chi)).$$

Hence,

$$\frac{<\lambda,\chi>}{||\lambda||} = \begin{cases} \mathrm{dist}(0, p_\lambda(\chi)) & \text{if } \theta \in [0,\pi/2] \\ -\mathrm{dist}(0, p_\lambda(\chi)) & \text{if } \theta \in [\pi/2,\pi] \end{cases}$$

Remark 5.20. We note that if we chose a different norm, then the quantity

$$\frac{<\lambda,\chi>}{||\lambda||}$$

may change, but for the moment we are only interested in the sign of this quantity which will not change and so we see that the choice of norm is not important for (semi)stability.

Let $\overline{\lambda - \mathrm{wt}_T(x)} := \{p_\lambda(\chi) : \chi \in \mathrm{wt}_T(x)\} = p_\lambda(\overline{\mathrm{wt}_T(x)})$; then these points all lie on the line spanned by λ and we have

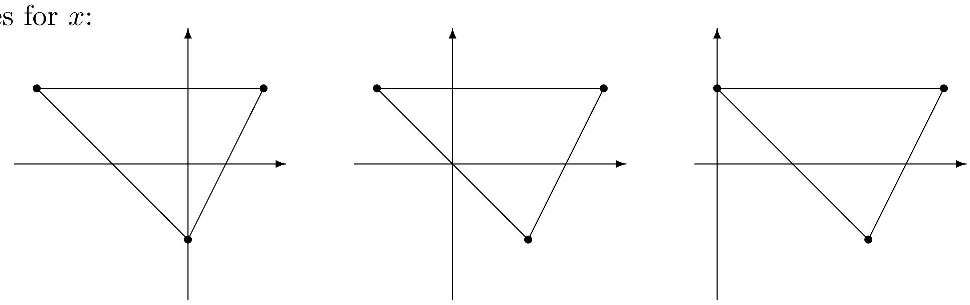
i) $\mu(x,\lambda) \geq 0$ and $\mu(x,\lambda^{-1}) \geq 0 \iff 0 \in \overline{\lambda - \mathrm{wt}_T(x)}$.
ii) $\mu(x,\lambda) > 0$ and $\mu(x,\lambda^{-1}) > 0 \iff 0 \in \mathrm{Int}(\overline{\lambda - \mathrm{wt}_T(x)})$.

Therefore:

Proposition 5.21. *(Hilbert–Mumford criterion for tori) Let $T = (\mathbb{G}_m)^r$ act linearly on a projective variety $X \subset \mathbb{P}^n$. Suppose $x \in X$ and pick $\tilde{x}$ lying over x. Then:*

i) $x \in X^{ss} \iff 0 \in \overline{\mathrm{wt}_T(x)}$.
ii) $x \in X^{s} \iff 0 \in \mathrm{Int}(\overline{\mathrm{wt}_T(x)})$.

Example 5.22. We suppose T is a two dimensional torus and draw some pictures of weight polytopes for x:



In the first picture x is stable, in the second picture x is semistable but not stable and in the third picture x is unstable.

Exercise 5.23. Consider the linear action of $G = \mathbb{G}_m^2$ on $\mathbb{P}^3$ corresponding to the representation $\mathbb{G}_m^2 \to \mathrm{GL}_4(k)$ given by

$$(s,t) \mapsto \mathrm{diag}(st, s^{-1}t, s^{-1}t^{-1}, st^{-1}).$$

Write down the weights χ for this action and by drawing the possible weight polytopes or otherwise determine which points are stable, semistable and unstable. Finally, give the GIT quotient for this action (perhaps by explicitly calculating the invariant functions).

Finally, let G be a reductive group which acts linearly on a projective variety $X \subset \mathbb{P}^n$. The image of any 1-PS of G is a commutative subgroup of G and so is contained in a maximal torus of G; hence

$$X_*(G) = \bigcup_{T \subset G} X_*(T)$$

where the union is over all maximal tori T of G. By the Hilbert–Mumford criterion we have

$$X^{G-(s)s} = \bigcap_{T \subset G} X^{T-(s)s}.$$

As any two maximal tori are conjugate we can fix a maximal torus T of G and if λ is a 1-PS of G, then $g\lambda g^{-1}$ is a 1-PS of T for some $g \in G$. Since

$$\mu(x, \lambda) = \mu(g \cdot x, g\lambda g^{-1})$$

and the second quantity can be evaluated by looking at the T-weight polytope of $g \cdot x$, we see that x is (semi)stable for the action of G if and only if $g \cdot x$ is (semi)stable for the action of T for all $g \in G$.

Theorem 5.24. *(Hilbert-Mumford criterion) If G is a reductive group which acts linearly on a projective variety $X \subset \mathbb{P}^n$ and T is a maximal torus of G, then*

i) $x \in X^{ss}(L) \iff 0 \in \overline{\mathrm{wt}_T(g \cdot x)}$ for all $g \in G$,
ii) $x \in X^{s}(L) \iff 0 \in \mathrm{Int}\overline{\mathrm{wt}_T(g \cdot x)}$ for all $g \in G$.

5.5. **Instability in GIT.** Let G be a reductive group acting on a projective variety X and suppose L is an ample linearisation of this action. If $x \in X - X^{ss}(L)$ is unstable, then, by the Hilbert–Mumford criterion, there exists a 1-PS λ such that $\mu^L(x, \lambda) < 0$. We would ideally like to see which 1-parameter subgroups(s) are most responsible for the instability of x, but

$$\mu^L(x, \lambda^n) = n\mu^L(x, \lambda)$$

and so the quantity $\mu^L(x, -)$ is unbounded. However if we pick a norm for 1-PSs such that $||\lambda^n|| = n||\lambda||$, then

$$\frac{\mu^L(x, \lambda)}{||\lambda||} = \frac{\mu^L(x, \lambda^n)}{||\lambda^n||}$$

and so we may instead try to work with this normalised version of the Hilbert–Mumford function.

As we are now interested in the value of this function, rather than just the sign, we see that the choice of norm is important (cf. Remark 5.20). We turn to the question of how to choose such a norm which is invariant under the conjugation action of G. As the conjugacy classes of G are equal to the Weyl group orbits in a maximal torus T, we can instead pick a norm on the set of 1-PSs of a fixed maximal torus T which is invariant under the action of the Weyl group $W = N_G(T)/T$. In particular, given any norm on $X_*(T) \cong \mathbb{Z}^n$, we can produce a norm which is W-invariant by averaging this norm over the finite group W.

Example 5.25. If $G = \mathrm{GL}_n$, then we can take the maximal torus T consisting of diagonal matrices with respect to the standard basis of k^n. Then under the natural identification $\mathbb{Z}^n \cong X_*(T)$ given by

$$(m_1, \dots, m_n) \mapsto \lambda(t) = \mathrm{diag}(t_0^m, \dots, t_n^m),$$

we claim that the dot product on $\mathbb{Z}^n$ gives an inner product on $X_*(T)$ (and hence also a norm) which is W-invariant. The Weyl group W is the symmetric group on n elements and its action (by conjugation) on T corresponds to permuting the diagonal entries. Hence the norm of a 1-PS λ and $\sigma \cdot \lambda$ for a permutation $\sigma \in W$ agree.

Remark 5.26. If G is a Lie group over k, then we can take the derivative of a 1-PS $\lambda : \mathbb{G}_m \to G$ to get an element $d\lambda : k \to \mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G and k is the Lie algebra of $\mathbb{G}_m$. Then to find a G-invariant inner product on the set of 1-PSs of G, it suffices to give a G-invariant inner product on $\mathfrak{g}$, where G acts on its Lie algebra by the adjoint representation. The Killing form $\kappa(-,-)$ is a G-invariant inner product on the Lie algebra $\mathfrak{g}$ and is given by

$$\kappa(A, B) = \mathrm{Tr}(\mathrm{ad}_A \circ \mathrm{ad}_B)$$

where Tr denotes the trace of an endomorphism and ad_A is the adjoint endomorphism of $\mathfrak{g}$ defined by $\mathrm{ad}_A(C) = [A, C]$.

For the rest of this section we fix a G-invariant norm $||-||$ on the set of conjugacy classes of 1-PSs of G.

Definition 5.27. For all $x \in X$, we define

$$M^L(x) := \inf_{\lambda \neq 0 \in X_*(G)} \frac{\mu^L(x,\lambda)}{||\lambda||}.$$

Then the Hilbert–Mumford criterion can be rephrased as

$$x \in X^{ss}(L) \iff M^L(x) \geq 0,$$
$$x \in X^{s}(L) \iff M^L(x) > 0.$$

Proposition 5.28. *Let G be a reductive group acting on a projective variety X and suppose L is an ample linearisation. The function $M^L(-)$ on X is bounded.*

Proof. We can assume without loss of generality that G acts linearly on a projective variety $X \subset \mathbb{P}^n$ (for example, see Remark 4.28). Fix a maximal torus T of G; then the action of T on $\mathbb{A}^{n+1}$ has a finite number of weights $\chi \in X^*(T)$. As all maximal tori are conjugate we have that

$$M^L(x) = \inf_{g \in G} \inf_{\lambda \neq 0 \in X_*(T)} \frac{\mu^L(g\cdot x,\lambda)}{||\lambda||}$$

However, it follows form Section 5.4 that

$$\left| \inf_{\lambda \neq 0 \in X_*(T)} \tfrac{\mu^L(x,\lambda)}{||\lambda||} \right| = \mathrm{dist}(0, \partial\, \overline{\mathrm{wt}_T(g\cdot x)})$$

where $\partial\, \overline{\mathrm{wt}_T(g\cdot x)}$ denotes the boundary of the T-weight polytope. Since there are only finitely many T-weights, there are only finitely many possible T-weight polytopes (which are by definition the convex hull of some non-empty subset of the T-weights) and so we see there are only finitely many possible values for $|M^L(x)|$. □

If x is unstable with respect to L, then $M^L(x) < 0$. We know this quantity is bounded below (in fact the above proposition really shows that M^L takes only finitely many values) and so we can ask which one parameter subgroups achieve the value of $M^L(x)$. This leads to Kempf's notion of adapted 1-PSs [23]:

Definition 5.29. A 1-PS λ for which

$$\frac{\mu^L(x,\lambda)}{||\lambda||} = M^L(x)$$

is said to be an adapted to x. We let $\wedge^L(x)$ denote the set of non-divisible 1-PSs which are adapted to x.

Lemma 5.30. *For $g \in G$, we have $\wedge^L(g.x) = g \wedge^L(x) g^{-1}$ and so $M^L(-)$ is G-invariant.*

Proof. This follows from the fact that the norm $||-||$ is invariant under the conjugation action of G and $\mu(x,\lambda) = \mu(g\cdot x, g\lambda g^{-1})$. □

Definition 5.31. For any 1-PS λ of G we define a parabolic subgroup

$$P(\lambda) := \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists in } G \right\}$$

of G.

We state without proof some properties of adapted 1-PSs due to Kempf [23] (see also [39]):

Theorem 5.32. *Let G be a reductive group acting on a projective variety X and suppose L is an ample linearisation. If $x \in X - X^{ss}(L)$ is an unstable point, then*

1) *$\wedge^L(x)$ is nonempty.*

2) *There is a parabolic subgroup* P_x^L *such that* $P_x^L = P(\lambda)$ *for all* $\lambda \in \wedge^L(x)$.
3) *All elements of* $\wedge^L(x)$ *are conjugate to each other by elements of* P_x^L.
4) *Let* $T \subset P_x^L$ *be a maximal torus of* G*, then there is a unique 1-PS of* T *which belongs to* $\wedge^L(x)$.
5) *If* $\lambda \in \wedge^L(x)$ *and* $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$*, then* $\lambda \in \wedge^L(x_0)$ *and* $M^L(x) = M^L(x_0)$.

Example 5.33. Suppose G is a two dimensional torus, then we may draw in $\mathbb{R}^2 \cong X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ the weight polytope $\overline{\mathrm{wt}(x)}$ of an unstable point x. The Hilbert–Mumford criterion (cf. Proposition 5.21) states that as x is unstable, the origin is disjoint from the weight polytope. Then

$$M^L(x) := \inf_\lambda \frac{\mu(x,\lambda)}{||\lambda||} = \inf_\lambda \left(-\min \left\{ \frac{<\lambda,\chi>}{||\lambda||} : \chi \in \mathrm{wt}(x) \right\} \right) < 0$$

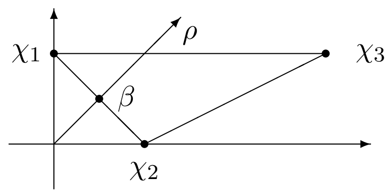
where the infimum is taken over all 1-PSs of G. If $p_\lambda(\chi)$ denotes the orthogonal projection of χ onto the line spanned by λ and θ denotes the angle between λ and χ, then

$$\frac{<\lambda,\chi>}{||\lambda||} = ||\chi|| \cos\theta = \delta(\chi,\lambda)\mathrm{dist}(0, p_\lambda(\chi)$$

where $\delta(\chi,\lambda) = 1$ if $\theta \in [0,\pi/2]$ and $\delta(\chi,\lambda) = -1$ otherwise. Therefore

$$M^L(x) = -\sup_\lambda \mathrm{dist}(0, p_\lambda(\overline{\mathrm{wt}(x)}))$$

where the supremum is taken over 1-PSs λ such that the projection $p_\lambda(\overline{\mathrm{wt}(x)})$ of the weight polytope onto the line spanned by λ is contained in the positive ray spanned by λ. In particular we see that the λ which achieves this value (i.e. is adapted to x) must be an integral point on the unique ray ρ through the origin which meets the weight polytope orthogonally.



As the vertices of the weight polytope are integral, the closest point β in $\partial\, \overline{\mathrm{wt}(x)}$ to the origin is actually a rational weight. In particular some positive multiple of β is integral and is thus corresponds to an adapted 1-PS to x.

5.6. **Hesselink's stratification of the null cone.** Following the work of Kempf on adapted 1-PSs, Hesselink considered a stratification of the null cone $X^{us} := X - X^{ss}$ by (conjugacy classes) of adapted 1-PSs [18] (see also the stratifications considered by Bilaynicki-Birula [3]). By a stratification we mean a finite decomposition of X^{us} for which there is a strict partial ordering $<$ on the index set such that the boundary of a given stratum is a union of higher strata (i.e. strata with larger indices). As we can have points x and y with $M^L(x) < M^L(y)$, and the same 1-PS λ being adapted to both x and y, we must also use the quantity $M^L(x)$ to index the stratification.

Let $<\lambda>$ be a conjugacy class of a 1-PS and $d > 0$, then we define

$$S_{d,<\lambda>} := \{x \in X : M^L(x) = -d \text{ and } g\lambda g^{-1} \in \wedge^L(x)\}$$

and can write

$$X^{us} = \bigsqcup_{d,<\lambda>} S_{d,<\lambda>}.$$

There is a partial order $<$ on the indices where $(d, <\lambda>) < (d', <\lambda'>)$ if $d < d'$.

If we fix a maximal torus T of G, then there is a representative from the conjugacy class $<\lambda>$ which is a 1-PS of T. We denote this 1-PS by λ and define

$$S_{d,\lambda} := \{x \in X : M^L(x) = -d \text{ and } \lambda \in \wedge^L(x)\}.$$

Then $S_{d,<\lambda>} = GS_{d,\lambda}$ and Hesselink refers to the subsets $S_{d,\lambda}$ as "blades".

By Theorem 5.32 v), these blades can be described by their limit sets

$$Z_{d,\lambda} := \{x \in X : M^L(x) = -d \text{ and } \lambda \in \wedge^L(x) \text{ and } \lambda(G_m) \subset G_x\}$$

and there is a retraction

$$p_{d,\lambda} : S_{d,\lambda} \to Z_{d,\lambda}$$

given by taking the limit of x under the action of $\lambda(t)$ as $t \to 0$. The limit set of $S_{d,<\lambda>}$ is then $Z_{d,<\lambda>} := GZ_{d,\lambda}$.

The conjugacy classes in G correspond to the Weyl group orbits in T; hence we can fix a representative $\lambda \in X_*(T)$ for each Weyl group orbit and write

$$X^{us} = \bigsqcup_{d,\lambda} GS_{d,\lambda}$$

where the union is over $d > 0$ and $\lambda \in X_*(T)/W$.

