6.4. **Proof of the Fundamental Theorem in GIT.** In order to complete our proof of the Hilbert–Mumford Theorem, it suffices to prove the following slightly weaker version of the Fundamental Theorem in GIT.

Theorem 6.19. *Let G be a reductive group acting linearly on $\mathbb{A}^n$ and let $z \in \mathbb{A}^n$ be a k-point. If 0 lies in the orbit closure of z, then there exists a 1-PS λ of G such that $\lim_{t\to 0}\lambda(t)\cdot z = 0$.*

Proof. Suppose that $0 \in \overline{G\cdot z}$; then we will split the proof into 6 steps.

Step 1. We claim there is an irreducible (but not complete and not necessarily smooth) curve $C_1 \subset G\cdot z$ which contains 0 in its closure. To prove the existence of this curve, we use an argument similar to Bertini's Theorem and obtain the curve by intersecting hyperplanes in a projective completion $\mathbb{P}^n$ of $\mathbb{A}^n$; the argument is given in Lemma 6.20 below.

Step 2. We claim that there is a smooth projective curve C, a rational map $p : C \dashrightarrow G$ and a k-point $c_0 \in C$ such that $\lim_{c\to c_0} p(c)\cdot z = 0$. To prove this claim, we consider the action morphism $\sigma_z : G \to \mathbb{A}^n$ given by $g \mapsto g\cdot z$ and find a curve C_2 in G which dominates C_1 under σ_z (see Lemma 6.21 below) and then let C be a projective completion of the normalisation $\tilde{C}_2 \to C_2$; then the rational map $p : C \dashrightarrow G$ is defined by the morphism $\tilde{C}_2 \to C_2 \to G$. Finally, as the morphism $\tilde{C}_2 \to C_1$ is dominant it extends to their smooth projective completions and, as 0 lies in the closure of C_1, we can take a preimage $c_0 \in C$ of zero under this extension. Then $\lim_{c\to c_0} p(c)\cdot z = \lim_{c\to c_0}\sigma_z(p(c)) = 0$.

Step 3. Since C is a smooth proper curve, the completion of the local ring $\mathcal{O}_{C,c_0}$ of the curve at c_0 is isomorphic to the formal power series ring $k[[t]]$, whose field of fractions is the field of Laurent series $k((t))$. As the rational map $p : C \dashrightarrow G$ is defined in a punctured neighbourhood of c_0, it induces a morphism

$$q : K := \operatorname{Spec} k((t)) \cong \operatorname{Spec}\operatorname{Frac}\hat{\mathcal{O}}_{C,c_0} \to \operatorname{Spec}\operatorname{Frac}\mathcal{O}_{C,c_0} \to G$$

such that $\lim_{t\to 0}[q(t)\cdot z] = 0$. In Step 5, we will relate this K-valued point of G to a 1-PS.

Step 4. Let $R := \operatorname{Spec} k[[t]]$ and $K := \operatorname{Spec} k((t))$; then there is a natural morphism $K \to R$ and so the R-valued points of G form a subgroup of the K-valued points (i.e. $G(R) \subset G(K)$) whose limit as $t \to 0$ exists. More precisely, the natural map $\operatorname{Spec} k \to R$ induces a morphism $G(R) \to G(k)$ given by taking the specialisation as $t \to 0$.

There is a morphism $K \to \mathbb{G}_m = \operatorname{Spec} k[s,s^{-1}]$ induced by the homomorphism $k[s,s^{-1}] \to k((t))$ given by $s \mapsto t$. For a 1-PS λ, we define its Laurent series expansion $<\lambda>\in G(K)$ to be the composition of the natural morphism $K \to \mathbb{G}_m$ with λ.

Step 5. We will use without proof the Cartan-Iwahori decomposition for G which states that every double coset in $G(K)$ for the subgroup $G(R)$ is represented by a Laurent series expansion $<\lambda>$ of 1-PS of G (for example, see [25] §2.1). Therefore, as $q \in G(K)$, there exists $l_i \in G(R)$ for $i = 1, 2$ and a 1-PS λ of G such that

$$l_1 \cdot q = <\lambda> \cdot l_2$$

and the 1-PS λ is non-trivial, as q is not an R-valued point of G.

