6. Criteria for (semi)stability

Let us suppose that we have a reductive group G acting on a projective scheme X with respect to an ample linearisation L. In order to determine the GIT semistable locus $X^{ss}(L) \subset X$, we need to calculate the algebra of G-invariant sections of all powers of L. In practice, there are very few examples in which one can compute these rings of invariants by hand (or even with the aid of a computer). In this section, we will give alternative criteria for determining the semistability of a point. The main references for the material covered in this section are [4], [25], [31] and [42].

We first observe that we can simplify our situation by assuming that $X \subset \mathbb{P}^n$ and the G-action is linear. Indeed, by replacing L by some power $L^{\otimes r}$, we get an embedding

$$X \subset \mathbb{P}^n = \mathbb{P}(H^0(X, L^{\otimes r})^*)$$

such that $\mathcal{O}_{\mathbb{P}^n}(1)|_X = L^{\otimes r}$ and G acts linearly on $\mathbb{P}^n$. Furthermore, by Remark 5.26, we have an agreement of (semi)stable sets $X^{(s)s}(L) = X^{(s)s}(L^{\otimes r})$.

6.1. A topological criterion.

Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then as G acts via $G \to \mathrm{GL}_{n+1}$, the action of G lifts to the affine cones $\tilde{X} \subset \mathbb{A}^{n+1}$. We let $R(X) = \mathcal{O}(\tilde{X})$ denote the homogeneous coordinate ring of X.

Proposition 6.1. *Let $x \in X(k)$ and choose a non-zero lift $\tilde{x} \in \tilde{X}(k)$ of x. Then:*

i) x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$;

ii) x is stable if and if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

Proof. i) If x is semistable, then there is a G-invariant homogeneous polynomial $f \in R(X)^G$ which is non-zero at x. We can view f as a G-invariant function on $\tilde{X}$ such that $f(\tilde{x}) \neq 0$. As invariant functions are constant on orbits and also their closures we see that $f(\overline{G \cdot \tilde{x}}) \neq 0$ and so there is a function which separates the closed subschemes $\overline{G \cdot \tilde{x}}$ and 0; therefore, these closed subschemes are disjoint.

For the converse, suppose that $\overline{G \cdot \tilde{x}}$ and 0 are disjoint. Then as these are both G-invariant closed subsets of the affine variety $\tilde{X}$ and G is geometrically reductive, there exists a G-invariant polynomial $f \in R(\tilde{X})^G$ which separates these subsets

$$f(\overline{G \cdot \tilde{x}}) = 1 \quad \text{and} \quad f(0) = 0$$

by Lemma 4.29. In fact, we can take f to be homogeneous: if we decompose f into homogeneous elements $f = f_0 + \cdots + f_r$, then as the action is linear, each f_i must be G-invariant and, in particular, there is at least one G-invariant homogeneous polynomial f_i which does not vanish on $\overline{G \cdot \tilde{x}}$. Hence, x is semistable.

ii) If x is stable, then $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G$ such that $x \in X_f$ and $G \cdot x$ is closed in X_f. Since $G_{\tilde{x}} \subset G_x$, the stabiliser of $\tilde{x}$ is also zero dimensional. We can view f as a function on $\tilde{X}$ and consider the closed subscheme

$$Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\}$$

of $\tilde{X}$. It suffices to show that $G \cdot \tilde{x}$ is a closed subset of Z. The projection map $\tilde{X} - \{0\} \to X$ restricts to a surjective finite morphism $\pi : Z \to X_f$. The preimage of the closed orbit $G \cdot x$ in X_f under π is closed and G-invariant and, as π is also finite, the preimage $\pi^{-1}(G \cdot x)$ is a finite number of G-orbits. Since π is finite, the finite number of G-orbits in the preimage of $G \cdot x$ all have dimension equal to $\dim G$, and so these orbits must be closed in the preimage (see Proposition 3.15). Hence $G \cdot \tilde{x}$ is closed in Z.