Remark 5.34. If we are working with a Lie group G, then we can fix a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra of a maximal torus T and only consider 1-PS λ of T which correspond to points in this positive Weyl chamber (see also Remark 5.26). As the Weyl group W permutes the positive Weyl chambers this is the same as fixing a representative in each Weyl group orbit.

Lemma 5.35. *There are only finitely many strata $S_{d,<\lambda>} = GS_{d,\lambda}$ which are nonempty.*

Proof. It suffices to show there are only finitely many limit sets $Z_{d,\lambda}$ which are nonempty. Since we have fixed a maximal torus T, there are only finitely many T weights and hence only finitely many possible T-weight polytopes for points (these are given by taking the convex hull of a subset of the T-weights). As the T-weight polytope uniquely determines the ray of 1-PSs which are adapted to a given point (for example, see Example 5.33), we see that there are only finitely many possible non-divisible 1-PSs λ of T which can be 1-PSs which are adapted to unstable points. In particular, there are only finitely many possible $Z_{d,\lambda}$. ☐

Theorem 5.36. *(Hesselink) There is a stratification*

$$X - X^{ss} = \bigsqcup_{d,\lambda} S_{d,<\lambda>}$$

into G-invariant locally closed subvarieties such that

$$\overline{S_{d,<\lambda>}} - S_{d,<\lambda>} \subset \bigsqcup_{(d',\lambda')>(d,\lambda)} S_{d',<\lambda'>}$$

Example 5.37. (3 points on $\mathbb{P}^1$) The group $G = \mathrm{SL}(2)$ acts on $\mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

In this example we consider the action of G on $X = (\mathbb{P}^1)^3$ by

$$g \cdot (p_1, p_2, p_3) = (g \cdot p_1, g \cdot p_2, g \cdot p_3)$$

where G acts on $\mathbb{P}^1$ as above. The Segre embedding $(\mathbb{P}^1)^3 \hookrightarrow \mathbb{P}^7$ allows us to realise $X = (\mathbb{P}^1)^3$ as a closed subvariety of projective space and with respect to this embedding the action is linear ((in fact the linearisation on X is given by the exterior product $\mathcal{O}_{\mathbb{P}^1}(1)^{\boxtimes 3}$). We fix a maximal torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\} \subset G$$

and note that any 1-PS of T is conjugate to a 1-PS of T. Moreover, as

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \text{ and } \lambda^{-1}(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

are conjugate, we see that any 1-PS of G is conjugate to λ^r for $r \geq 1$. In particular, there is only one conjugacy class of non-divisible 1-PSs given by $< \lambda >$. To calculate Hesselink's stratification, we can therefore find which values of d give non-empty $Z_{d,\lambda}$. As $Z_{d,\lambda}$ is contained in the fixed point locus of $\lambda(\mathbb{G}_m)$, we see that $p = (p_1, p_2, p_3)$ is in $Z_{d,<\lambda>}$ only if each p_i is fixed by λ; that is, p_i is either $[1:0]$ or $[0:1]$. This gives 2^3 possible choices and we list the points p for which $\mu(p, \lambda) < 0$:

(1) If $p = ([1:0],[1:0],[1:0])$, then $\mu(p, \lambda) = -3$.
(2) If $p = ([1:0],[1:0],[0:1])$ or $([1:0],[0:1],[1:0])$ or $([0:1],[1:0],[1:0])$, then $\mu(p, \lambda) = -1$.

We use the natural norm, so that $||\lambda|| = 1$. Then there are two unstable strata indexed by $(1, <\lambda>) < (3, <\lambda>)$:

$$S_{1,<\lambda>} = \{(p_1, p_2, p_3) : \text{ exactly 2 of the 3 } p_i \text{ agree}\}$$

and

$$S_{3,<\lambda>} = \left\{(p, p, p) : p \in \mathbb{P}^1\right\}.$$

Hence the semistable locus consists of mutually distinct points (p_1, p_2, p_3).

Exercise 5.38. Consider the linear action of $G = \mathbb{G}_m$ on $\mathbb{P}^2$ given by the representation $\mathbb{G}_m \to \mathrm{GL}_3(k)$

$$t \mapsto \mathrm{diag}(t, 1, t^{-1}).$$

as in Exercise 5.18. By considering the weight polytopes of points, write down the adapted 1-PSs for all unstable points and write down Hesselink's stratification by adapted 1-PSs. We note that as the group is commutative, the conjugacy class of a 1-PS consists of that single 1-PS and the blades of a stratum are equal to the whole stratum. Finally note that the different unstable strata correspond to the different possible unstable weight polytopes in $\mathbb{R} \cong X^*(\mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$.

Exercise 5.39. Write down Hesselink's stratification for the linear action of $G = \mathbb{G}_m^2$ on $\mathbb{P}^3$ given in Exercise 5.23.

6. Examples

6.1. **Projective hypersurfaces.** Given a homogeneous degree d polynomial F in $n+1$ variables $x_0, \dots, x_n$, we associate to F a projective degree d hypersurface in $\mathbb{P}^n$ as follows. If F is irreducible then the associated (irreducible) hypersurface is the set of zeros of F which is an irreducible closed subvariety of $\mathbb{P}^n$ of codimension 1. If F is reducible, then the associated (reducible) hypersurface is a union of irreducible subvarieties of $\mathbb{P}^n$ of codimension 1 (whose points are equal to the zeros of F) counted with multiplicities. For example, we can consider the d-fold point in $\mathbb{P}^1$ as a degree d reducible hypersurface defined by $F(x_0, x_1) = x_0^d$.

The space $k[x_0, \dots, x_n]_d$ of such polynomials is an affine space of dimension $\binom{n+d}{d}$. As any nonzero scalar multiple of F defines the same hypersurface, we are really interested in the projectivisation of this space

$$X_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d).$$

To avoid some difficulties associated with fields of positive characteristic we assume that the characteristic of k is coprime to d.

Definition 6.1. A point p in $\mathbb{P}^n$ is a singular point of a projective hypersurface defined by a polynomial $F \in k[x_0, \dots, x_n]_d$ if

$$F(p) = 0 \text{ and } \frac{\partial F}{\partial x_i}(p) = 0 \text{ for } i = 0, \dots, n.$$

We say a hypersurface is non-singular or smooth if it has no singular points.

By using the Euler formula

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = d\, F$$

and the fact that d is coprime to the characteristic of k, we see that $p \in \mathbb{P}^n$ is a singular point of F if and only if all partial derivatives $\partial F/\partial x_i$ vanish at p.

We recall that the resultant polynomial of a collection of polynomials is a function in the coefficients of these polynomials which vanishes if and only if these polynomials all have a common root. Given $F \in k[x_0, \dots, x_n]_d$, we define the discriminant $\Delta(F)$ of F to be the resultant of the polynomials $\partial F/\partial x_i$. In fact Δ is a homogeneous polynomial on $X_{n,d}$ and is non-zero at F if and only if F defines a smooth hypersurface.

The natural action of $\mathrm{SL}(n+1)$ on $\mathbb{P}^n$ corresponding to a change of homogeneous coordinates, gives an action of $\mathrm{SL}(n+1)$ on the space of hypersurfaces $X_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$ by $g \cdot F(x) = F(g^{-1}x^t)$ where $x = (x_0, \dots, x_n)$. We say two hypersurfaces are projectively equivalent if they are in the same orbit for this action. We want to construct a 'moduli space' for degree d hypersurfaces in $\mathbb{P}^n$ up to projective equivalence as a quotient for this action of $\mathrm{SL}(n+1)$ on $X_{d,n}$ using GIT. We use the natural linearisation $\mathcal{O}_{X_{d,n}}(1)$ on $X_{d,n}$ and consider the GIT quotient

$$X_{d,n}^{ss} \to X_{d,n}//\mathrm{SL}(n+1)$$

which is a good quotient and is a compactification of the geometric quotient

$$X^s \to X^s/\mathrm{SL}(n+1).$$

We want to determine the semistable and stable locus for this action using the tools we've developed above. For small values of (d, n) we shall see that this is possible, although as both values get larger the problem becomes increasingly difficult. We've already seen that there is one $\mathrm{SL}(n+1)$-invariant homogeneous polynomial on $X_{d,n}$: the discriminant Δ.

Example 6.2. If $d = 1$, then $X_{1,n} \cong \mathbb{P}^n$ and as the only $\mathrm{SL}(n+1)$-invariant homogeneous polynomials are the constants:

$$k[x_0, \dots, x_n]^{\mathrm{SL}(n+1)} = k,$$

there are no semistable points for the action of $\mathrm{SL}(n+1)$ on $X_{1,n}$. In particular, the discriminant Δ is constant on $X_{1,n}$. Alternatively, as the action of $\mathrm{SL}(n+1)$ on $\mathbb{P}^n$ is transitive to show $X_{1,n}^{ss} p(\mathbb{P}^n)^{ss} = \phi$, it suffices to show a single point $x = [1 : 0 : \dots : 0] \in \mathbb{P}^n$ is unstable. For this, one can use the Hilbert-Mumford criterion: it is easy to check that if $\lambda(t) = \mathrm{diag}(t, t^{-1}, 1, \dots, 1)$, then $\mu(x, \lambda) < 0$.

For $d > 1$, the discriminant is a non-constant $\mathrm{SL}(n+1)$-invariant homogeneous polynomials on $X_{d,n}$ and as its nonzero for all smooth hypersurfaces we have:

Proposition 6.3. *For $d > 1$, every smooth degree d hypersurface in $\mathbb{P}^n$ is semistable for the action of* $\mathrm{SL}(n+1)$ *on $X_{d,n}$.*

To determine whether a semistable point is stable we can check whether its stabiliser subgroup is finite.

Example 6.4. If $d = 2$, then we are considering the space $X_{2,n}$ of quadric hypersurfaces in $\mathbb{P}^n$. Given $F = \sum_{i,j} a_{ij} x_i x_j \in k[x_0, \dots, x_n]_2$, we can associate to F a symmetric $(n+1) \times (n+1)$ matrix $B = (b_{ij})$ where $b_{ij} = b_{ji} = a_{ij}$ and $b_{ii} = 2a_{ii}$. This procedure defines an isomorphism between $X_{2,n}$ and the space $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$ where $\mathrm{Sym}_{(n+1)\times(n+1)}(k)$ denotes the space of symmetric $(n+1) \times (n+1)$ matrices. The discriminant Δ on $X_{n,d}$ corresponds to the determinant on $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$; thus F is smooth if and only if its associated matrix is invertible. In fact if F corresponds to a matrix B of rank $r+1$, then F is projectively equivalent to the quadratic form

$$x_0^2 + \dots + x_r^2.$$

As all non-singular quadratic forms $F(x_0, \dots, x_n)$ are equivalent to $x_0^2 + \dots x_n^2$ (after a change of coordinates), we see that these points cannot be stable: the stabiliser of $x_0^2 + \dots x_n^2$ is equal to the special orthogonal group $\S 0(n+1)$ which is positive dimensional. Moreover, the discriminant generates the ring of invariants (for example, see [36] Example 4.2) and so the semistable locus is just the set of non-singular quadratic forms. In this case, the GIT quotient consists of a single point and this represents the fact that all non-singular quadratic forms are projectively equivalent to $x_0^2 + \dots x_n^2$.

The projective automorphism group of a hypersurface is the subgroup of the automorphism group $\mathrm{PGL}(n+1)$ of $\mathbb{P}^n$ which leaves this hypersurface invariant. For $d > 2$, it is a well-known fact that the projective automorphism group of any irreducible degree d hypersurface is finite. As $\mathrm{PGL}(n+1)$ is a quotient of $\mathrm{SL}(n+1)$ by a finite subgroup, this implies the stabiliser subgroup of a point in $X_{d,n}$ corresponding to an irreducible hypersurface is finite dimensional. For smooth hypersurfaces, we have the following result (cf. [31] §4.3):

Proposition 6.5. *For $d > 2$, every degree d smooth hypersurface is stable.*

To determine the (semi)stable points for the action of $\mathrm{SL}(n+1)$ on $X_{d,n}$, we can use the Hilbert–Mumford criterion. Any 1-PS of $\mathrm{SL}(n+1)$ is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & & \\ & t^{r_1} & & \\ & & \ddots & \\ & & & t^{r_n} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^n r_i = 0$ and $r_0 \geq r_1 \geq \cdots \geq r_n$. Then given $F = \sum a_I x_I \in k[x_0, \dots, x_n]$ where $I = (m_0, \dots, m_n)$ is a tuple of non-negative integers which sum to d and $x_I = x_0^{m_0} x_1^{m_1} \dots x_n^{m_n}$, we have

$$\begin{aligned} \mu(F, \lambda) &= -\min\{-\sum r_i m_i : I = (m_0, \dots, m_n) \text{ and } a_I \neq 0\} \\ &= \max\{\sum r_i m_i : I = (m_0, \dots, m_n) \text{ and } a_I \neq 0\}. \end{aligned}$$

For general (d, n), there is not always a nice description of the semistable locus. However for certain small values, we shall see that this has a nice description. In Section 6.2 below we discuss the case when $n = 1$; in this case, a degree d hypersurface corresponds to d unordered points (counted with multiplicity) on $\mathbb{P}^1$. Then in Section 6.3 we discuss the case when $(d, n) = (3, 2)$; that is, cubic curves in the projective plane $\mathbb{P}^2$. Both of these classical examples were studied by Hilbert and can also be found in [31] and [36].

6.2. **Binary forms of degree d.** A binary form of degree d is a degree d homogeneous polynomial in 2 variables x, y. The set of zeros of a binary form F determine d points (counted with multiplicity) in $\mathbb{P}^1$. In this section we study the action of $\mathrm{SL}(2)$ on

$$X_{d,1} = \mathbb{P}(k[x, y]_d) \cong \mathbb{P}^d \cong \mathrm{Sym}^d \mathbb{P}^1.$$

Our aim is to describe the (semi)stable locus and the GIT quotient. One method to determine the semistable and stable locus is to compute the ring of invariants $R(X_{d,1})^{\mathrm{SL}(2)}$ for this action. For general d, the ring of invariants is still unknown today, which shows how difficult it can be in general to produce generators for the ring of invariants. However, for some low values of d the ring of invariants is known and the computations of the generators goes back to the work of Hilbert.

Remark 6.6. If $d = 1$, then this corresponds to the action of $SL(2)$ on $\mathbb{P}^1$, for which there are no semistable points as the only invariant functions are constant (see also Example 6.2).

Therefore, we assume $d \geq 2$ and use the Hilbert–Mumford criterion for semistability. We consider the maximal torus $T \subset \mathrm{SL}(2)$ given by

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}.$$

By the Hilbert–Mumford criterion, $F \in X_{d,1}$ is semistable if and only if $\mu(F, \lambda) \geq 0$ for all 1-PSs λ. A 1-PS of G is conjugate to a 1-PS of T of the form

$$\lambda(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$$

for an integer $r \geq 1$. If $F(x, y) = \sum_i a_i x^{d-i} y^i$, then

$$\lambda(t) \cdot F(x, y) = \sum t^{r(2i-d)} a_i x^{d-i} y^i.$$

Therefore

$$\mu(F,\lambda) = \min\{r(d-2i) : a_i \neq 0\} = r(d-2i_0)$$

where i_0 is the largest integer for which $a_i \neq 0$. Hence

(1) $\mu(F,\lambda) \geq 0$ if and only if $i_0 \leq d/2$ if and only if $[0:1]$ occurs with multiplicity at most $d/2$.

(2) $\mu(F,\lambda) > 0$ if and only if $i_0 < d/2$ if and only if $[0:1]$ occurs as a root with multiplicity strictly less than $n/2$.

For a general 1-PS λ' we can write $\lambda = g^{-1}\lambda' g$, then

$$\mu(F,\lambda') = \mu(g\cdot F,\lambda).$$

If F has roots $p_1,\dots,p_d \in \mathbb{P}^1$, then $g\cdot F$ has roots $g\cdot p_1,\dots,g\cdot p_d$. As $SL(2)$ acts transitively on $\mathbb{P}^1$, we deduce the following result.

Proposition 6.7. *Let $F \in X_{d,1}$; then:*

i) *F is semistable if and only if all its roots have multiplicity less than or equal to $d/2$.*

ii) *F is stable if and only if all its roots have multiplicity strictly less than $d/2$. In particular, if d is odd then $X^{ss}_{d,1} = X^s_{d,1}$ and the GIT quotient is a projective variety which is a geometric quotient of the space of stable degree d hypersurfaces in $\mathbb{P}^1$.*

Remark 6.8. In particular, this example shows how the Hilbert–Mumford criterion allows us to calculate the semistable set even though the ring of invariants is unknown. It is also possible to calculate Hesselink's stratification for this GIT problem. As we saw in Example 5.37, every (non-divisible) 1-PS of SL(2) is conjugate to

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and hence the stratification is indexed by pairs (e,λ) for positive numbers e and the 1-PS λ above.