Step 6. Let $g_i := l_i(0) \in G$; then following the equality in Step 5, we have

$$0 = g_1 \cdot 0 = \lim_{t\to 0} l_1(t) \cdot \lim_{t\to 0}(q(t)\cdot z) = \lim_{t\to 0}\left[(<\lambda>\cdot l_2)(t)\cdot z\right].$$

We claim that $\lim_{t\to 0}\lambda(t)\cdot g_2\cdot z = 0$ and so $\lambda' := g_2^{-1}\lambda g_2$ is a 1-PS of G with $\lim_{t\to 0}\lambda'(t)\cdot z = 0$, which would complete the proof of the theorem. To prove the claim, we use the fact that the action of the 1-PS λ on $V = \mathbb{A}^n$ decomposes into weight spaces V_r for $r \in \mathbb{Z}$. Since $l_2 \in G(R)$ and $g_2 = \lim_{t\to 0} l_2(0)$, we can write $l_2(t)\cdot z = g_2\cdot z + \epsilon(t)$, where $\epsilon(t)$ only involves strictly positive powers of t. Then with respect to the weight space decomposition, we have

$$g_2\cdot z + \epsilon(t) = \sum_{r\in\mathbb{Z}} (g_2\cdot z)_r + \epsilon(t)_r.$$

Since $\lim_{t\to 0}[(<\lambda>\cdot l_2)(t)\cdot z] = 0$, it follows that $(g_2\cdot z)_r = 0$ for $r \leq 0$, which proves the claim and completes our proof. □

Lemma 6.20. *With the notation and assumptions of the previous theorem, there exists an irreducible curve $C_1 \subset G \cdot z$ which contains the origin in its closure.*

Proof. Fix an embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ and let $p \in \mathbb{P}^n$ denote the image of the origin. Let Y denote the closure of $G \cdot z$ in $\mathbb{P}^n$. We claim there exists a complete curve C_1' in Y containing the point $p \in \mathbb{P}^n$ and which is not contained entirely in the boundary $Z := Y - G \cdot z$. Assuming this claim, we obtain the desired curve $C_1 \subset G \cdot z$, by removing points of C_1' that lie in Z. To prove the claim, let $d = \dim Y$; then we can assume $d > 1$ as otherwise Y is already a curve. Then also $n > 1$. In the following section, we will see that hyperplanes in $\mathbb{P}^n$ are parametrised by $\mathbb{P}^n = \mathbb{P}(k[x_0, \dots, x_n]_1)$ and the space of hyperplanes containing p is a closed codimension 1 subspace $\mathcal{H}_p \subset \mathbb{P}^n$. Let $\mathcal{H}$ be the non-empty open subset of the product of $(d-1)$-copies of $\mathcal{H}_p$ consisting of hypersurfaces $(H_1, \dots, H_{d-1})$ such that

(1) $\cap_i H_i \cap Y$ is a curve (generically, $\dim \cap_{i=1}^{d-1} H_i \cap Y = \dim Y - (d-1) = 1$ and so this is a non-empty open condition), and
(2) $\cap_i H_i \cap Y$ is not entirely contained in Z (this is also a non-empty open condition, as $Z \subsetneq Y$ is a closed subscheme).