Conversely suppose that $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$; then $0 \notin \overline{G \cdot \tilde{x}} = G \cdot \tilde{x}$ and so x is semistable by i). As x is semistable there is a non-constant G-invariant homogeneous polynomial f such that $f(x) \neq 0$. As above, we consider the finite surjective morphism

$$\pi : Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\} \to X_f.$$

Since $\pi(G\cdot\tilde{x}) = G\cdot x$ and π is finite, x has zero dimensional stabiliser group and $G\cdot x$ is closed in X_f. Since this holds for all f such that $f(x) \neq 0$, it follows that $G\cdot x$ is closed in $X^{ss} = \cup_f X_f$. Hence x is stable by Lemma 5.9. □

6.2. **The Hilbert–Mumford Criterion.** Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$ as above. In this section, we give a numerical criterion which can be used to determine (semi)stability of a point x.

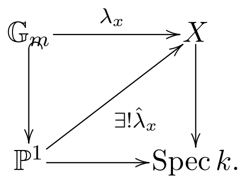
Following the topological criterion above, we see that to determine semistability, it is important to understand the closure of an orbit. One way to study the closure of an orbit is by using 1-parameter subgroups of G.

Definition 6.2. A 1-parameter subgroup (1-PS) of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

Fix $x \in X(k)$ and a 1-PS $\lambda : \mathbb{G}_m \to G$. Then we let $\lambda_x : \mathbb{G}_m \to X$ be the morphism given by

$$\lambda_x(t) = \lambda(t)\cdot x.$$

We have a natural embedding of $\mathbb{G}_m = \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{P}^1$ given by $t \mapsto [1 : t]$. Since X is projective, it is proper over $\operatorname{Spec} k$ and so, by the valuative criterion for properness, the morphism $\lambda_x : \mathbb{G}_m \to X$ extends uniquely to a morphism $\hat{\lambda}_x : \mathbb{P}^1 \to X$:



We use suggestive notation for the specialisations of this extended morphism at the zero and infinity points of $\mathbb{P}^1$:

$$\lim_{t\to 0}\lambda(t)\cdot x := \hat{\lambda}_x([1:0]) \quad \text{and} \quad \lim_{t\to\infty}\lambda(t)\cdot x := \hat{\lambda}_x([0:1]).$$

In fact, we can focus on the specialisation at zero, as

$$\lim_{t\to\infty}\lambda(t)\cdot x = \lim_{t\to 0}\lambda^{-1}(t)\cdot x.$$

Let $y := \lim_{t\to 0}\lambda(t)\cdot x$; then y is fixed by the action of $\lambda(\mathbb{G}_m)$; therefore, on the fibre over y of the line bundle $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_X$, the group $\lambda(\mathbb{G}_m)$ acts by a character $t \mapsto t^r$.

Definition 6.3. We define the *Hilbert-Mumford weight* of the action of the 1-PS λ on $x \in X(k)$ to be

$$\mu^{\mathcal{O}(1)}(x,\lambda) = r$$

where r is the weight of the $\lambda(\mathbb{G}_m)$ on the fibre $\mathcal{O}(1)_y$ over $y := \lim_{t\to 0}\lambda(t)\cdot x$.

From this definition, it is not so straight forward to compute this Hilbert–Mumford weight; therefore, we will rephrase this in terms of the weights for the action on the affine cone. Recall that $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual of the tautological line bundle on $\mathbb{P}^n$. Let $\mathbb{A}^{n+1}$ be the affine cone over $\mathbb{P}^n$; then $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the blow up of $\mathbb{A}^{n+1}$ at the origin. Pick a non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$. Then we can consider the morphism

$$\lambda_{\tilde{x}} := \lambda(-)\cdot\tilde{x} : \mathbb{G}_m \to \tilde{X}$$

which may no longer extend to $\mathbb{P}^1$, as $\tilde{X}$ is not proper. If it extends to zero (or infinity), we will denote the limits by

$$\lim_{t\to 0}\lambda(t)\cdot\tilde{x} \quad (\text{or} \lim_{t\to\infty}\lambda(t)\cdot\tilde{x}).$$

Any point in the boundary $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} - \lambda_{\tilde{x}}(\mathbb{G}_m)$ must be equal to either of these limit points.