Example 6.9. If $d=2$, then the semistable locus corresponds to forms F with two distinct roots and the stable locus is empty. Given any two distinct points (p_1,p_2) on $\mathbb{P}^1$, there is a mobius transformation taking these points to any other two distinct points (q_1,q_2). However this mobius transformation is far from unique; in fact given points p_3 distinct from (p_1,p_2) and q_3 distinct from (q_1,q_2), there is a unique mobius transformation taking p_i to q_i. Hence all semistable points have positive dimensional stabilisers and so can never be stable (cf. Example **??**). As the action on the semistable locus is transitive, the GIT quotient is just the point $\operatorname{Spec} k$.

Example 6.10. If $d=3$, then the stable locus (which coincides with the stable locus) consists of forms with 3 distinct roots (cf. Example 5.37). We recall that given any 3 distinct points (p_1,p_2,p_3) on $\mathbb{P}^1$, there is a unique mobius transformation taking these points to any other 3 distinct points. Hence the GIT quotient is the projective variety $\mathbb{P}^0 = \operatorname{Spec} k$. In fact, the SL(2)-invariants have a single generator: the discriminant

$$\Delta(\sum a_i x^{n-i}y^i) := 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3$$

which is zero if and only if there is a repeated root.

Example 6.11. If $d=4$, then we are considering binary quartics. In this case the semistable locus is the set of forms F with at most 2 repeated roots and the stable locus is the set of points in which all 4 roots are distinct. The strictly semistable points have either one or two double roots and correspond to two orbits. The orbit consisting of one double root is not closed and its closure contains the orbit of points with two double roots (imagine choosing a family of mobius transformations h_t that sends (p,p,q,r) to $(1,1,0,t)$, then as $t\to 0$ we see the point $(1,1,0,0)$ lies in this orbit closure). There are two independent generators for the SL(2)-invariants of binary quartics (called the I and J invariants - for example, see [36] or [10] where they are called S and T) and the good quotient is $\varphi : X^{ss}_{4,1} \to \mathbb{P}^1$. The two strictly semistable orbits both represent the point at infinity in the GIT quotient so that the map $X^s_{4,1} \to \mathbb{A}^1$ is a good quotient.

Remark 6.12. For $d = 5, 6$ and 8, the ring of invariants is also known. The case for $d = 5$ and 6 are classical results from invariant theory, whereas for $d = 8$, the result is more recent and due to [44]. For all other values of d, the rings of invariants is still today unknown. However, thanks to the Hilbert–Mumford criterion, it is still possible for us to calculate the semistable and stable locus as above.

6.3. **Plane cubics.** We now consider the case of degree 3 hypersurfaces in $\mathbb{P}^2$; that is, plane cubic curves. We write a degree 3 homogeneous polynomial F in variables x_0, x_1, x_2 as

$$F = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x_0^{3-i-j} x_1^i x_2^j.$$

We want to describe all irreducible and reducible plane cubic curves up to projective equivalence; that is, describe the quotient for the action of SL(3) on $X_{3,2}$. For simplicity we assume that the characteristic of k is not equal to 2 or 3.

If F is reducible, then it is either the union of an irreducible conic with a line or a union of three lines. In the first case the line can either meet the conic at two distinct points or a single point (so that the line is tangent to the conic). In fact as any irreducible conic in $\mathbb{P}^2$ is projectively equivalent to $x_0x_2 + x_1^2 = 0$ and the projective automorphism of this conic act transitively on the set of tangents to this conic and also on the set of lines meeting the conic at two distinct points, we have that a reducible cubic of this form is either defined by

$(x_0x_2 + x_1^2)x_1$ (the line meets the conic in two distinct points), or

$(x_0x_2 + x_1^2)x_2$ (the line meets the conic tangentially).

In the second case of three lines, there are four possibilities: one line occurring with multiplicity three; a union of a double line with another distinct line; a union of three lines meeting in a single point; a union of three lines with no common intersection. In these cases the plane cubic conic is projectively equivalent to

$$x_1^3 \quad \text{or} \quad x_1^2(x_1 + x_2) \quad \text{or} \quad x_1x_2(x_1 + x_2) \quad \text{or} \quad x_0x_1x_2$$

respectively.

Definition 6.13. A singular point at p of cubic curved defined by $F(x_0, x_1, x_2)$ is a triple point if all second order partial derivatives $\partial^2 F/\partial x_i \partial x_j$ vanish at p; otherwise we say p is a double point.

As we saw above all reducible cubics contain a singular point: the cubics defined by x_1^3, $x_1^2(x_1 + x_2)$ and $x_1x_2(x_1 + x_2)$ all contain a triple point at $p = [1 : 0 : 0]$; the cubic defined by $x_0x_1x_2$ contains three double points; the cubic defined by $(x_0x_2 + x_1^2)x_1$ contains two double points and the cubic defined by $(x_0x_2 + x_1^2)x_0$ contains a double point (with a single tangent direction).

There are two possible types of double points on an irreducible plane cubic:

- nodes (or ordinary double points); that is, a double point where the curve intersects itself in two branches which have distinct tangents.
- cusps; that is, a double point which is not given by a self intersection point of the curve (so there is a single tangent direction at this point).

Example 6.14. Let $F_1(x_0, x_1, x_2) = x_0x_2^2 + x_1^3 + x_1^2x_0$ and $F_2(x_0, x_1, x_2) = x_0x_2^2 + x_1^3$. Then these cubics are irreducible and have a singular point at $p = [1 : 0 : 0]$. The point p is a double point which is a node of the first cubic corresponding to F_1 and a cusp of the second cubic corresponding to F_2.

Exercise 6.15. Let F be as above and consider the point $p = [1 : 0 : 0] \in \mathbb{P}^2$. Then p is a point of the curve C defined by F if and only if $a_{00} = 0$. In addition to this verify that:

i) p is a singular point of F if and only if $a_{00} = a_{10} = a_{01} = 0$.

ii) p is a triple point of F if and only if $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{02} = 0$.

Finally, recall that if $p = [p_1 : p_2 : p_3]$ is a double point of F, then its tangent lines are defined by the equation

$$\sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}(x_i - p_i)(x_j - p_j) = 0.$$

Show is $p = [1 : 0 : 0]$ is a double point of F, then its tangent lines are defined by

$$a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 = 0.$$

Then write down the tangent planes for the cubics defined by F_1 and F_2 in Example 6.14 and verify that for the node there are two distinct tangent lines whereas for the cusp the two lines coincide.

Remark 6.16. If p is a singular point of F, then $g \cdot p$ is a singular point of $g \cdot F$. Moreover, p is a double (resp. triple) point of F if and only if $g \cdot p$ is a double (resp. triple) point of $g \cdot F$.

We use the Hilbert–Mumford criterion to give a complete description of the (semi)stable locus. Any 1-PS of $\mathrm{SL}(3, k)$ is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & \\ & t^{r_1} & \\ & & t^{r_2} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^{2} r_i = 0$ and $r_0 \geq r_1 \geq r_2$. It is easy to calculate that

$$\mu(F, \lambda) = \max\{(3 - i - j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.$$

Lemma 6.17. *A plane cubic curve C defined by F is semistable if and only if it has no triple point and no double point with a unique tangent. A plane cubic curve C is stable if and only if it is smooth.*

Proof. If F is unstable (that is, not semistable), then by the Hilbert–Mumford criterion there is a 1-PS λ of SL(3) such that $\mu(F, \lambda) < 0$. For some $g \in G$, the 1-PS $\lambda' := g\lambda g^{-1}$ is of the form $\lambda(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ for integers $r_0 \geq r_1 \geq r_2$ which satisfy $\sum r_i = 0$. Then

$$\mu(g \cdot F, \lambda') = \mu(F, \lambda) < 0$$

and if we write $F' := gF = \sum_{i,j} a'_{ij} x_0^{3-i-j} x_1^i x_2^j$, then the relations between the r_i imply that $a'_{00} = a'_{10} = a'_{20} = a'_{11} = 0$. Thus $p = [1 : 0 : 0]$ is a singular point of F' by Exercise 6.15 and $g^{-1} \cdot p$ is a singular point of $F = g^{-1} \cdot F'$. Moreover, if $a_{02} = 0$ also then this point p is a triple point and if $a_{02} \neq 0$ then this is a double point with a single tangent.

Conversely if $F = \sum a_{ij} x_0^{3-i-j} x_1^i x_2^j$ has a double point with a unique tangent or triple point, then we can assume without loss of generality (by using the action of SL(3)) that this point is $p = [1 : 0 : 0]$ and that $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$. Then if $\lambda(t) = \mathrm{diag}(t^3, t^{-1}, t^{-2})$, we see

$$\mu(F, \lambda) \leq \max\{-3, -4, -5, -6, -1\} < 0.$$

Therefore F is semistable if and only if it has no triple point or double point with a unique tangent.

If F has a singular point p, then we can assume without loss of generality that $p = [1 : 0 : 0]$ and so $a_{00} = a_{10} = a_{01} = 0$. Then if $\lambda(t) = \mathrm{diag}(t^2, t^{-1}, t^{-1})$ we have

$$\mu(F, \lambda) \leq \max\{-3, 0\} \leq 0$$

and so F is not stable.

Thus it remains to show that if F is not stable then F is not smooth. Without loss of generality, using the Hilbert–Mumford criterion and the action of SL(3) we can assume that $\mu(F, \lambda) \leq 0$ for $\lambda(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ where $r_0 \geq r_1 \geq r_2$ and $\sum r_i = 0$. It follows from this relation and inequalities that $a_{00} = a_{10} = 0$. If also $a_{01} = 0$, then $p = [1 : 0 : 0]$ is a singular point as required. If $a_{01} \neq 0$, then

$$0 \geq \mu(F, \lambda) \geq 2r_0 + r_2. \tag{1}$$

The inequalities between the r_i imply that we must have equality in (1) and so $r_1 = r_0$ and $r_2 = -2r_0$. Then

$$\mu(F,\lambda) = \max\{(3-3j)r_0 : a_{ij} = 0\} \leq 0$$

where $r_0 > 0$ and so this implies $a_{20} = a_{30} = 0$. In particular, x_2 divides F and so F is a reducible, and hence singular, cubic. □

Hence there are three strictly semistable orbits: nodal cubics, cubics which are a union of a conic and a (non-tangential) line, cubics which are the union of three distinct lines with no common intersection. The orbit consisting of nodal cubics contains in its closure the other two orbits. In particular the compactification of the geometric quotient $X^s_{3,2} \to X^s_{3,2}/\mathrm{SL}(3)$ of smooth cubics is given by adding a single point corresponding to these three orbits. In fact, the geometric quotient is $\mathbb{A}^1$ and its compactification, which is a good quotient of $X^{ss}_{3,2}$, is $\mathbb{P}^1$ (for example, see [10] Chapter 10). The unstable orbits can also be listed: cuspidal cubics, cubics which are the union of a conic and a tangent line, cubics which are the union of three lines with a common intersection, cubics which are the union of a double line with a distinct line and cubics which are given by a single line with multiplicity three.

7. Symplectic geometry

In this section we cover the basics that we need from symplectic geometry. Good references for the material in this section are [8], [27], [40] and [49].

7.1. Symplectic vector spaces.

Definition 7.1. A symplectic form ω on a real vector space V is a skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$ which is non-degenerate (that is, if $v \in V$ and for all $u \in V$ we have $\omega(v,u) = 0$ then $v = 0$). We call (V,ω) a symplectic vector space.

Exercise 7.2. Show that the form ω defines an isomorphism $V \cong V^*$.

Remark 7.3. The notion of a skew-symmetric bilinear form on V agrees with the notion of a 2-form on V (that is, a section of $\wedge^2(T^*V)$) where we identify the cotangent space T^*_vV at v with V.

Exercise 7.4. Show that a real vector space admits a symplectic form only if it is even dimensional. It may help to consider the determinant of the $n \times n$ real matrix $A = (\omega(e_i, e_j))$ associated to ω with respect to a basis $e_1, \dots, e_n$ of V.

Example 7.5. The standard symplectic vector space is the pair $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the standard symplectic form whose associated matrix with respect to a basis $x_1, \dots, x_n, y_1, \dots, y_n$ is

$$A = \left(\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array} \right).$$

As a 2-form we may write this as

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i.$$

Example 7.6. Any complex vector space V with Hermitian inner product $H : V \times V \to \mathbb{C}$ can be viewed as a symplectic vector space by taking the imaginary part of the Hermitian inner product as a symplectic form on V. In particular if $V = \mathbb{C}^n$ then we can use the standard Hermitian inner product on $\mathbb{C}^n$

$$H(z,w) = \sum_{k=1}^n z_k \overline{w_k}$$

which has imaginary part

$$\omega(z,w) = \sum_{k=1}^n \mathrm{Re}(w_k)\mathrm{Im}(z_k) - \mathrm{Re}(z_k)\mathrm{Im}(w_k).$$

Let $z_1, \dots, z_n$ be a basis of $\mathbb{C}^n$ and write $z_k = x_k + iy_k$; then this basis gives a natural identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. With respect to this basis we have that

$$\omega = \sum_{k=1}^{n} dy_k \wedge dx_k = \frac{1}{2i} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k$$

We note that this 'natural' symplectic form on $\mathbb{C}^n$ is equal to minus the standard symplectic form on $\mathbb{R}^{2n}$ (although we can also identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ so that the symplectic forms on $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ agree).

Definition 7.7. Let W be a subspace of a symplectic vector space (V, ω); then we define the symplectic orthogonal of W to be

$$W^\omega := \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

We say W is

i) symplectic if $\omega|_W$ is non-degenerate.
ii) isotropic if $\omega|_W \equiv 0$.
iii) coisotropic if $W^\omega \subset W$.
iv) Lagrangian if $W^\omega = W$.

Example 7.8. Let x_1, x_2, y_1, y_2 denote the standard basis of the standard symplectic vector space $(\mathbb{R}^4, \omega)$ of dimension 4. Then

i) $W = \mathrm{Span}(x_1, y_1)$ is symplectic.
ii) $W = \mathrm{Span}(x_1)$ is isotropic.
iii) $W = \mathrm{Span}(x_1, x_2, y_1)$ is coisotropic.
iv) $W = \mathrm{Span}(x_1, x_2)$ is Lagrangian.

Exercise 7.9. Let W be a subspace of a symplectic vector space (V, ω); then show

i) $\dim W + \dim W^\omega = \dim V$.
ii) W is symplectic if and only if $W^\omega \cap W = \{0\}$ if and only if $V = W^\omega \oplus W$.
iii) W is isotropic if and only if $W \subset W^\omega$.
iv) W is Lagrangian if and only if W is both isotropic and coisotropic if and only if W is isotropic and $\dim W = \dim V/2$.

7.2. **Symplectic manifolds.**

Definition 7.10. A symplectic manifold is a pair (X, ω) where X is a real manifold and ω is a closed non-degenerate 2-form on X which we call the symplectic form.

A 2-form is a smooth section of the second exterior power of the cotangent bundle of X; or equivalently, skew-symmetric bilinear forms $\omega_x : T_xX \times T_xX \to \mathbb{R}$ which vary smoothly with $x \in X$. If we take local coordinates $(x_1, \dots, x_n)$ on X, then we can locally write ω as

$$\omega = \sum_{1 \leq i,j \leq n} f_{i,j} \, dx_i \wedge dx_j.$$

The exterior derivative of a 2-form ω is a 3-form $d\omega$ given locally by

$$d\left(\sum_{1 \leq i,j \leq n} f_{i,j} \, dx_i \wedge dx_j \right) = \sum_{k=1}^{n} \sum_{1 \leq i,j \leq n} \frac{\partial f_{i,j}}{\partial x_k} \, dx_k \wedge dx_i \wedge dx_j$$

and we say that ω is a closed form if $d\omega = 0$. A 2-form ω is non-degenerate if and only if the bilinear forms ω_x are non-degenerate for all $x \in X$. In particular ω_x allows us to identify $T_xX \cong T_x^*X$ by sending ζ to $\omega_x(\zeta, -)$ and combining these isomorphisms over all x in X we get an isomorphism

$$\mathrm{Vect}(X) := \Gamma(TX) \cong \Gamma(T^*X) =: \Omega^1(X)$$

between the spaces of smooth sections of the tangent and cotangent bundles.