Hence, $\mathcal{H}$ is a non-empty open subset of $(\mathcal{H}_p)^{d-1}$, which has dimension $(n-1)(d-1) > 0$, and so the desired curve exists: we take $C_1' := \cap_i H_i \cap Y$, for $(H_1, \dots, H_{d-1}) \in \mathcal{H}_p \neq \emptyset$. □

Lemma 6.21. *With the notation and assumptions of the previous theorem, there exists a curve $C_2 \subset G$ that dominates the curve $C_1 \subset G \cdot z$ under the action morphism $\sigma_z : G \to G \cdot z$.*

Proof. Let η be the generic point of C_1. As η is not a geometric point and the above arguments about the existence of curves requires an algebraically closed field, we pick a geometric point $\overline{\eta}$ over η corresponding to a choice of an algebraically closed finite field extension of $k(C_1)$. We let $\sigma_z^{-1}(C_1)_\eta$ and $\sigma_z^{-1}(C_1)_{\overline{\eta}}$ be the base change of the preimage to $k(C_1)$ and its fixed algebraic closure. Then by Lemma 6.20, there exists a curve $C_2' \subset \sigma_z^{-1}(C_1)_{\overline{\eta}}$. The curve C_2' maps to a curve $C_2 \subset \sigma_z^{-1}(C_1)_\eta$ under the finite map $\sigma_z^{-1}(C_1)_{\overline{\eta}} \to \sigma_z^{-1}(C_1)_\eta$. By construction, C_2 is a curve in $\sigma_z^{-1}(C_1) \subset G$ which dominates C_1 under σ_z. □

7. Moduli of projective hypersurfaces

In this section, we will consider the moduli problem of classifying hypersurfaces of a fixed degree d in a projective space $\mathbb{P}^n$ up to linear change of coordinates on $\mathbb{P}^n$; that is, up to the action of the automorphism group PGL_{n+1} of $\mathbb{P}^n$. To avoid some difficulties associated with fields of positive characteristic, we assume that the characteristic of k is coprime to d.

7.1. The moduli problem.

A non-zero homogeneous degree d polynomial F in $n+1$ variables $x_0, \dots, x_n$ determines a projective degree d hypersurface $(F = 0)$ in $\mathbb{P}^n$. If F is irreducible then the associated hypersurface is an irreducible closed subvariety of $\mathbb{P}^n$ of codimension 1. If F is reducible, then the associated hypersurface is a union of irreducible subvarieties of $\mathbb{P}^n$ of codimension 1 counted with multiplicities. For example, the polynomial $F(x_0, x_1) = x_0^d$ gives a degree d reducible hypersurface in $\mathbb{P}^1$: the d-fold point.

Hypersurfaces of degree d in $\mathbb{P}^n$ are parametrised by points in the space $k[x_0, \dots, x_n]_d - \{0\}$ of non-zero degree d homogeneous polynomials in $n+1$ variables. This variety has dimension

$$\binom{n+d}{d}.$$

As any non-zero scalar multiple of a homogeneous polynomial F defines the same hypersurface, the projectivisation of this space

$$Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$$

is a smaller dimensional parameter space for these hypersurfaces.

The automorphism group PGL_{n+1} of $\mathbb{P}^n$ acts naturally on $Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$ as follows. The linear representation $\mathrm{GL}_{n+1} \to \mathrm{GL}(k^{n+1})$ given by acting by left multiplication induces a linear action of GL_{n+1} on $\mathbb{P}^n$. Consequently, there is an induced GL_{n+1}-action on the homogeneous coordinate ring $R(\mathbb{P}^n) = k[x_0, \dots, x_n]$ which preserves the graded pieces $k[x_0, \dots, x_n]_d$. This determines a linear action of GL_{n+1} on $\mathbb{P}(k[x_0, \dots, x_n]_d)$ by

$$(g \cdot F)(p) = F(g^{-1} \cdot p)$$

for $g \in \mathrm{GL}_{n+1}$, $F \in k[x_0, \dots, x_n]_d$ and $p \in \mathbb{A}^{n+1}$ (we note that the inverse here makes this a left action). This descends to an action

$$\mathrm{PGL}_{n+1} \times \mathbb{P}(k[x_0, \dots, x_n]_d) \to \mathbb{P}(k[x_0, \dots, x_n]_d).$$