The action of the 1-PS $\lambda(\mathbb{G}_m)$ on the affine cone $\mathbb{A}^{n+1}$ is linear, and so diagonalisable by Proposition 3.12; therefore, we can pick a basis $e_0, ..., e_n$ of k^{n+1} such that

$$\lambda(t)\cdot e_i = t^{r_i}e_i \quad \text{for } r_i \in \mathbb{Z}.$$

We call the integers r_i the λ-weights of the action on $\mathbb{A}^{n+1}$. For $x \in X(k)$ we can pick $\tilde{x} \in \tilde{X}(k)$ lying above this point and write $\tilde{x} = \sum_{i=0}^n x_i e_i$ with respect to this basis; then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i$$

and we let $\lambda\text{-wt}(x) := \{r_i : x_i \neq 0\}$ be the λ-weights of x (note that this does not depend on the choice of lift $\tilde{x}$).

Definition 6.4. We define the Hilbert-Mumford weight of x at λ to be

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\}.$$

We will soon show that this definition agrees with the above definition. However, we first note some useful properties of the Hilbert–Mumford weight.

Exercise 6.5. Show that the Hilbert–Mumford weight has the following properties.

(1) $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t\to 0} t^{\mu}\lambda(t) \cdot \tilde{x}$ exists and is non-zero.
(2) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive n.
(3) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G$.
(4) $\mu(x, \lambda) = \mu(y, \lambda)$ where $y = \lim_{t\to 0} \lambda(t) \cdot x$.

Lemma 6.6. *The two definitions of the Hilbert–Mumford weight agree:*

$$\mu^{\mathcal{O}(1)}(x, \lambda) = \mu(x, \lambda).$$

Proof. Pick a non-zero lift $\tilde{x}$ in the affine cone which lies over x. Then we assume that we have taken coordinates on $\mathbb{A}^{n+1}$ as above so that the action of $\lambda(t)$ is given by

$$\lambda(t) \cdot \tilde{x} = \lambda(t) \cdot (x_0, \ldots, x_n) = (t^{r_0} x_0, \ldots, t^{r_n} x_n).$$

Since $\mu(x, \lambda) + r_i \geq 0$ for all i such that $x_i \neq 0$, with equality for at least one i with $x_i \neq 0$, we see that

$$\tilde{y} := \lim_{t\to 0} t^{\mu(x,\lambda)} \lambda(t) \cdot \tilde{x} = (y_0, \ldots, y_n)$$

exists and is non-zero. More precisely, we have

$$y_i = \begin{cases} x_i & \text{if } r_i = -\mu(x, \lambda) \\ 0 & \text{else.} \end{cases}$$

Therefore, $\lambda(t) \cdot \tilde{y} = t^{-\mu(x,y)}\tilde{y}$. Furthermore, $\tilde{y}$ lies over $y := \lim_{t\to 0} \lambda(t) \cdot x$ and the weight of the λ-action on $\tilde{y}$ is $-\mu(x, y)$. Since $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the blow up of $\mathbb{A}^{n+1}$ at 0, we see that $-\mu(x, \lambda)$ is the weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}(-1)_y$. Hence, the weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}(1)_y$ is $\mu(x, \lambda)$ and this completes the proof of the claim. □

From the second definition of the Hilbert–Mumford weight, we easily deduce the following lemma.

Lemma 6.7. *Let λ be a 1-PS of G and let $x \in X(k)$. We diagonalise the $\lambda(\mathbb{G}_m)$-action on the affine cone as above and let $\tilde{x} = \sum_{i=0}^n x_i e_i$ be a non-zero lift of x.*

i) $\mu(x, \lambda) < 0 \iff \tilde{x} = \sum_{r_i > 0} x_i e_i \iff \lim_{t\to 0} \lambda(t) \cdot \tilde{x} = 0$.
ii) $\mu(x, \lambda) = 0 \iff \tilde{x} = \sum_{r_i \geq 0} x_i e_i$ *and there exists* $r_i = 0$ *such that* $x_i \neq 0 \iff \lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ *exists and is non-zero.*
iii) $\mu(x, \lambda) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ *and there exists* $r_i < 0$ *such that* $x_i \neq 0 \iff \lim_{t\to 0} \lambda(t) \cdot \tilde{x}$ *does not exist.*

Remark 6.8. We can use λ^{-1} to study $\lim_{t\to\infty} \lambda(t) \cdot \tilde{x}$ as

$$\lim_{t\to 0} \lambda^{-1}(t) \cdot \tilde{x} = \lim_{t\to\infty} \lambda(t) \cdot \tilde{x}.$$