Remark 7.11. There is an associated $n \times n$ skew-symmetric matrix $A = (f_{ij})$ of smooth functions $f_{i,j} : X \to \mathbb{R}$ associated to ω where $f_{i,j} = -f_{j,i}$ are locally determined by expressing ω with respect to local coordinates $(x_1, \dots, x_n)$ as

$$\omega = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j.$$

As ω is non-degenerate, its associated matrix A is invertible and so it follows that $n = \dim_{\mathbb{R}} X$ is even (cf. Exercise 7.4).

Example 7.12. Any symplectic vector space is trivially a symplectic manifold: the symplectic form $\omega : V \times V \to \mathbb{R}$ corresponds to a 'constant' two form where we make the natural identification $T_v V \cong V$ and define $\omega_v := \omega : V \times V \to \mathbb{R}$.

(1) If we take coordinates $x_1, \dots, x_n, y_1, \dots y_n$ on $\mathbb{R}^{2n}$, then the standard symplectic form ω_0 on this vector space can be expressed as a 2-form as

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

We evaluate this 2-form at $u, v \in V$ by

$$\omega_0(u, v) = \sum_{i=1}^{n} dx_i(u)dy_i(v) - dx_i(v)dy_i(u)$$

and one can check that $\omega_0(x_i, x_j) = 0$, $\omega_0(x_i, y_j) = \delta_{ij}$ and so on.

(2) For the symplectic form $\omega = \mathrm{Im}H$ on $\mathbb{C}^n$ where H denotes the standard Hermitian inner product on $\mathbb{C}^n$, with respect to the standard coordinates $z_k = x_k + iy_k$ for $k = 1, \dots, n$ we have

$$\omega = \sum_{k=1}^{n} dy_k \wedge dx_k = \frac{1}{2i}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k = \frac{1}{2i}\partial\overline{\partial}\sum_{k=1}^{n} |z_k|^2$$

where ∂ and $\overline{\partial}$ are the dolbeault operators defined by the splitting of the exterior derivative $d = \partial + \overline{\partial}$ into a holomorphic and antiholomorphic part (coming from the additional complex structure we have on $\mathbb{C}^n$).

Example 7.13. Let Y be an n-dimensional real manifold, then its cotangent space $X = T^*Y$ is a real manifold of dimension $2n$ and we may equip it with a symplectic form ω as follows. We can take local coordinates $(y_1, \dots, y_n)$ at a point y in Y and induced local coordinates $(y_1, \dots, y_n, \zeta_1, \dots, \zeta_n)$ at a point $x = (y, \zeta)$ in $X = T^*Y$. There is a universal 1-form α on X where $\alpha_x : T_x X \to \mathbb{R}$ is the composition

$$\alpha_x = \zeta \circ d_x\pi : T_x X \to T_y Y \to \mathbb{R}$$
$$\eta \mapsto \zeta(d_x\pi(\eta))$$

where $d_x\pi : T_x X \to T_y Y$ is the derivative of the projection $\pi : X = T^*Y \to Y$ at $x \in X$. With respect to the local coordinates given above we have

$$\alpha = \sum_{k=1}^{n} \zeta_k dy_k.$$

We define a 2-form $\omega := -d\alpha$ on X which is given locally by

$$\omega = \sum_{k=1}^{n} -d\zeta_k \wedge dy_k = \sum_{k=1}^{n} dy_k \wedge d\zeta_k.$$

The 2-form is closed and non-degenerate and so defines a symplectic from on the cotangent bundle of Y. We note that the appearance of the minus sign is so that when $Y = \mathbb{R}^n$, the symplectic form on $X = T^*Y \cong \mathbb{R}^{2n}$ is equal to the standard symplectic form on this vector space.

Example 7.14. Complex projective space $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is a smooth complex (and so also a smooth real) manifold. In this exercise we shall describe an atlas for $\mathbb{P}^n$. Later on we shall see that $\mathbb{P}^n$ is symplectic (in fact it is actually a Kähler manifold) by giving it a symplectic form ω_{FS} called the Fubini–Study form. The open sets $U_i = \{[z_0 : \cdots : z_n] : z_i \neq 0\}$ cover $\mathbb{P}^n$ and we define charts $\varphi_i : U_i \to \mathbb{C}^n$ by

$$\varphi_i[z_0 : \cdots : z_n] = \left(\frac{z_0}{z_i}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

where the notation hat signifies that we omit this entry. It is easy to see that φ_i is a homeomorphism and so it remains to check that the transition functions $\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth. Without loss of generality we assume $i < j$ and so

$$\varphi_i(U_i \cap U_j) = \{(a_0, \dots, \hat{a}_i, \dots, a_j, \dots, a_n) \in \mathbb{C}^n : a_j \neq 0\}$$

and

$$\psi_{ij}(a_0, \dots, \hat{a}_i, \dots, a_j, \dots, a_n) = \left(\frac{a_0}{a_j}, \dots, \frac{1}{a_j}, \dots, \frac{\hat{a}_j}{a_j}, \dots, \frac{a_n}{a_j}\right)$$

is smooth.

Example 7.15. We now construct a symplectic form ω_{FS}, the 'Fubini–Study' form, on complex projective space $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ which comes from the standard Hermitian inner product H on $\mathbb{C}^{n+1}$. Let $S^{2n+1} = \{p = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} : ||p||^2 := H(p,p) = 1\}$ denote the sphere of (real) dimension $2n+1$ in $\mathbb{C}^{n+1}$; then we have two constructions of $\mathbb{P}^n$:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* = S^{2n+1}/S^1.$$

The idea is that we want to use the imaginary part of H (which is a symplectic form on $\mathbb{C}^{n+1}$) to obtain a symplectic form on the quotient. In fact this will be our first example of the Marsden–Weinstein–Meyer theorem which allows us to construct symplectic forms on manifolds realised as such quotients. In order for our symplectic form to descend to the quotient we need it to be invariant under the group action with respect to which we are quotienting. Unfortunately, the Hermitian inner product is not invariant under the natural scaling action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$; that is, it is not the case that

$$H(cu, cv) = H(u, v) \text{ for all } c \in \mathbb{C}^*.$$

However, this is true if we replace $\mathbb{C}^*$ by $S^1 \cong U(1)$; that is, H is S^1-invariant (or in terms of the language we shall introduce later on, the action of S^1 on $(\mathbb{C}^{n+1}, \mathrm{Im}H)$ is 'symplectic'). Therefore we use the construction $\mathbb{P}^n = S^{2n+1}/S^1$. We let $\omega_{S^{2n+1}}$ denote the restriction of the symplectic form $\mathrm{Im}H$ on $\mathbb{C}^n$ to $S^{2n+1} \subset \mathbb{C}^n$ (we note that the sphere S^{2n+1} is not symplectic as it has odd (real) dimension but we can still restrict the symplectic form nevertheless). For $p \in S^{2n+1}$, the tangent space $T_{[p]}\mathbb{P}^n$ fits into a short exact sequence

$$0 \to T_p(S^1 \cdot p) \to T_p(S^{2n+1}) \to T_{[p]}\mathbb{P}^n \to 0 \tag{2}$$

where the final map is the derivative $d_p\pi$ of the projection $\pi : S^{2n+1} \to \mathbb{P}^n$. We want to define the Fubini–Study form ω_{FS} on $\mathbb{P}^n$ so that $\pi^*\omega_{FS} = \omega_{S^{2n+1}}$; i.e.

$$\pi^*\omega_{FS,[p]}(\zeta, \xi) := \omega_{FS,[p]}(d_p\pi(\zeta), d_p\pi(\xi)) = \omega_{S^{2n+1},p}(\zeta, \xi). \tag{3}$$

To check that this is well-defined, using the short exact sequence (2), it remains to check that $\omega_{S^{2n+1}}(\zeta, \xi) = 0$ if either ζ or ξ are tangent vectors in $T_p(S^1 \cdot p)$. However as $T_p(S^1 \cdot p) \cong 2\pi i p\mathbb{R}$ and $(T_pS^{2n+1})^\omega = 2\pi i p\mathbb{R}$, we see that this is the case. Therefore (3) defines a 2-form ω_{FS} on $\mathbb{P}^n$. The fact that this 2-form is closed and non-degenerate follows from the fact that $\mathrm{Im}H$ is closed and non-degenerate (we shall see a proof of this when we give the proof of the Marsden–Weinstein–Meyer theorem).

Finally we describe the Fubini–Study form on a chart $\varphi_0 : U_0 \cong \mathbb{C}^n$. We factor the inverse of this homeomorphism via the sphere S^{2n+1}:

$$\varphi_0^{-1} : \mathbb{C}^n \to S^{2n+1} \to U_0$$

$$z = (z_1, \dots, z_n) \mapsto \frac{1}{\sqrt{1 + ||z||^2}}(1, z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$$

and let $f = (f_0, \dots, f_n) : \mathbb{C}^n \to S^{2n+1} \subset \mathbb{C}^{n+1}$ denote the first map in this factorisation. Then $(\varphi_0^{-1})^* \omega_{FS}|_{U_0} = (\pi \circ f)^* \omega_{FS}|_{U_0} = f^* \omega_{S^{2n+1}}|_{\pi^{-1}(U_0)}$; that is,

$$(\varphi_0^{-1})^* \omega_{FS}|_{U_0} = \frac{1}{2i} \sum_{k=0}^{n} df_k \wedge d\overline{f_k}$$

which after carefully writing down df_k and $d\overline{f_k}$ in terms of the coordinates z_j on $\mathbb{C}^n$ becomes

$$(\varphi_0^{-1})^* \omega_{FS}|_{U_0} = \frac{1}{2i} \left(\frac{1}{1 + ||z||^2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k + \frac{1}{(1+||z||^2)^2} (\sum_{k=1}^{n} \overline{z}_k dz_k) \wedge (\sum_{j=1}^{n} z_j d\overline{z}_j) \right)$$

or, with respect to the Dolbeault operators,

$$(\varphi_0^{-1})^* \omega_{FS}|_{U_0} = \frac{1}{2i} \partial \overline{\partial} \log(1 + ||z||^2).$$

Therefore at $[p] = [1 : 0 : 0 \cdots : 0] \in \mathbb{P}^n$ we have

$$\omega_{FS,[p]} = \frac{1}{2i} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k$$

with respect to the local coordinates $(z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$ at $[p]$.

Remark 7.16. One can alternatively 'glue' the Fubini–Study on $\mathbb{P}^n$ by pulling back 'the Fubini–Study form' on $\mathbb{C}^n$ via the charts $\varphi : U_i \cong \mathbb{C}^n$. Of course one must check that on the overlaps we get the same form and it is for this reason that we must use the Fubini–Study form on $\mathbb{C}^n$ rather than the standard symplectic form on $\mathbb{C}^n$ (which is equal to the imaginary part of the standard Hermitian inner product on $\mathbb{C}^n$). For a description of the Fubini–Study form on $\mathbb{P}^n$ and how this gluing procedure works, see for example [8].

It follows from this example that any smooth closed complex subvariety of $\mathbb{P}^n$ is symplectic with symplectic form obtained by pulling back the Fubini–Study form on $\mathbb{P}^n$.

Example 7.17. Let be G be a compact and connected real Lie group; its Lie algebra $\mathfrak{g}$ is a vector space that is by definition the tangent space to G at the identity. The adjoint action of G is a representation of G on its Lie algebra $\mathfrak{g}$ which we write as $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ where Ad_g is the derivative at the identity of the map $G \to G$ corresponding to conjugation by g. We can also consider the coadjoint representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ which is the dual representation to the adjoint representation. We note that in order to get a left action we define $(\mathrm{Ad}^*)_g = (\mathrm{Ad}_{g^{-1}})^*$ so that $(\mathrm{Ad}^*)_h (\mathrm{Ad}^*)_g = (\mathrm{Ad}^*)_{hg}$. Let $\mathrm{Der}(\mathfrak{g})$ denote the Lie algebra of $\mathrm{GL}(\mathfrak{g})$ which is equal to the derivation algebra of $\mathfrak{g}$; then the infinitesimal version of the adjoint representation of G is a representation of the Lie algebra $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by taking the derivative of Ad. More precisely, for $A, B \in \mathfrak{g}$ we have

$$\mathrm{ad}_A B := \frac{d}{dt} \mathrm{Ad}_{\exp(tA)}(B)|_{t=0} = [A, B].$$

We call the representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$ and the coadjoint representation of $\mathfrak{g}$ is the dual of this representation, which we denote by ad^*.

Let $\eta \in \mathfrak{g}^*$ and $\mathcal{O} \subset \mathfrak{g}^*$ denote the coadjoint orbit of η for the action of G on $\mathfrak{g}^*$; thus $\mathcal{O}$ is the image of G under the map

$$\mathrm{Ad}^*_{-}(\eta) : G \to \mathfrak{g}^* \quad g \mapsto \mathrm{Ad}^*_g(\eta)$$

and the kernel of this map is the stabiliser G_η of η for this action. The tangent space $T_\eta \mathcal{O}$ is then the image of $\mathfrak{g}$ under the map

$$\mathrm{ad}^*_{-}(\eta) : \mathfrak{g} \to \mathfrak{g}^* \quad A \mapsto \mathrm{ad}^*_A(\eta)$$

and the kernel is the isotropy group $\mathfrak{g}_\eta := \{A \in \mathfrak{g} : \mathrm{ad}^*_A(\eta) \cdot B = 0 \, \forall \, B \in \mathfrak{g}\}$. In particular, we have a short exact sequence

$$0 \to \mathfrak{g}_\eta \to \mathfrak{g} \to T_\eta \mathcal{O} \to 0.$$

We use the Lie bracket $[-,-]$ to define a symplectic form ω on the coadjoint orbit $\mathcal{O}$ of η by

$$\omega_\eta(\mathrm{ad}^*_A\eta, \mathrm{ad}^*_B\eta) = \eta \cdot [A, B]$$

where $\cdot : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural evaluation map. To check this is well-defined we must check that $\omega_\eta(\mathrm{ad}^*_A\eta, \mathrm{ad}^*_B\eta) = 0$ if either A or B belong to the isotropy group $\mathfrak{g}_\eta$; this follows immediately as if say $A \in \mathfrak{g}_\eta$, then $0 = \mathrm{ad}^*_A(\eta) \cdot B = \eta \cdot \mathrm{ad}_{-A}(B) = -\eta \cdot [A, B]$ for all $B \in \mathfrak{g}$. It is easy to see from this description that ω is non-degenerate.

Exercise 7.18. Prove that the form defined above on the coadjoint orbit $\mathcal{O}$ is closed.

Corollary 7.19. *Every coadjoint orbit associated to a compact and connected Lie group is even dimensional.*

7.3. **Morphisms in symplectic geometry.** We need a notion of morphisms between symplectic manifolds so that we can construct a symplectic category. The initial notion is given by symplectomorphisms:

Definition 7.20. A symplectomorphism $f : (M, \omega) \to (M', \omega')$ of symplectic manifolds is a diffeomorphism $f : M \to M'$ such that $f^*\omega' = \omega$.

Theorem 7.21. *(Darboux) Any symplectic manifold of dimension $2n$ is locally symplectomorphic to $\mathbb{R}^{2n}$ equipped with the standard symplectic form.*

As a symplectomorphism is a diffeomorphism, there are only symplectomorphisms between manifolds of the same dimension. In particular, this notion of morphism is rather restrictive. As we are interested in construction symplectic quotients we want to allow morphisms between manifolds of different dimensions. A more general notion of morphisms is given by using Lagrangian correspondences.

Definition 7.22. A submanifold L of a symplectic manifold X is Lagrangian if $2 \dim L = \dim X$ and $i^*\omega = 0$ where $i : L \hookrightarrow X$ is the inclusion. Equivalently, L is Lagrangian if for all $x \in L$ the vector space T_xL is a Lagrangian subspace of T_xX; that is,

$$(T_xL)^{\omega_x} = \{\eta \in T_xX : \omega_x(\eta, \zeta) = 0 \,\forall\, \zeta \in T_xL\} = T_xL.$$

A Lagrangian correspondence between symplectic manifolds (X_1, ω_1) and (X_2, ω_2) is a Lagrangian submanifold L_{12} of $(X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ where $-\omega_1 \boxplus \omega_2 := -\pi_1^*\omega_1 + \pi_2^*\omega$.