One may expect that a moduli space for degree d hypersurfaces in $\mathbb{P}^n$ is given by a categorical quotient of this action and we will soon show that this is the case, by proving that $Y_{d,n}$ parametrises a family with the local universal property. However, the PGL_{n+1}-action on $Y_{n,d}$ is not linear, but the actions of GL_{n+1} and SL_{n+1} are both linear. Since we have a surjection $\mathrm{SL}_{n+1} \to \mathrm{PGL}_{n+1}$ with finite kernel, the SL_{n+1}-orbits are the same as the PGL_{n+1}-orbits, and the only small changes is that for SL_{n+1} there is now a global finite stabiliser group, but from the perspective of GIT finite groups do not matter. Therefore, we will work with the SL_{n+1}-action.

To prove the tautological family over $Y_{d,n}$ has the local universal property in order to apply Proposition 3.35, we need to introduce a notion of families of hypersurfaces. Let us start formulating a reasonable notion of families of hypersurfaces. One natural idea for a family of hypersurfaces over S is that we have a closed subscheme $X \subset S \times \mathbb{P}^n$ such that $X_s = X \cap \{s\} \times \mathbb{P}^n$ is a degree d hypersurface. For $S = \mathbb{A}^r = \operatorname{Spec} k[z_1, \dots, z_r]$, this is given by $H \in k[z_1, \dots, z_r, x_0, \dots, x_n]$ which is homogeneous of degree d in the variables $x_0, \dots, x_n$ and is non-zero at each point $s \in S$. In this case, a family of hypersurfaces is given by a degree d homogeneous polynomial in $n+1$ variables with coefficients in $\mathcal{O}(S)$. In fact, we can take this as a local definition for our families and generalise this notion to allow coefficients in an arbitrary line bundle L over S.

Definition 7.1. A *family of degree d hypersurfaces in $\mathbb{P}^n$ over S* is a line bundle L over S and a tuple of sections

$$\sigma := (\sigma_{i_0 \dots i_n} : i_j \geq 0, \sum_{j=0}^{n} i_j = d)$$

of L such that for each k-point $s \in S$, the polynomial

$$F(L, \sigma, s) := \sum_{i_0 \dots i_n} \sigma_{i_0 \dots i_n}(s) x_0^{i_0} \dots x_n^{i_n}$$

is non-zero.

We note that to make sense of this final sentence, we must trivialise L locally at s. Then the tuple of constants $\sigma(s)$ are determined up to multiplication by a non-zero scalar. In particular, we can determine whether $F(L, \sigma, s)$ is non-zero and the associated hypersurface is uniquely determined. We denote the family by (L, σ) and the hypersurface over a k-point s by $(L, \sigma)_s : F(L, \sigma, s) = 0$.

Definition 7.2. We say *two families (L, σ) and (L', σ') of degree d hypersurfaces in $\mathbb{P}^n$ over S are equivalent over S* if there exists an isomorphism $\phi : L \to L'$ of line bundles and $g \in \mathrm{GL}_{n+1}$ such that $\phi \circ \sigma = g \cdot \sigma'$.

We note that with this definition of equivalence the families (L, σ) and $(L, \lambda\sigma)$ are equivalent for any non-zero scalar λ.

Exercise 7.3. Show that $Y_{d,n} = \mathbb{P}(k[x_0, \dots, x_n]_d)$ parametrises a tautological family of degree d hypersurfaces in $\mathbb{P}^n$ with the local universal property. Deduce that any coarse moduli space for hypersurfaces is a categorical quotient of SL_{n+1} acting on $Y_{d,n}$ as above.

Since SL_{n+1} is reductive, we can take a projective GIT quotient of the action on $Y_{d,n}$ which is a good (and categorical) quotient of the semistable locus $Y_{d,n}^{ss}$. There are now two problems to address:

(1) determine the (semi)stable points in $Y_{d,n}$;
(2) geometrically interpret (semi)stability of points in terms of properties of the corresponding hypersurfaces.

For small values of d and n, we shall see that it is possible to give a full solution to the above two problems, although as both values get larger the problem becomes increasingly difficult.