Then it follows that

i) $\mu(x, \lambda^{-1}) < 0 \iff \tilde{x} = \sum_{r_i < 0} x_i e_i \iff \lim_{t\to\infty} \lambda(t) \cdot \tilde{x} = 0$.

ii) $\mu(x,\lambda^{-1}) = 0 \iff \tilde{x} = \sum_{r_i \leq 0} x_i e_i$ and there exists $r_i = 0$ such that $x_i \neq 0 \iff \lim_{t\to\infty} \lambda(t)\cdot\tilde{x}$ exists and is non-zero.

iii) $\mu(x,\lambda^{-1}) > 0 \iff \tilde{x} = \sum_{r_i} x_i e_i$ and there exists $r_i > 0$ such that $x_i \neq 0 \iff \lim_{t\to\infty} \lambda(t)\cdot\tilde{x}$ does not exist.

Following the discussion above and the topological criterion (see Proposition 6.1), we have the following results for (semi)stability with respect to the action of the subgroup $\lambda(\mathbb{G}_m) \subset G$.

Lemma 6.9. *Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Suppose $x \in X(k)$; then*

i) x is semistable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x,\lambda) \geq 0$ and $\mu(x,\lambda^{-1}) \geq 0$.
ii) x is stable for the action of $\lambda(\mathbb{G}_m)$ if and only if $\mu(x,\lambda) > 0$ and $\mu(x,\lambda^{-1}) > 0$.

Proof. For i), by the topological criterion x is semistable for $\lambda(\mathbb{G}_m)$ if and only if $0 \notin \overline{\lambda(\mathbb{G}_m)\cdot\tilde{x}}$, where and $\tilde{x} \in \tilde{X}(k)$ is a point lying over x. Since any point in the boundary $\overline{\lambda(\mathbb{G}_m)\cdot\tilde{x}} - \lambda(\mathbb{G}_m)\cdot\tilde{x}$ is either

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{x} \quad \text{or} \quad \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x},$$

it follows from Lemma 6.7 that x is semistable if and only if

$$\mu(x,\lambda) \geq 0 \quad \text{and} \quad \mu(x,\lambda^{-1}) \geq 0.$$

For ii), by the topological criterion x is stable for $\lambda(\mathbb{G}_m)$ if and only if $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$ and $\lambda(\mathbb{G}_m)\cdot\tilde{x}$ is closed. The orbit is closed if and only if the boundary is empty; that is, if and only if both limits

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{x} \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)\cdot\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x}$$

do not exist, i.e.

$$\mu(x,\lambda) > 0 \quad \text{and} \quad \mu(x,\lambda^{-1}) > 0.$$

Furthermore, if these inequalities hold, then $\lambda(\mathbb{G}_m)$ cannot fix $\tilde{x}$ (as otherwise the above limits would both exist) and so we must have that $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$. ☐

Exercise 6.10. Let $\mathbb{G}_m$ act on $\mathbb{P}^2$ by $t\cdot[x:y:z] = [tx : y : t^{-1}z]$. For every point $x \in \mathbb{P}^2$ and the 1-PS $\lambda(t) = t$, calculate $\mu(x,\lambda^{\pm 1})$ and then by using Lemma 6.9 above or otherwise, determine X^s and X^{ss}.

If x is (semi)stable for G, then it is (semi)stable for all subgroups H of G as every G-invariant function is also H-invariant. Hence, for a k-point x, we have

$$x \text{ is semistable} \implies \mu(x,\lambda) \geq 0 \,\forall \text{ 1-PS } \lambda \text{ of } G,$$

$$x \text{ is stable} \implies \mu(x,\lambda) > 0 \,\forall \text{ 1-PS } \lambda \text{ of } G.$$

The Hilbert-Mumford criterion gives the converse to these statements; the idea is that because G is reductive it has enough 1-PSs to detect points in the closure of an orbit (see Theorem 6.13 below).

Theorem 6.11. *(Hilbert–Mumford Criterion) Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X(k)$, we have*

$$x \in X^{ss} \iff \mu(x,\lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s} \iff \mu(x,\lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

Remark 6.12. A 1-PS is *primitive* if it is not a multiple of any other 1-PS. By Exercise 6.5 ii), it suffices to check the Hilbert–Mumford criterion for primitive 1-PSs of G.