Example 7.23. We view Lagrangian correspondences as morphisms in the symplectic category.

1. The identity morphism on (X, ω) is given by the diagonal $\Delta_X \subset (X \times X, -\omega \boxplus \omega)$.
2. Given any symplectomorphism $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$, the graph $\Gamma(\phi) \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ is a Lagrangian submanifold. Therefore, the notion of symplectic correspondence generalises that of symplectomorphisms.

As we want to view Lagrangian correspondences as morphisms between symplectic manifold we need to define the composition of two Lagrangian correspondence. Given Lagrangian submanifolds $L_{12} \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ and $L_{23} \subset (X_2 \times X_3, -\omega_2 \boxplus \omega_3)$, we define the composition $L_{13} = L_{23} \circ L_{12}$ to be the Lagrangian submanifold $L_{13} := \pi_{13} L_{12} \times_{X_2} L_{23}$ of $(X_1 \times X_3, -\omega_1 \boxplus \omega_3)$ given by

$$L_{13} = \{(x_1, x_3) \in X_1 \times X_3 : \exists x_2 \in X_2 \text{ such that } (x_1, x_2) \in L_{12} \text{ and } (x_2, x_3) \in L_{23}\}.$$

Then, following [49], we define morphisms in the symplectic category to consist of chains of symplectic correspondences.

7.4. **Hamiltonian vector fields.** Let X be a smooth manifold of dimension n. We denote the (infinite dimensional) Lie algebra of smooth vector fields on X by $\mathrm{Vect}(X)$, where the Lie bracket $[-,-]$ on $\mathrm{Vect}(X)$ is given by the commutator, and we denote the differential graded algebra of smooth forms on X by

$$\Omega^*(X) = \oplus_{r \geq 0} \Omega^r(X)$$

where $\Omega^r(X) = \Gamma(\wedge^r T^*X)$ and the differential $d : \Omega^r(X) \to \Omega^{r+1}(X)$ is given by the exterior derivative:

$$d\left(\sum_{I=(i_1,\dots,i_r)} f_I dx_I\right) = \sum_{j=1}^{n}\sum_{I} \frac{df_I}{dx_j} dx_j \wedge dx_I.$$

We recall that for a vector field ζ we have the Lie derivative $L_\zeta : \Omega^r(X) \to \Omega^r(X)$ defined by

$$L_\zeta = d \circ \iota_\zeta + \iota_\zeta \circ d$$

where $\iota_\zeta : \Omega^r(X) \to \Omega^{r-1}(X)$ is given by contracting with ζ:

$$\iota_\zeta \alpha(\zeta_1, \dots, \zeta_{r-1}) = \alpha(\zeta, \zeta_1, \dots \zeta_n).$$

For $f \in \Omega^0(X) = \mathcal{C}^\infty(X)$ and a vector field ζ on X, this satisfies

$$L_\zeta(f) = df(\zeta).$$

Now suppose (X, ω) is a symplectic manifold. Given a smooth function $H : X \to \mathbb{R}$, we can construct an associated vector field ζ_H using the duality defined by the symplectic form ω: recall that there is $\zeta_H \in \mathrm{Vect}(X)$ which corresponds under the duality defined by ω to $dH \in \Omega^1(X)$; that is,

$$\iota_{\zeta_H}\omega = dH.$$

Definition 7.24. A vector field ζ on a symplectic manifold (X, ω) is Hamiltonian if $\zeta = \zeta_H$ for some smooth function $H : X \to \mathbb{R}$ (or equivalently, $\iota_\zeta\omega$ is exact). A vector field ζ is symplectic if $L_\zeta\omega = 0$ (or equivalently, $\iota_\zeta\omega$ is closed).

As exact forms are closed, every Hamiltonian vector field is a symplectic vector field.

8. Actions in symplectic geometry

In this section we consider actions of Lie groups on symplectic manifolds. We shall assume basic familiarity with Lie groups and Lie algebras (for example, see [6], [9] and [13]). Good references for the results in this section on actions in symplectic geometry are [2],[8], [27], [48] and [50].

8.1. **Symplectic actions.** Let K be a real Lie group and X be a real smooth manifold. A smooth action of K on X is an action $K \times X \to X$ which is smooth. In particular this gives a group homomorphism from K to the group of diffeomorphisms of X

$$K \to \mathrm{Diff}(X)$$
$$k \mapsto (x \mapsto k \cdot x)$$

which we call the action homomorphism.

Definition 8.1. Let (X, ω) be a symplectic manifold and let K be a lie group which acts smoothly on X. We say the action is symplectic if K acts by symplectomorphisms i.e. the image of the action map $K \to \mathrm{Diff}(X)$ is contained in the subgroup $\mathrm{Sympl}(X, \omega)$ of symplectomorphisms of X.

8.2. **Hamiltonian actions.** The infinitesimal version of the Lie group K is its Lie algebra $\mathfrak{K}$ which is by definition the tangent space at the identity of G. For A in $\mathfrak{K}$, we may consider the associated real 1-PS $\exp(-A) : \mathbb{R} \to K$ which induces an diffeomorphism $X \to X$ given by $x \mapsto \exp(tA) \cdot x$. If we fix $x \in X$ and $A \in \mathfrak{K}$ then we can take the derivative of the smooth map $\exp(-A) \cdot x : \mathbb{R} \to X$ at $0 \in \mathbb{R}$ to get a linear map $d_0 \exp(-A) \cdot x : \mathbb{R} \to T_xX$ whose evaluation at $1 \in \mathbb{R}$ we refer to as the infinitesimal action of A on x:

$$A_x := (d_0 \exp(-A) \cdot x)(1) = \frac{d}{dt} \exp(tA) \cdot x|_{t=0} \in T_xX.$$

Letting x vary, we get a vector field A_X on X such that $A_{X,x}$ is the infinitesimal action A_x as above. The resulting Lie algebra homomorphism

$$\mathfrak{K} \to \mathrm{Vect}(X)$$

$$A \mapsto A_X$$

obtained by differentiating the action homomorphism is referred to as the infinitesimal action homomorphism.

Definition 8.2. Let K be a Lie group acting on a symplectic manifold (X, ω). The action is

i) infinitesimally symplectic if A_X is a symplectic vector field for all $A \in \mathfrak{K}$.
ii) weakly Hamiltonian if A_X is a Hamiltonian vector field for all $A \in \mathfrak{K}$.

Given a symplectic action of K on X for which the exponential map $\mathfrak{K} \to K$ is surjective (for example, this is the case if K is compact and connected), we have for all $k \in K$ the map $k : X \to X$ given by $x \mapsto k \cdot x$ is a symplectomorphism i.e. $k^*\omega = \omega$. On differentiating the condition $\exp(tA)^*\omega = \omega$, we get $L_{A_X}\omega = 0$ (or equivalently $d\iota_{A_X}\omega = 0$). Hence in this case, the symplectic action is infinitesimally symplectic.

For a weakly Hamiltonian action the infinitesimal action $\mathfrak{K} \to \mathrm{Vect}(X)$ can be pointwise lifted to a map $\mathfrak{K} \to \mathcal{C}^\infty(X)$ as a Hamiltonian vector field corresponds (under ω) to the exterior derivative of a smooth function on X. Of course in general, this lift is non-unique as there may be several smooth functions with the same exterior derivative. We can define a Lie algebra structure on $\mathcal{C}^\infty(X)$ using ω by

$$\{F, H\}(x) := \omega_x(\zeta_{F,x}, \zeta_{H,x})$$

where ζ_F is the vector field which corresponds under the duality defined by ω to the 1-form dF. Then we can ask if there is a way to lift the infinitesimal action so that the lift is a Lie algebra homomorphism.

Definition 8.3. If we have a symplectic action of a Lie group K on (X, ω) such that the infinitesimal action $\mathfrak{K} \to \mathrm{Vect}(X)$ can be lifted to a Lie algebra homomorphism $\mathfrak{K} \to \mathcal{C}^\infty(X)$, then we say the action is Hamiltonian. The map $\phi : \mathfrak{K} \to \mathcal{C}^\infty(X)$ is called the comoment map.

8.3. **Moment map.** Hamiltonian actions can also be described by using a moment (or momentum) map. As the name suggest the moment(um) map first arose in classical mechanics (for a description of the moment map from the viewpoint of classical mechanics see the notes of Butterfield [7]).

Definition 8.4. A smooth map $\mu : X \to \mathfrak{K}^*$ is called a moment map if it is K-equivariant with respect to the action of the given K on X and the coadjoint action of K on $\mathfrak{K}^*$ and in addition

$$d\mu_x(\zeta) \cdot A = \omega_x(A_x, \zeta) \tag{4}$$

for all $x \in X$, $\zeta \in T_xX$ and $A \in \mathfrak{K}$.

Remark 8.5. There is still no consistent choice of sign conventions for the moment map and so often a minus sign may appear in the condition (4) used to define a moment map.

A comoment map ϕ defines a moment map μ by $\mu(x) \cdot A = \phi(A)(x)$ for $x \in X$ and $A \in \mathfrak{K}$.

Remark 8.6. The moment map is not necessarily unique (see Example 8.7 below), although for certain groups we will see that it is unique (cf. part (1) of Example 8.8).

8.4. **Examples of moment maps.**

Example 8.7. Let $K = \mathrm{U}(n)$ be the unitary group of $n \times n$ matrices and consider its standard representation on $\mathbb{C}^n$. The infinitesimal action is given by

$$A_x = \frac{d}{dt}\exp(tA) \cdot x|_{t=0} = Ax$$

for $x \in \mathbb{C}^n$ and A a skew-Hermitian matrix in $\mathfrak{u}(n)$. Let $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denote the standard Hermitian inner product on $\mathbb{C}^n$: if we write $z \in \mathbb{C}^n$ as a row vector $(z_1, \dots z_n)$, then

$$H(z, v) = z\overline{v}^t = \overline{v}z^t = \mathrm{Tr}(z^t\overline{v}) = \mathrm{Tr}(\overline{v}^t z)$$

where v^t denotes the transpose of a matrix. We take the symplectic inner product $\omega : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ equal to the imaginary part of this Hermitian inner product:

$$\omega(z,v) = \frac{1}{2i}\left(H(z,v) - \overline{H(z,v)}\right) = \frac{1}{2i}\left(H(z,v) - H(v,z)\right).$$

The Hermitian inner product is $\mathrm{U}(n)$-invariant (that is, $H(Az, Av) = H(z,v)$ for all unitary matrices $A \in \mathrm{U}(n)$) and so it follows that the action of $\mathrm{U}(n)$ on $(\mathbb{C}^n, \omega)$ is symplectic. In fact this action is Hamiltonian and there is a canonical choice of moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^*$ which is defined by

$$\mu(z) \cdot A = \frac{1}{2}\omega(Az, z) = \frac{1}{2i}H(Az, z)$$

for $A \in \mathfrak{u}(n)$. The second equality follows from the fact that

$$H(Az,v) + H(z,Av) = \mathrm{Tr}(z^t A^t \overline{v}) + \mathrm{Tr}(z^t \overline{A}\overline{v}) = \mathrm{Tr}(z^t(A^t + \overline{A})\overline{v}) = \mathrm{Tr}(0) = 0 \tag{5}$$

for all $z, v \in \mathbb{C}^n$ and $A \in \mathfrak{u}(n)$. We shall now carefully check that this is a moment map. Firstly it is $\mathrm{U}(n)$-equivariant:

$$\mu(k \cdot z) \cdot A = \frac{1}{2}\omega(Ak \cdot z, k \cdot z) = \frac{1}{2}\omega(k^{-1}Ak \cdot z, \cdot z) = \mu(z) \cdot k^{-1}Ak = (Ad^*)_k \mu(x) \cdot A$$

where $A \in \mathfrak{u}(n)$, $z \in \mathbb{C}^n$ and $k \in \mathrm{U}(n)$. To verify condition (4), we may identify $T_z\mathbb{C}^n \cong \mathbb{C}^n$ and then this condition becomes

$$d\mu_z(v) \cdot A = \omega(Az, v)$$

for $v \in \mathbb{C}^n \cong T_z\mathbb{C}^n$ and $A \in \mathfrak{u}(n)$. We can verify that

$$\begin{aligned} d\mu_z(v) \cdot A &:= \frac{d}{dt}\mu(z + tv) \cdot A|_{t=0} \\ &= \frac{1}{2}\frac{d}{dt}\omega(A(z+tv), z+tv)|_{t=0} = \frac{1}{2}\left[\omega(Az,v) + \omega(Av,z)\right] \\ &= \frac{1}{4i}\left[H(Az,v) - H(v,Az) + H(Av,z) - H(z,Av)\right] \\ &= \frac{1}{2i}\left[H(Az,v) - H(v,Az)\right] =: \omega(Az,v) \end{aligned}$$

where the equality on the final line comes from the relation given at (5). We note that the moment map for this symplectic action is not unique, although it is unique up to addition by an element η of $\mathfrak{u}(n)^*$ which is fixed by the coadjoint action $\mathrm{U}(n) \to \mathrm{GL}(\mathfrak{u}(n))$ (we call such η a central element). Every character of $\mathrm{U}(n)$ is a power of the determinant $\det : \mathrm{U}(n) \to \mathrm{S}^1$ whose derivative is give by the trace $\mathrm{Tr} : \mathfrak{u}(n) \to \mathrm{Lie}\,\mathrm{S}^1 \cong 2\pi i\mathbb{R}$. Hence such a central element η of $\mathfrak{u}(n)^*$ must be equal to $ci\mathrm{Tr}$ for some $c \in \mathbb{R}$ and the associated moment map is

$$\mu_\eta(z) \cdot A = \frac{1}{2i}H(Az,z) + \eta \cdot A.$$

Example 8.8. In fact if K is a Lie group which acts on $\mathbb{C}^n$ by a representation $\rho : K \to \mathrm{U}(n)$, then we can write down the moment map μ_K for the action of K on $\mathbb{C}^n$ using ρ and the moment map $\mu_{\mathrm{U}(n)} : \mathbb{C}^n \to \mathfrak{u}(n)^*$ for the $\mathrm{U}(n)$-action on $\mathbb{C}^n$ constructed in Example 8.7. The moment map for the action of K is given by

$$\mu_K = \rho^* \mu_{\mathrm{U}(n)}$$

where $\rho^* : \mathfrak{u}(n)^* \to \mathfrak{K}^*$ is the dual to the inclusion $\rho : \mathfrak{K} \to \mathfrak{u}(n)$. We consider the following special cases:

(1) If $K = \mathrm{SU}(n)$ acts on $\mathbb{C}^n$ by the standard inclusion into $\mathrm{U}(n)$, then its moment map is given by

$$\mu(z) \cdot A = \frac{1}{2}\omega(Az, z) = \frac{1}{2i}H(Az, z)$$

for $A \in \mathfrak{su}(n)$. However, there are no non-zero central elements of $\mathfrak{su}(n)$ which we can use to shift the moment map by and so this moment map is unique.

(2) If $K = (S^1)^n$ acts on $\mathbb{C}^n$ via the representation
$$(t_1, \ldots, t_n) \mapsto \mathrm{diag}(t_1, \ldots, t_n),$$
then the moment map is given by
$$\mu(z) \cdot (A_1, \ldots, A_n) = \frac{1}{2}\omega(\mathrm{diag}(A_1, \ldots, A_n)z, z)$$
for $A_k \in \mathrm{Lie} S^1 \cong 2\pi i\mathbb{R}$. If we write $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $A_k = 2\pi i a_k$ for real numbers a_k then
$$\mu(z) \cdot (2\pi i a_1, \ldots, 2\pi i a_n) = \pi \sum_{k=1}^{n} a_k |z_k|^2.$$
Of course the group $K = (S^1)^n$ is commutative and so every element in central and so it follows that all elements in $\mathfrak{K} \cong (2\pi i\mathbb{R})^n$ are central. We can shift the standard moment map by any n-tuple $(c_1, \ldots, c_n)$ of real numbers to get
$$\mu(z) \cdot (2\pi i a_1, \ldots, 2\pi i a_n) = \pi \sum_{k=1}^{n} (a_k |z_k|^2 + c_k).$$

(3) We may also consider $K = (S^1)^n$ acting on $\mathbb{C}^n$ via the representation
$$(t_1, \ldots, t_n) \mapsto \mathrm{diag}(t_1^{r_1}, \ldots, t_n^{r_n})$$
for integers r_k. In this case the moment map (shifted by real numbers c_i) is given by
$$\mu(z) \cdot (2\pi i a_1, \ldots, 2\pi i a_n) = \pi \sum_{k=1}^{n} (a_k r_k |z_k|^2 + c_k).$$

Exercise 8.9. Consider the action of $K = U(m)$ on the space of $l \times m$-matrices over the complex numbers $M_{l\times m}(\mathbb{C}) \cong \mathbb{C}^{lm}$ given by $k \cdot M = Mk^{-1}$ where we take the natural symplectic structure given by the imaginary part of the standard Hermitian inner product on $\mathbb{C}^{lm}$. Then if $M \in M_{l\times m}$ and $A \in \mathfrak{u}(m)$ show
$$\mu(M) \cdot A = \frac{i}{2} \mathrm{Tr}(MAM^*)$$
is a moment map for this action.

So far the only examples of moment maps that we have seen are for affine spaces. As we will eventually be interested in comparing projective GIT quotients with symplectic quotients, we should of course verify that these sorts of actions are Hamiltonian. Often we will be in the situation of a Lie group K acting on a smooth complex projective variety $X \subset \mathbb{P}^n$ via a representation $K \to \mathrm{U}(n+1)$. In order to construct a moment map for such an action it suffices to construct a moment map for the standard action of $\mathrm{U}(n+1)$ on $\mathbb{P}^n$.

Example 8.10. Consider $U(n+1)$ acting on complex projective space $\mathbb{P}^n$ by acting on its affine cone $\mathbb{C}^{n+1}$ in the standard way. The symplectic form on $\mathbb{P}^n$ is the Fubini-Study form ω_{FS} constructed in Example 7.15 from the standard Hermitian inner product H on $\mathbb{C}^{n+1}$. It is easy to see this from is $U(n+1)$-invariant (that is, the action is symplectic). As the Fubini-Study form is preserved by the action of $\mathrm{U}(n+1)$ and the unitary group is compact and connected, the action is infinitesimally symplectic so that $d\iota_{A_X}\omega_{\mathrm{FS}} = 0$. Moreover, as $H^1(\mathbb{P}^n) = 0$, every closed 1-form is exact and so the action is weakly Hamiltonian. It turns out that this action is Hamiltonian and we can write down a moment map explicitly as follows. Let $p = (p_0, \ldots, p_n) \in \mathbb{C}^{n+1} - \{0\}$, then we claim that
$$\mu([p]) \cdot A = \frac{\mathrm{Tr} p^* A p}{2i||p||^2}$$
defines a moment map where $[p] \in \mathbb{P}^n$ and $A \in \mathfrak{u}(n+1)$ and p^* denotes the complex conjugate transpose. We leave the $U(n+1)$-equivariance of μ for the reader to check. As the action of $U(n+1)$ on $\mathbb{P}^n$ is transitive, we need only verify the condition (4) at a single point $[p] = [1 : 0 :$

$\cdots : 0] \in \mathbb{P}^n$. Following Example 7.15, we can identify $T_{[p]}\mathbb{P}^n$ with the orthogonal space to p in $\mathbb{C}^{n+1}$ with respect to the standard Hermitian product on $\mathbb{C}^{n+1}$:

$$T_{[p]}\mathbb{P}^n = T_p S^{2n+1}/T_p(S^1 \cdot p) \cong \{(0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}\} \cong \mathbb{C}^n.$$

With respect to the coordinates $(z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$ at $[p]$, the Fubini-Study form can be expressed locally as

$$\omega_{\mathrm{FS},[p]} = \frac{1}{2i}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k;$$

that is, for $v, w \in T_{[p]}\mathbb{P}^n$, we have

$$\omega_{\mathrm{FS},[p]}(v,w) = \mathrm{Im}H(v,w) = \frac{1}{2i}[H(v,w) - H(w,v)].$$

Let $v \in T_{[p]}\mathbb{P}^n$ and $A \in \mathfrak{u}(n+1)$; then

$$\begin{aligned} d_{[p]}\mu(v)\cdot A &:= \frac{d}{dt}\mu(p+tv)\cdot A|_{t=0} = \frac{1}{2i}\frac{d}{dt}\frac{\mathrm{Tr}((p+tv)^*A(p+tv))}{||p+tv||^2}|_{t=0} \\ &= \frac{1}{2i}\frac{\mathrm{Tr}(v^*Ap + p^*Av)||p||^2 - (pv^* + vp^*)\mathrm{Tr}(p^*Ap)}{||p||^4} \\ &\overset{(a)}{=} \frac{1}{2i}[H(Ap,v) - H(v,Ap)] \\ &= \omega_{\mathrm{FS},[p]}(Ap, v) \end{aligned}$$

where (a) follows as $H(p,v) = p^*v = 0$ for $v \in T_{[p]}\mathbb{P}^n$.

Exercise 8.11. Let K be a compact and connected Lie group; then a coadjoint orbit $\mathcal{O} \subset \mathfrak{K}^*$ for the action of K on $\mathfrak{K}^*$ has a symplectic form ω by Example 7.17. Describe the infinitesimal action for the natural action of K on $\mathcal{O}$ and show that the inclusion $\mu : \mathcal{O} \hookrightarrow \mathfrak{K}^*$ is a moment map for this action.

9. Symplectic quotients

Given an action of a Lie group K on (X, ω) we can ask whether a quotient exists. The topological quotient always exists, but it will only be a manifold if the action is free and proper. Even if the action is free and proper, the resulting quotient manifold may have odd dimension over $\mathbb{R}$ and so will not admit a symplectic form. Hence, the topological quotient X/K does not in general provide a suitable quotient in symplectic geometry.

In this section we define the symplectic reduction associated to a Hamiltonian action and show that in nice cases it is a symplectic manifold and also has a universal property amongst all symplectic quotients. The central theorem in this section is a result of Marsden and Weinstein [27] and Meyer [27] on the symplectic reduction; although other good references are [8], [27] and [50].

9.1. Properties of moment maps.

Suppose K is a Lie group acting on a symplectic manifold (X, ω). For $x \in X$, we let $K \cdot x := \{kx : k \in K\}$ denote the orbit of x and $K_x = \{k : k \cdot x = x\}$ denote the stabiliser of x. Then $\mathfrak{K}_x := \{A \in \mathfrak{K} : A_x = 0\}$ is the Lie algebra of K_x.

If the action is Hamiltonian, then there is an associated moment map $\mu : X \to \mathfrak{K}^*$ and one can naturally ask what other properties of the action are encoded by the moment map.

Lemma 9.1. *Suppose we have a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{K}^*$. Then for all $x \in X$:*

i) $\ker d\mu_x = (T_x(K \cdot x))^{\omega_x} := \{\zeta \in T_xX : \omega_x(\eta, \zeta) = 0\, \forall \eta \in T_x(K\cdot x)\}$,
ii) $\mathrm{Im} d\mu_x = \mathrm{Ann}\mathfrak{K}_x := \{\eta \in \mathfrak{K}^* : \eta \cdot A = 0\, \forall A \in \mathfrak{K}_x\}$.

Proof. i) The tangent space $T_x(K \cdot x)$ to the orbit $K \cdot x$ at x consists of tangent vectors $\gamma'(0)$ where $\gamma : \mathbb{R} \to K \cdot x$ is a smooth curve such that $\gamma(0) = x$. We recall for each $A \in \mathfrak{K}$, we can define a smooth curve $\gamma_A(t) = \exp(tA) \cdot x$ such that $\gamma_A(0) = x$ and whose associated tangent vector $\gamma'(0) = A_x$ is the infinitesimal of A on x; therefore $T_x(K \cdot x) = \{A_x : A \in \mathfrak{K}\}$. A tangent vector $\zeta \in T_xX$ is in the kernel of $d_x\mu$ if and only if for all $A \in \mathfrak{K}$ we have

$$0 = d_x\mu(\zeta) \cdot A = \omega_x(A_x, \zeta);$$

that is, if and only if $\zeta \in T_x(K \cdot x)^{\omega_x} := \{\zeta \in T_xX : \omega_x(\zeta, \eta) = 0 \, \forall \eta \in T_x(K \cdot x)\}$.

ii) By definition, an element $A \in \mathfrak{K}$ belongs to $\mathfrak{K}_x$ if and only if the infinitesimal action of A at x is trivial i.e. $A_x = 0 \in T_xX$. If $\eta = d_x\mu(\zeta) \in \mathrm{Im} d_x\mu$ where $\zeta \in T_xX$, then for all $A \in \mathfrak{K}$ we have that

$$\eta \cdot A = d_x\mu(\zeta) \cdot A = \omega_x(A_x, \zeta).$$

In particular, if $A \in \mathfrak{K}_x$, then $A_x = 0$ and so we see that $\eta \in \mathrm{Ann}\mathfrak{K}_x$. Hence we have an inclusion $\mathrm{Im} d_x\mu \subset \mathrm{Ann}\mathfrak{K}_x$ and we count dimensions to verify these vector spaces are equal. Suppose the kernel of $d_x\mu$ has dimension n and T_xX has dimension d, so that the image $\mathrm{Im} d_x\mu$ has dimension $d - n$. By i), the kernel is equal to $T_x(K \cdot x)^{\omega_x}$ and so its symplectic orthogonal $T_x(K \cdot x)$ has dimension $d - n$. By the orbit stabiliser theorem:

$$\dim \mathfrak{K} = \dim \mathfrak{K}_x + \dim T_x(K \cdot x). \tag{6}$$

The inclusion $\mathrm{Ann}\mathfrak{K}_x \subset \mathfrak{K}^*$ induces a short exact sequence

$$0 \to \mathrm{Ann}\mathfrak{K}_x \to \mathfrak{K}^* \to \mathfrak{K}_x^* \to 0$$

and so

$$\dim \mathrm{Ann}\mathfrak{K}_x = \dim \mathfrak{K} - \dim \mathfrak{K}_x$$

which in turn is equal to $\dim T_x(K \cdot x) = d - n$ by (6). Therefore, the inclusion of vector spaces $\mathrm{Im} d_x\mu \subset \mathrm{Ann}\mathfrak{K}_x$ of the same dimension must be an equality. □

We recall that the action of K on X is free if all stabilisers K_x are trivial. We say an action is locally free at x if the stabiliser K_x is finite.

Corollary 9.2. *Suppose we have a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{K}^*$. Then:*

i) *The action is locally free at $x \in X$ if and only if $\mathfrak{K}_x = 0$, or equivalently, if and only if $d\mu_x$ is onto i.e. x is a regular point of μ.*
ii) *Let η be an element in $\mathfrak{K}^*$ which is fixed by the coadjoint action of K. Then at every point of $\mu^{-1}(\eta)$ the K-action is locally free if and only if η is a regular value of μ.*
iii) *If η is a regular value of μ, then $\mu^{-1}(\eta) \subset X$ is a closed submanifold of codimension equal to the dimension of $\mathfrak{K}$ and $T_x\mu^{-1}(\eta) = \ker d\mu_x$ for all $x \in \mu^{-1}(\eta)$. Moreover, $T_x\mu^{-1}(\eta)$ and $T_x(K \cdot x)$ are orthogonal with respect to the symplectic form ω_x on T_xX.*

Proof. i) The stabiliser K_x of a point x is finite if and only if its Lie algebra $\mathfrak{K}_x = 0$ is zero. By Lemma 9.1 ii), we have that $\mathrm{Im} d_x\mu = \mathrm{Ann}\mathfrak{K}_x$ which is equal to $\mathfrak{K}^*$ if and only if $\mathfrak{K}_x = 0$. Hence the action is locally free at x if and only if $d_x\mu$ is surjective i.e. x is a regular point of μ. Then ii) follows from i) and the definition of regular value. For iii), we use the preimage theorem for smooth manifolds: if $\mu : X \to \mathfrak{K}^*$ is a smooth map of smooth manifolds, then the preimage of a regular value is a closed submanifold of dimension $\dim X - \dim \mathfrak{K}^*$. As $\mu|_{\mu^{-1}(\eta)} = \eta$ is constant, $d_x\mu = 0$ on $T_x\mu^{-1}(\eta)$ for all $x \in \mu^{-1}(\eta)$. Therefore $T_x\mu^{-1}(\eta) \subset \ker d_x\mu$. Since η is a regular value, $d_x\mu$ is surjective and so

$$\dim \ker d_x\mu = \dim T_xX - \dim \mathfrak{K}^* = \dim X - \dim \mathfrak{K}^* = \dim \mu^{-1}(\eta);$$

thus $T_x\mu^{-1}(\eta) = \ker d_x\mu$. The final statement of iii) follows from Lemma 9.1. □

9.2. **Symplectic reduction.** We suppose as above that we have a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with associated moment map $\mu : X \to \mathfrak{K}^*$. We want to construct a quotient of the K-action on X (or a submanifold of X on which the action is free) in the symplectic category.

Given $\eta \in \mathfrak{K}^*$ which is fixed by the coadjoint action, equivariance of μ implies that the preimage $\mu^{-1}(\eta)$ is preserved by the action of K. We define the symplectic reduction at η to be the orbit space

$$X_\eta^{\mathrm{red}} := \mu^{-1}(\eta)/K.$$

This orbit space was considered by Marsden and Weinstein [26] and Meyer [28] as a possible symplectic quotient. In general this quotient is just a topological space: its topology is the weakest topology for which the quotient map $\mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is continuous. If η is a regular value of μ, then the preimage $\mu^{-1}(\eta)$ is a submanifold of X of dimension equal to $\dim X - \dim(K)$. However, the action of K on $\mu^{-1}(\eta)$ may not be free and hence the symplectic reduction will be an orbifold rather than a manifold (recall that if η is a regular value we know the action of K is locally free at each point and so the stabilisers are all finite). If η is a regular value and the action of K on $\mu^{-1}(\eta)$ is free and proper, then the symplectic reduction is a manifold of dimension $\dim X - 2\dim K$. In this situation, we shall shortly see that there is a natural symplectic form on $\mu^{-1}(\eta)/K$ (this is a theorem of Marsden and Weinstein [26] and Meyer [28]).

Remark 9.3. If η is a regular value of μ, but is not necessarily fixed by the coadjoint action, then we can instead consider the symplectic reduction

$$X_\eta^{\mathrm{red}} = \mu^{-1}(\eta)/K_\eta$$

where $K_\eta = \{k \in K : \mathrm{Ad}^* k\eta = \eta\}$ is the stabiliser group of η for the coadjoint action.

There is one point of particular interest which is always fixed by the coadjoint action, namely the origin $0 \in \mathfrak{K}^*$ and so we may consider the symplectic reduction

$$X_0^{\mathrm{red}} := \mu^{-1}(0)/K.$$

We shall often write simply X^{red} for the reduction at 0 and refer to this as the symplectic reduction. If 0 is a regular value of μ and the action of K on $\mu^{-1}(0)$ is free and proper, then this is a manifold whose dimension is $\dim X - 2\dim(K)$.

9.3. **Marsden-Weinstein-Meyer Theorem.**

Theorem 9.4. *Given a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{K}^*$ and a regular value $\eta \in \mathfrak{K}^*$ of the moment map μ which is fixed by the coadjoint action. If the action of K on $\mu^{-1}(\eta)$ is free and proper, then*

i) *The symplectic reduction $X_\eta^{\mathrm{red}} = \mu^{-1}(\eta)/K$ is a smooth manifold of dimension* $\dim X - 2\dim K$.
ii) *There is a unique symplectic form ω^{red} on X_η^{red} such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$ where $i : \mu^{-1}(\eta) \hookrightarrow X$ denotes the inclusion and $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ the quotient map.*

Remark 9.5. The assumption that the action of K on $\mu^{-1}(\eta)$ is free and proper is only required so that the quotient is a manifold. In Section 10, we will consider symplectic quotients constructed by actions of compact Lie groups in which case the action is always proper. Of course, if η is a regular value, we already know that the action is locally free and so the topological quotient is at least an orbifold.