It follows from the topological criterion given in Proposition 6.1 and also from Lemma 6.9, that the Hilbert–Mumford criterion is equivalent to the following fundamental theorem in GIT.

Theorem 6.13. *[Fundamental Theorem in GIT] Let G be a reductive group acting on an affine space $\mathbb{A}^{n+1}$. If $x \in \mathbb{A}^{n+1}$ is a closed point and $y \in \overline{G\cdot x}$, then there is a 1-PS λ of G such that $\lim_{t\to 0} \lambda(t)\cdot x = y$.*

The proof of the above fundamental theorem relies on a decomposition theorem of Iwahori which roughly speaking says there is an abundance of 1-PSs of reductive groups [17]. The proof of this theorem essentially follows from ideas of Mumford [25] §2.1 and we delay the proof until the end of this section.

Example 6.14. We consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ as in Example 5.8. As the group is a 1-dimensional torus, we need only calculate $\mu(x, \lambda)$ and $\mu(x, \lambda^{-1})$ for $\lambda(t) = t$ as was the case in Lemma 6.9. Suppose $\tilde{x} = (x_0, \dots, x_n)$ lies over $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n$. Then

$$\lim_{t \to 0} \lambda(t) \cdot \tilde{x} = (t^{-1}x_0, tx_1 \dots, tx_n)$$

exists if and only if $x_0 = 0$. If $x_0 = 0$, then $\mu(x, \lambda) = -1$ and otherwise $\mu(x, \lambda) > 0$. Similarly

$$\lim_{t \to 0} \lambda^{-1}(t) \cdot \tilde{x} = (tx_0, t^{-1}x_1 \dots, t^{-1}x_n)$$

exists if and only if $x_1 = \cdots = x_n = 0$. If $x_1 = \cdots = x_n = 0$, then $\mu(x, \lambda) = -1$ and otherwise $\mu(x, \lambda) > 0$. Therefore, the GIT semistable set and stable coincide:

$$X^{ss} = X^s = \{[x_0 : \cdots : x_n] : x_0 \neq 0 \text{ and } (x_1, \dots, x_n) \neq 0\} \subset \mathbb{P}^n.$$

6.3. **The Hilbert–Mumford Criterion for ample linearisations.** In this section we consider the following more general set up: suppose X is a projective scheme with an action by a reductive group G and ample linearisation L.

Definition 6.15. The *Hilbert–Mumford weight* of a 1-PS λ and $x \in X(k)$ with respect to L is

$$\mu^L(x, \lambda) := r$$

where r is the weight of the $\lambda(\mathbb{G}_m)$-action on the fibre L_y over the fixed point $y = \lim_{t\to 0} \lambda(t) \cdot x$.

Remark 6.16. We note that when $X \subset \mathbb{P}^n$ and the action of G is linear that this definition is consistent with the old definition; that is,

$$\mu^{\mathcal{O}_{\mathbb{P}^n}(1)|_X}(x, \lambda) = \mu(x, \lambda).$$

Exercise 6.17. Fix $x \in X$ and a 1-PS λ of G; then show $\mu^\bullet(x, \lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ is a group homomorphism where $\mathrm{Pic}^G(X)$ is the group of G-linearised line bundles on X.

Theorem 6.18. *(Hilbert–Mumford Criterion for ample linearisations) Let G be a reductive group acting on a projective scheme X and L be an ample linearisation of this action. Then, for $x \in X(k)$, we have*

$$x \in X^{ss}(L) \iff \mu^L(x, \lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G,$$
$$x \in X^{s}(L) \iff \mu^L(x, \lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

Proof. (Assuming Theorem 6.11) As L is ample, there is $n > 0$ such that $L^{\otimes n}$ is very ample. Then since

$$\mu^{L^{\otimes n}}(x, \lambda) = n\mu^L(x, \lambda)$$

it suffices to prove the statement for L very ample. If L is very ample then it induces a G-equivariant embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $L \cong i^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then we can just apply the first version of the Hilbert–Mumford criterion (cf. Theorem 6.11 and Remark 6.16). □