Before we prove this result we need a few lemmas:

Lemma 9.6. *With the assumptions of the above theorem, for all $x \in \mu^{-1}(\eta)$, the subspace $T_x(K \cdot x)$ of T_xX is an isotropic subspace.*

Proof. We recall that $T_x(K\cdot x)$ is an isotropic subspace of the symplectic vector space (T_xX, ω_x) if $\omega|_{T_x(K\cdot x)} \equiv 0$ or, equivalently, if $T_x(K\cdot x) \subset T_x(K\cdot x)^{\omega_x}$. The subspaces $\ker d_x\mu = T_x\mu^{-1}(\eta)$ and $T_x(K\cdot x)$ of T_xX are symplectic orthogonal complements with respect to ω_x for $x \in \mu^{-1}(\eta)$ by Corollary 9.2. As η is fixed by the coadjoint action, this implies $\mu^{-1}(\eta)$ is K-invariant and so $K \cdot x \subset \mu^{-1}(\eta)$. Therefore

$$T_x(K\cdot x) \subset T_x\mu^{-1}(\eta) = T_x(K\cdot x)^{\omega_x}$$

which completes the proof that $T_x(K\cdot x)$ is an isotropic subspace of (T_xX, ω_x). □

Lemma 9.7. *Let I be an isotropic subspace of a symplectic vector space (V, ω). Then ω induces a unique symplectic form ω' on the quotient I^ω/I.*

Proof. We define
$$\omega'([v],[w]) = \omega(v,w)$$
and check this definition is well defined:
$$\begin{aligned}\omega'(v+i, w+j) &= \omega(v,w) + \omega(i,w) + \omega(v,j) + \omega(i,j)\\ &= \omega(v,w) + 0 + 0 + 0\end{aligned}$$
for $i, j \in I$. The non-degeneracy of ω' follows from that of ω: if $[u] \in I^\omega/I$ and $\omega'([u],[v]) = 0$ for all $v \in I^\omega/I$, then $\omega(u,v) = 0$ for all $v \in I^\omega$ and so $u \in (I^\omega)^\omega = I$ i.e. $[u] = 0$. □

Proof. (Marsden-Weinstein-Meyer Theorem) The preimage theorem shows that $\mu^{-1}(\eta)$ is a closed smooth submanifold of X of dimension $\dim X - \dim K$. Furthermore, as K acts on $\mu^{-1}(\eta)$ freely and properly, the quotient X_η^{red} is a smooth manifold of dimension $\dim X - 2\dim K$. We shall construct a non-degenerate 2-form ω^{red} on X_η^{red} such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$, by constructing symmetric forms ω_p^{red} on $T_pX_\eta^{\mathrm{red}}$ for all $p \in X_\eta^{\mathrm{red}}$. Let $p = \pi(x)$ where $\pi : \mu^{-1}(\eta) \to X_\eta^{\mathrm{red}}$; then we have a short exact sequence of vector spaces
$$0 \to T_x(K\cdot x) \to T_x\mu^{-1}(\eta) \to T_pX_\eta^{\mathrm{red}} \to 0.$$
By Lemma 9.6, the subspace $T_x(K \cdot x)$ is isotropic whose symplectic orthogonal complement (with respect to ω_x) is $T_x(\mu^{-1}(\eta))$. It then follows from Lemma 9.7 that there is a canonical symplectic form ω_p^{red} on
$$T_x(K\cdot x)^\omega/T_x(K\cdot x) = T_x\mu^{-1}(\eta)/T_x(K\cdot x) \cong T_pX_\eta^{\mathrm{red}}.$$
By construction this is a non-degenerate 2-form such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$ and so it remains to check that this symplectic form is closed. As the exterior derivative d commutes with pullback we have that
$$\pi^* d\omega^{\mathrm{red}} = d\pi^*\omega^{\mathrm{red}} = di^*\omega = i^*d\omega = 0.$$
The pullback map on 3-forms
$$\pi^* : \Omega^3(X_\eta^{\mathrm{red}}) \to \Omega^3(\mu^{-1}(\eta))$$
is injective as π is surjective and hence $d\omega^{\mathrm{red}} = 0$. □

Example 9.8. Consider the action of $\mathrm{U}(1) \cong S^1$ on $\mathbb{C}^n$ by multiplication $s \cdot (a_1, \dots, a_n) = (sa_1, \dots, sa_n)$. We can take the standard symplectic form on $\mathbb{C}^n$ and use the Killing form on $\mathfrak{u}(1)$ to identify $\mathfrak{u}(1)^* \cong \mathfrak{u}(1) \cong \mathbb{R}$ and write the moment map for this action as
$$\mu(x_1, \dots, x_n) = \frac{1}{2}\left(\sum_{k=1}^n |x_k|^2 - C\right)$$
where C is any real number. If we take $C = 1$, then
$$\mu^{-1}(0) = S^{2n-1} = \{(x_1, \dots, x_n) : \sum |x_k|^2 = 1\}$$
and the symplectic reduction is $\mu^{-1}(0)/S^1 = S^{2n-1}/S^1 = \mathbb{P}^{n-1}$. The canonical symplectic form ω induced from the standard symplectic form on $\mathbb{C}^n$ (by which we mean the imaginary part of the standard Hermitian inner product H on $\mathbb{C}^n$) is then the Fubini–Study form that we constructed in Example 7.15. In fact, if you revisit our construction of the Fubini–Study form given in Example 7.15, then you will now see that it was actually constructed using the Marsden–Weinstein–Meyer theorem.

Example 9.9. Consider the action of $K = U(m)$ on the space of $l \times m$-matrices over the complex numbers $M_{l\times m}(\mathbb{C}) \cong \mathbb{C}^{lm}$ as in Example 8.9 where $l > m$. We recall that the moment map is given by
$$\mu(M)\cdot A = \frac{i}{2}\mathrm{Tr}(MAM^*)$$

for $M \in M_{l\times m}$ and $A \in \mathfrak{u}(m)$. If we use the Killing form on $\mathfrak{u}(m)$ then we can identify $\mathfrak{u}(m)^* \cong \mathfrak{u}(m)$ and view the moment map as a morphism $\mu : M_{l\times m} \to \mathfrak{u}(m)$ given by

$$\mu(M) = \frac{i}{2} M^* M.$$

Let $\eta = iI_m/2$ denote the skew-Hermitian matrix which is an (imaginary) scalar multiple of the identity matrix I_m; then clearly η is fixed by the adjoint action of $U(m)$ on $\mathfrak{u}(m)$. The preimage $\mu^{-1}(\eta) = \{M \in M_{l\times m} : M^*M = I_m\}$ consists of $l \times m$ matrices whose m columns are linearly independent and define a length m unitary frame of $\mathbb{C}^l$. The symplectic reduction $\mu^{-1}(\eta)/U(m)$ is the grassmannian $\mathrm{Gr}(m, l)$ of m-planes in $\mathbb{C}^l$.

There is a more general version of the Marsden-Weinstein-Meyer Theorem which allows us to take reductions at points which are not fixed by the coadjoint action:

Proposition 9.10. *Given a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{K}^*$ and an orbit $\mathcal{O}$ for the coadjoint action of K on $\mathfrak{K}^*$. If the orbit consists of regular values of μ and the action of K on $\mu^{-1}(\mathcal{O})$ is free and proper, then the symplectic reduction $X^{\mathrm{red}}_{\mathcal{O}} = \mu^{-1}(\mathcal{O})/K$ is a symplectic manifold of dimension* $\dim X + \dim \mathcal{O} - 2\dim K$.

Proof. The assumption that every point of $\mathcal{O}$ is a regular value of the moment map means that the preimage $\mu^{-1}(\mathcal{O})$ is a closed submanifold of X of dimension $\dim X + \dim \mathcal{O} - \dim K$. Recall from Example 7.17, that the coadjoint orbit has a natural symplectic form which we denote by $\omega_{\mathcal{O}}$. Consider the natural action of K on the product $(X \times \mathcal{O}, -\omega \boxplus \omega_{\mathcal{O}})$, for which the moment map $\mu' : X \times \mathcal{O} \to \mathfrak{K}^*$ is given by

$$\mu'(x, \eta) = -\mu(x) + \eta.$$

The proposition follows by applying the original version of the Marsden-Weinstein-Meyer Theorem to the regular value 0 of μ' and the fact that $\mu^{-1}(\mathcal{O}) \cong (\mu')^{-1}(0)$. $\square$

Remark 9.11. If $\eta \in \mathfrak{K}^*$ is not fixed by the coadjoint action of K on $\mathfrak{K}^*$, then

$$K_\eta = \{k \in K : Ad^*_k \eta = \eta\}$$

acts on $\mu^{-1}(\eta)$. Then the symplectic reduction $X^{\mathrm{red}}_{\mathcal{O}_\eta}$ constructed above for the coadjoint orbit $\mathcal{O}_\eta$ of η is equal to the quotient $\mu^{-1}(\eta)/K_\eta$.

Remark 9.12. Suppose X is a Kähler manifold (i.e. it has a complex structure which is compatible with the symplectic structure) and the action of the Lie group K preserves this complex structure (as well as preserving the symplectic structure). If the action is Hamiltonian with moment map μ and K acts freely and properly on $\mu^{-1}(0)$ where 0 is a regular value of the moment map, then the symplectic reduction $\mu^{-1}(0)/X$ also has a Kähler structure (i.e. the almost complex structure induced on the quotient is integrable and compatible with the induced symplectic form).

9.4. **Universality of the symplectic reduction.** We recall that the notion of morphism in the symplectic category is given by chains of Lagrangian correspondences (see Definition 7.22 and [49]). Given a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{K}^*$, the aim of this section is to show (under the assumptions of the Marsden-Weinstein-Meyer theorem) that the symplectic reduction $(X^{\mathrm{red}}_0 = \mu^{-1}(0)/K, \omega^{\mathrm{red}})$ is a universal quotient of the K-action on (X, ω) in the symplectic category. We recall that the symplectic form ω^{red} is the unique symplectic form such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$ where $i : \mu^{-1}(0) \hookrightarrow X$ denotes the inclusion and $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ the quotient map. First of all we need to construct a morphism between (X, ω) and $(X^{\mathrm{red}}_0, \omega^{\mathrm{red}})$.

Lemma 9.13. *Under the set up of the Marsden-Weinstein-Meyer Theorem with regular value $\eta = 0$, if L_μ denotes the image of the map $i \times \pi : \mu^{-1}(0) \to X \times \mu^{-1}(0)/K$, then L_μ is a Lagrangian submanifold of $(X \times \mu^{-1}(0)/K, -\omega \boxplus \omega^{red})$.*

Proof. As L_μ is diffeomorphic to $\mu^{-1}(0)$ we have

$$\dim L_\mu = \dim X - \dim K = \frac{1}{2}\dim X \times \mu^{-1}(0)/K.$$

If $j : L_\mu \hookrightarrow X \times \mu^{-1}(0)/K$, then $j^*\omega \equiv 0$ if and only if $-i^*\omega + \pi^*\omega^{\mathrm{red}} = 0$ which holds by the Marsden-Weinstein-Meyer theorem. □

In particular, this lemma gives a morphism in the symplectic category (X, ω) and $(X_0^{\mathrm{red}}, \omega^{\mathrm{red}})$. However, we want this morphism to be K-equivariant and so we should define what it means for a Lagrangian correspondence to be K-invariant (or in fact more generally K-equivariant):

Definition 9.14. A Lagrangian correspondence between symplectic manifold with Hamiltonian K-actions which is given by a Lagrangian submanifold $L \subset (X_1 \times X_2, -\omega_1 \boxplus \omega_2)$ is said to be K-equivariant if L is a K-invariant submanifold and if $\mu_{12}(L) = 0$ where μ_{12} denotes the moment map for the K-action on $(X_1 \times X_2, -\omega_1 \boxplus \omega_2)$. A K-invariant Lagrangian correspondence between a symplectic manifold (X_1, ω_1) with Hamiltonian K-action and a symplectic manifold (X_2, ω_2) is then a K-equivariant Lagrangian correspondence when we give (X_2, ω_2) the trivial K-action.

It is immediate that:

Lemma 9.15. *The Lagrangian correspondence defined by L_μ in Lemma 9.13 is K-invariant.*

Proposition 9.16. *Given a Hamiltonian action of a Lie group K on a symplectic manifold (X, ω) for which the assumptions of the Marsden-Weinstein-Meyer Theorem hold for $\eta = 0$, we let L_μ denote the K-invariant Lagrangian correspondence between (X, ω) and $(X_0^{\mathrm{red}}, \omega^{red})$ given above. Then every other K-invariant symplectic morphism from (X, ω) to a symplectic manifold (Y, ω') with trivial K-action factors through the morphism corresponding to the K-invariant Lagrangian L_μ.*

Proof. It suffices to prove the result when the morphism from (X, ω) to (Y, ω') is given by a single K-invariant Lagrangian correspondence i.e. a Lagrangian submanifold $L' \subset (X \times Y, -\omega \boxplus \omega')$ which is preserved by the action of K and on which $\mu_{XY} : X \times Y \to \mathfrak{K}^*$ is zero. As the K-action on Y is trivial, so is the moment map μ_Y and so μ_{XY} is the projection $X \times Y$ followed by $\mu = \mu_X : X \to \mathfrak{K}^*$. In particular $L' \subset \mu^{-1}(0) \times Y$. To show L' factors through L_μ, it suffices to produce a Lagrangian correspondence L'' between $\mu^{-1}(0)/K$ and Y such that $L' = L'' \circ L_\mu$. One checks that $L'' := L'/K \subset \mu{-}1(0)/K \times Y$ is the Lagrangian submanifold we require for this correspondence. □

10. Kempf-Ness theorem

If K is a real compact Lie group, we recall that its complexification $G := K_{\mathbb{C}}$ is a complex Lie group which contains K and the Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{K}$ of K ($\mathfrak{g} = \mathfrak{K} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{K} \oplus i\mathfrak{K}$). We shall make use of the following standard result about complex reductive groups (for a proof see, for example, [38] Theorem 2.7):

Theorem 10.1. *The operation of complexification defines a one-to-one correspondence between isomorphism classes of compact real Lie groups and complex reductive groups.*

Example 10.2. We list a few examples of pairs (G, K) consisting of a complex reductive group G and its maximal compact torus: $((\mathbb{C}^*)^r, (S^1)^r)$, $(\mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n))$ and $(\mathrm{SL}(n, \mathbb{C}), \mathrm{SU}(n))$.

Suppose G is a complex reductive group acting on a smooth complex projective variety $X \subset \mathbb{P}^n$ via a representation $\rho : G \to \mathrm{GL}_{n+1}(\mathbb{C})$. Let K be the maximal compact subgroup of G, so that G is the complexification of K. Then K also acts on $\mathbb{P}^n$ and as K is compact, we can choose coordinates on $\mathbb{P}^n$ so ρ restricts to a unitary representation $\rho : K \to U(n+1)$ of K; that is, the associated Fubini–Study form ω for this choice of coordinates is the preserved by the action of K. Therefore the action of K on both X and $\mathbb{P}^n$ is symplectic. In this case we know that the action is Hamiltonian and we can explicitly write down a moment map $\mu : X \to \mathfrak{K}^*$ (by restricting to X the $U(n)$-moment map constructed on $\mathbb{P}^n$ in Example 8.10 and then composing with the projection $\rho^* : \mathfrak{u}(n)^* \to \mathfrak{K}^*$). In nice cases (cf. the Marsden–Weinstein–Meyer theorem), the symplectic reduction $\mu^{-1}(0)/K$ at $0 \in \mathfrak{K}^*$ is a symplectic manifold of dimension $\dim X - 2\dim K = \dim X - \dim G$.

Alternatively, one can consider the geometric invariant theory (GIT) quotient $X//G$ which in nice cases is an orbit space for the G-action on a (Zariski) open subset of X and has dimension $\dim X - \dim G$. In this section our aim is to compare the symplectic reduction with the GIT quotient and in particular to prove the Kempf–Ness Theorem [22] below (see also [31] §8, [48] and [50]). We recall that a semistable point is polystable if and only if its orbit is closed in X^{ss} and we denote the polystable locus by X^{ps}. We have inclusions $X^s \subset X^{ps} \subset X^{ss}$ and in nice cases all three subsets coincide and the GIT quotient is a geometric quotient of the G-action on $X^{ss} = X^s$.

Theorem 10.3 (Kempf–Ness theorem). *Let $G = K_{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ and suppose its maximal compact subgroup K is connected and acts symplectically on X (where the restriction of the Fubini–Study form on $\mathbb{P}^n$ is used to give X its symplectic structure). Let $\mu : X \to \mathfrak{K}^*$ denote the associated moment map; then:*

i) $G\mu^{-1}(0) = X^{ps}$.
ii) *If $x \in X$ is polystable, then its orbit $G \cdot x$ meets $\mu^{-1}(0)$ in a single K-orbit.*
iii) *$x \in X$ is semistable if and only if its orbit closure $\overline{G \cdot x}$ meets $\mu^{-1}(0)$.*

Before giving the proof we give a corollary:

Corollary 10.4. *The inclusion $\mu^{-1}(0) \subset X^{ss}$ induces a homeomorphism*

$$\mu^{-1}(0)/K \to X//G.$$

Proof. We first show as sets these are isomorphic. The GIT quotient $X//G$ as a set is the semistable set X^{ss} modulo S-equivalence where x_1 and x_2 are S-equivalent if and only if their orbit closures meet in X^{ss} (cf. Corollary 4.30). By Lemma 4.16, the closure of every semistable orbit contains a unique polystable orbit, and so the GIT quotient as a set is isomorphic to the set of G-orbits in X^{ps}. By part i) of the Kempf-Ness theorem, every polystable orbit meets the level set $\mu^{-1}(0)$ in a unique K-orbit and so we get the required bijection of sets. As $\mu^{-1}(0)$ is a closed subset of a compact space, it is compact and so is the symplectic reduction $\mu^{-1}(0)/K$. The inclusion $\mu^{-1}(0) \subset X^{ss}$ induces a continuous bijection from a compact space to a Hausdorff space and so is a homeomorphism. □

Remark 10.5. In fact one can construct a continuous inverse to the map $\mu^{-1}(0)/K \to X//G$ by using the gradient flow of the norm square of the moment map. For more details on the norm square of the moment map and the Morse-type stratifications induced by the gradient flow see [25] and [35].

Example 10.6. Consider the linear action of $G = \mathbb{C}^*$ on complex projective space $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n]$$

as in Example 4.13. In Example 4.13 we saw that the semistable locus and stable locus coincide and are given by

$$X^s = X^{ss} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \ldots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

and the GIT quotient

$$\varphi : X^{ss} = \mathbb{A}^n - \{0\} \dashrightarrow X//G \cong \mathbb{P}^{n-1}$$

is a geometric quotient. As G is a complex reductive group it is equal to the complexification of its maximal compact subgroup $K = U(1) \cong S^1$. We use the Fubini–Study form to consider $X = \mathbb{P}^n$ as a symplectic manifold (X, ω) as in Example 7.15. The action is Hamiltonian with moment map $\mu : X \to \mathfrak{K}^*$ formed from the natural representation $U(1) \to U(n+1)$ associated to this action and the standard moment map for the action of $U(n+1)$ given in Example 8.10. By using the Killing form to identify $\mathfrak{K}^* \cong \mathfrak{K}$ and using the natural identification $\mathfrak{K} = \mathrm{Lie} S^1 \cong 2\pi i\mathbb{R}$, we can instead consider the induced map $\tilde{\mu} : X \to 2\pi i\mathbb{R}$ given by

$$\tilde{\mu}([x_0 : \cdots : x_n]) = \frac{-|x_0|^2 + |x_1|^2 + \cdots |x_n|^2}{2i \sum_{i=0}^n |x_i|}.$$

Then $\mu^{-1}(0) = \tilde{\mu}^{-1}(0)$ which is equal to

$$\{[x_0 : \cdots : x_n] : |x_0| = \sum_{i=1}^n |x_i|\} \cong \{(x_1, \ldots, x_n) \in \mathbb{C}^n \cong \mathbb{P}^n_{x_0 \neq 0} : \sum_{i=1}^n |x_i|^2 = 1\} \cong S^{2n-1}.$$

In particular we see that $\mu^{-1}(0) \subset X^{ss}$. The symplectic reduction

$$\mu^{-1}(0)/K \cong S^{2n-1}/S^1 \cong \mathbb{P}^{n-1} \cong X//G$$

agrees with the GIT quotient for the action of the complexified group G.

10.1. **The proof of the Kempf-Ness theorem.** For the proof of the theorem, we shall assume for simplicity that $X = \mathbb{P}^n$ as the version for general X follows from the version for $\mathbb{P}^n$. We let $H : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ be the K-invariant Hermitian inner product on $\mathbb{C}^{n+1}$ such that the Fubini–Study form ω on $\mathbb{P}^n$ is constructed via the projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ from the imaginary part of H; that is, if $v \in \mathbb{C}^{n+1} - \{0\}$ and we identify $T_{[v]}\mathbb{P}^n$ with the orthogonal space to v in $\mathbb{C}^{n+1}$, then

$$\omega_{\mathrm{FS},[v]}(u, w) = \frac{\omega_{\mathbb{C}^{n+1}}(u, w)}{||v||^2} = \frac{1}{2i||v||^2}[H(u, w) - H(w, u)].$$

Then the moment map $\mu : \mathbb{P}^n \to \mathfrak{K}^*$ is given by

$$\mu([v]) \cdot A = \frac{\mathrm{Tr}\overline{v}^t A v}{2i||v||^2} = \frac{1}{2}\omega_{\mathrm{FS},[v]}(Av, v).$$

We let $|| - ||$ denote the norm associated to the Hermitian inner product. Then as in [22], we consider the non-negative function

$$p_v : G \to \mathbb{R}$$
$$g \mapsto ||g \cdot v||^2.$$

We recall that a point $g \in G$ is a critical point of p_v if the derivative $d_g p_v$ of p_v at g is zero. The relationship between this function and the moment map is given by the following lemma.

Lemma 10.7. *Let $v \in \mathbb{C}^{n+1} - \{0\}$ and $p_v : G \to \mathbb{R}$ be as above; then*

i) *p_v is constant on K.*
ii) *Let e denote the identity of G; then $p_v(g) = p_{g\cdot v}(e)$ and so $d_g p_v = d_e p_{g\cdot v} : \mathfrak{g} \to \mathbb{R}$.*
iii) *If $A \in \mathfrak{K}$, then $d_e p_v(iA) = -4||v||^2 \mu([v]) \cdot A$ and so $d_e p_v = 0$ (that is, e is a critical point of p_v) if and only if $\mu([v]) = 0$.*
iv) *Moreover, $g \in G$ is a critical point of p_v if and only if $\mu(g \cdot [v]) = 0$.*
v) *The second derivatives of p_v are non-negative and so p_v is convex. In particular every critical point of p_v is a minimum.*

Proof. i) holds as the Hermitian inner product is K-invariant and ii) is immediate. As $\mathfrak{g} = \mathfrak{K} \oplus i\mathfrak{K}$ and p_v is constant on K, to study the first order derivatives of p_v we need only calculate $d_e p_v(iA)$ for $A \in \mathfrak{K}$. In this case we have

$$\begin{aligned}
d_e p_v(iA) &:= \frac{d}{dt}||\exp(itA) \cdot v||^2|_{t=0} \\
&= H(\frac{d}{dt}\exp(itA) \cdot v|_{t=0}, v) + H(v, \frac{d}{dt}\exp(itA) \cdot v|_{t=0}) \\
&= H(iAv, v) + H(v, iAv) = i[H(Av, v) - H(v, Av)] \\
&\overset{(5)}{=} 2iH(Av, v) \\
&= -4||v||^2 \mu([v]) \cdot A.
\end{aligned}$$

Therefore e is a critical point of p_v if and only if $\mu([v]) = 0$ and more generally g is a critical point of p_v if and only if e is a critical point of $p_{g\cdot v}$ which is if and only if $\mu(g \cdot [v]) = 0$. For v), the second derivative of p_v at e is given by:

$$\begin{aligned}\frac{d^2}{dt^2}p_v(\exp(itA))|_{t=0} &= \frac{d}{dt}H(iA\exp(itA)\cdot v, \exp(itA)\cdot v) + H(\exp(itA)\cdot v, iA\exp(itA)\cdot v)|_{t=0} \\ &= H((iA)^2v, v) + H(iAv, iAv) + H(iAv, iAv) + H(v, (iA)^2 v) \\ &= 2H(Av, Av) - [H(A^2v, v) + H(v, A^2v)] \\ &\overset{(5)}{=} 2H(Av, Av) \geq 0.\end{aligned}$$

It follows from this calculation and part ii) that all second derivatives of p_v are non-negative. Hence p_v is convex and every critical point is a minimum. $\square$

Lemma 10.8. *The norm function $||-||^2 : \mathbb{C}^{n+1} \to \mathbb{R}$ is proper. Furthermore, if $G\cdot v$ is a closed orbit where $v \in \mathbb{C}^{n+1} - \{0\}$, then p_v achieves its minimum at some $g \in G$ and $\mu([g\cdot v]) = 0$.*

Proof. The norm is continuous and the preimage of a bounded set is bounded so it is proper. If $G\cdot v$ is a closed orbit, then $||G\cdot v||^2$ is also closed as the norm is a proper function on a manifold. If $c = \inf_g p_v(g)$, then

$$c \in \overline{||G\cdot v||^2} = ||G \cdot v||^2$$

and so there is $g \in G$ for which $||g\cdot v||^2 = c$. Finally by Lemma 10.7 above, this minimum g is a critical point of p_v and so $\mu(g\cdot[v]) = 0$. $\square$

Remark 10.9. As G is the complexification of K, every element $g \in G$ has a Cartan decomposition $g = k\exp(iA)$ for $k \in K$ and $A \in \mathfrak{K}$ (for example every invertible matrix $M \in \mathrm{GL}_n(\mathbb{C})$ has a decomposition $A = UH$ as a product of a unitary matrix U and a Hermitian matrix H).

Lemma 10.10. *Let $v \in \mathbb{C}^{n+1} - \{0\}$. If $G\cdot v$ is not closed, then $p_v : G \to \mathbb{R}$ does not attain a minimum.*

Proof. Firstly, if the orbit closure of v in $\mathbb{C}^{n+1}$ contains zero then $0 = \inf_g p_v(g)$ and p_v does not attain its minimum. Hence we suppose the orbit closure does not contain zero. Since $p_v(g) = p_{g\cdot v}(e)$ and the action of $U(n+1)$ on $\mathbb{P}^n$ is transitive, it suffices to show that e is not a minimum of p_v. As the orbit is not closed, there is a nonzero point $u \in \overline{G\cdot v} - G\cdot v$. It is a fundamental result of GIT (see Theorem 5.10) that there is a 1-PS $\lambda(t) : \mathbb{C}^* \to G$ such that $u = \lim_{t\to 0}\lambda(t)\cdot v$ and moreover we can assume (by conjugating λ) that $\lambda(S^1) \subset K$.

We can diagonalise the action of λ to get a decomposition

$$V := \mathbb{C}^{n+1} = \oplus_{r\in\mathbb{Z}} V_r$$

into weight spaces $V_r := \{v \in V : \lambda(t)\cdot v = t^r v\}$ and this decomposition is orthogonal with respect to the Hermitian inner product H on $V = \mathbb{C}^{n+1}$ by the assumption that the image of S^1 under λ is contained in K. We can write $v = \sum_r v_r$ with respect to the above orthogonal decomposition and the assumption that $\lim_{t\to 0}\lambda(t)\cdot v$ exists means that $v_r = 0$ for $r < 0$. If $v_r = 0$ for all $r > 0$, then $\lim_{t\to 0}\lambda(t)\cdot v = v \in G\cdot v$. Hence, there is at least one $r > 0$ for which $v_r \neq 0$. Let

$$A = \frac{d}{dt}\lambda(\exp(2\pi it))|_{t=0} \in \mathfrak{K};$$

then the infinitesimal action is given by $Av_r = 2\pi i r v_r$ and

$$\begin{aligned}\mu([v])\cdot A = \frac{H(Av, v)}{2i||v||^2} &= \frac{1}{2i||v||^2}\sum_{r,s} H(2\pi i r v_r, v_s) \\ &= \frac{\pi}{||v||^2}\sum_{r\geq 0} rH(v_r, v_r) > 0.\end{aligned}$$

By part ii) of Lemma 10.7, we conclude that e is not a critical value of p_v and p_v does not obtain a minimum at e. $\square$

Proof. (Kempf-Ness Theorem) i) If $x \in \mathbb{P}^n$, then we let $v \in \mathbb{C}^{n+1} - \{0\}$ be a point lying over x. We recall that x is semistable if the origin is not contained in the closure of $G \cdot v$ and x is polystable if $G \cdot v$ is closed. By Lemma 10.7, $g \cdot x \in \mu^{-1}(0)$ if and only if p_v attains a minimum at g and this is if and only if $G \cdot v$ is closed by Lemmas 10.8 and 10.10); that is x is polystable.

ii) Clearly if x is polystable and $\mu(x) = 0$, then $\mu(K \cdot x) = 0$ by equivariance of μ. If x and y are both points in the same polystable G-orbit for which $\mu(x) = \mu(y) = 0$, then for ii) it remains to show that x and y belong to the same K-orbit. We can write $x = gy$ and let $v, u \in \mathbb{C}^{n+1}$ be points such that $[v] = x$ and $[u] = y$; then p_u and p_v both attain their minimum at e and $p_u(e) = p_v(e) = p_u(g)$. We use a Cartan decomposition and write $g = k \exp(iA)$. As the Hermitian inner product is K-invariant, we have

$$p_u(g) = ||g \cdot u||^2 = ||\exp(iA) \cdot u||^2 = p_u(\exp(iA)).$$

By Lemma 10.7 iii), p_u is convex and so $p_u(\exp(itA)) \leq p_u(\exp(iA)) = p_u(e)$ for $t \in [0, 1]$. But $p_u(e)$ is the unique global minimum of the convex function p_u and so this must be an equality; that is, p_u is not strictly convex along $\exp(itA)$ or, equivalently,

$$0 = \frac{d^2}{dt^2} p_u(\exp(itA))|_{t=0} = \frac{\pi}{||u||^2} H(Au, Au).$$

Therefore the infinitesimal action $A_y = Au$ is zero and $iA \in i\mathfrak{K}_y \subset \mathfrak{g}_y = \operatorname{Lie} G_y$. In particular x and y belong to the same K-orbit:

$$x = g \cdot y = k \exp(iA) \cdot y = k \cdot y.$$

For iii), we note that for every semistable orbit $G \cdot x$, there is a unique polystable orbit $G \cdot y$ in $\overline{G \cdot x}$ which is closed in X^{ss} by Lemma 4.16. By part i) of the Kempf-Ness theorem, this polystable orbit meets $\mu^{-1}(0)$. Conversely, if $\overline{G \cdot x}$ meets the level set $\mu^{-1}(0)$ in a point y, then $G \cdot y$ is polystable by i). Hence x is also semistable by the openness of the GIT semistable set $X^{ss} \subset X$. □

Remark 10.11. Many gauge theoretic moduli spaces also have an algebraic description; for example, the gauge theoretic moduli space of flat unitary connections on a rank n degree zero vector bundle on a compact Riemann surface can be seen as the moduli space of semistable algebraic rank n degree zero vector bundles on the associated smooth projective complex algebraic curve by a theorem of Narasimhan–Seshadri and Donaldson. The gauge theoretic constructions often arise as (infinite dimensional) symplectic reductions, whereas the algebraic description appear as a GIT quotients. In this case, we can view such correspondences as infinite dimensional analogues of the Kempf–Ness theorem. For a brief overview of this area, see [31] §8 and for further details, see [1].

Corollary 10.12. *The origin is a regular value of μ if and only if $X^{ss} = X^s$.*

Proof. The origin is not a regular value of μ if and only if there is $x \in \mu^{-1}(0)$ for which $d_x\mu : T_xX \to \mathfrak{K}^*$ is not surjective. The derivative $d_x\mu : T_xX \to \mathfrak{K}^*$ is not surjective if and only if there is nonzero $A \in \mathfrak{K}$ such that $0 = d_x\mu \cdot A = \omega_x(A_x, -)$ and this is if and only if $A_x = 0$; that is, $A \in \mathfrak{K}_x$. Hence the origin is not a regular value of μ if and only if $\exp(tA) \subset G_x$ for some $x \in \mu^{-1}(0)$ and nonzero $A \in \mathfrak{K}$.

If 0 is not a regular value of the moment map μ, then by above there is $x \in \mu^{-1}(0)$ with positive dimensional stabiliser. By the Kempf-Ness theorem x is polystable, but x is not stable as its stabiliser is positive dimensional.

Conversely, if x is semistable but not stable, then we can assume without loss of generality that x is polystable but not stable (cf. Lemma 4.16). In this case x has positive dimensional stabiliser. By the Kempf-Ness theorem, we can also assume without loss of generality that $x \in \mu^{-1}(0)$. As x has positive dimensional stabiliser, the argument above shows 0 is not a regular value of the moment map. □

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