7.2. Singularities of hypersurfaces.

Definition 7.4. A point p in $\mathbb{P}^n$ is a *singular point* of a projective hypersurface defined by a polynomial $F \in k[x_0, \dots, x_n]_d$ if

$$F(\tilde{p}) = 0 \text{ and } \frac{\partial F}{\partial x_i}(\tilde{p}) = 0 \text{ for } i = 0, \dots, n,$$

where $\tilde{p} \in \mathbb{A}^{n+1} - \{0\}$ is a lift of $p \in \mathbb{P}^n$. We say a hypersurface is *non-singular (or smooth)* if it has no singular points.

Remark 7.5.

(1) By using the Euler formula

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = d\, F$$

and the fact that d is coprime to the characteristic of k, we see that $p \in \mathbb{P}^n$ is a singular point of F if and only if all partial derivatives $\partial F/\partial x_i$ vanish at p.

(2) If we consider F as a function $F : \mathbb{A}^{n+1} \to k$, then we can consider its derivative $d_{\tilde{p}}F : T_{\tilde{p}}\mathbb{A}^{n+1} \to T_{\tilde{p}}k \cong k$ at $\tilde{p} \in \mathbb{A}^{n+1} - \{0\}$. The corresponding point $p \in \mathbb{P}^n$ is a singular point of F if and only if this derivative $d_{\tilde{p}}F$ is zero.

(3) Let $\sigma_g : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ denote the action of $g \in G$. By the chain rule, we have $d_{g\cdot\tilde{p}}(g\cdot F) = d_{\tilde{p}}F \circ d_{g\cdot\tilde{p}}\sigma_{g^{-1}}$, where $d_p\sigma_{g^{-1}}$ is invertible (as σ is an action). Hence $d_{g\cdot\tilde{p}}(g \cdot F) = 0$ if and only if $d_{\tilde{p}}F$; in other words p is a singular point of the hypersurface $F = 0$ if and only if $g \cdot p$ is a singular point of the hypersurface $g \cdot F = 0$ for any $g \in G$.

The resultant polynomial of a collection of polynomials is a function in the coefficients of these polynomials which vanishes if and only if these polynomials all have a common root; for the existence of the resultant and how to compute it, see [7] Chapter 13 1.A.

Definition 7.6. For a polynomial $F \in k[x_0, \dots, x_n]_d$, we define the *discriminant* $\Delta(F)$ of F to be the resultant of the polynomials $\partial F/\partial x_i$.

Then Δ is a homogeneous polynomial in $R(Y_{d,n})$ and is non-zero at F if and only if F defines a smooth hypersurface. It follows from Remark 7.5 that Δ is SL_{n+1}-invariant.

Example 7.7. If $d = 1$, then $Y_{1,n} \cong (\mathbb{P}^n)^\vee$ and as the only SL_{n+1}-invariant homogeneous polynomials are the constants:

$$k[x_0, \dots, x_n]^{\mathrm{SL}_{n+1}} = k,$$

there are no semistable points for the action of SL_{n+1} on $Y_{1,n}$. In particular, the discriminant Δ is constant on $Y_{1,n}$. Alternatively, as the action of SL_{n+1} on $\mathbb{P}^n$ is transitive, to show $Y_{1,n}^{ss} \cong (\mathbb{P}^n)^{ss} = \emptyset$, it suffices to show a single point $x = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$ is unstable. For this, one can use the Hilbert-Mumford criterion: it is easy to check that if $\lambda(t) = \mathrm{diag}(t, t^{-1}, 1, \dots, 1)$, then $\mu(x, \lambda) < 0$.

For $d > 1$, the discriminant is a non-constant SL_{n+1}-invariant homogeneous polynomial on $Y_{d,n}$ and as it is non-zero for all smooth hypersurfaces we have:

Proposition 7.8. *For $d > 1$, every smooth degree d hypersurface in $\mathbb{P}^n$ is semistable for the action of SL_{n+1} on $Y_{d,n}$.*

To determine whether a semistable point is stable we can check whether its stabiliser subgroup is finite.

Example 7.9. If $d = 2$, then we are considering the space $Y_{2,n}$ of quadric hypersurfaces in $\mathbb{P}^n$. Given $F = \sum_{i,j} a_{ij}x_ix_j \in k[x_0, \dots, x_n]_2$, we can associate to F a symmetric $(n + 1) \times (n + 1)$ matrix $B = (b_{ij})$ where $b_{ij} = b_{ji} = a_{ij}$ and $b_{ii} = 2a_{ii}$. This procedure defines an isomorphism between $Y_{2,n}$ and the space $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$ where $\mathrm{Sym}_{(n+1)\times(n+1)}(k)$ denotes the space of symmetric $(n+1)\times(n+1)$ matrices. The discriminant Δ on $Y_{2,n}$ corresponds to the determinant on $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(k))$; thus F is smooth if and only if its associated matrix is invertible. In fact if F corresponds to a matrix B of rank $r+1$, then F is projectively equivalent to the quadratic form

$$x_0^2 + \cdots + x_r^2.$$

As all non-singular quadratic forms $F(x_0, \dots, x_n)$ are equivalent to $x_0^2 + \cdots + x_n^2$ (after a change of coordinates), we see that these points cannot be stable: the stabiliser of $x_0^2 + \cdots + x_n^2$ is equal to the special orthogonal group $\mathrm{SO}(n+1)$ which is positive dimensional. Moreover, the discriminant generates the ring of invariants (for example, see [31] Example 4.2) and so the semistable locus is just the set of non-singular quadratic forms. In this case, the GIT quotient consists of a single point and this represents the fact that all non-singular quadratic forms are projectively equivalent to $x_0^2 + \cdots + x_n^2$.

The projective automorphism group of a hypersurface is the subgroup of the automorphism group PGL_{n+1} of $\mathbb{P}^n$ which leaves this hypersurface invariant. For $d > 2$, the projective automorphism group of any irreducible degree d hypersurface is finite; this is a classical but non-trivial result (see [20] Lemma 14.2). As PGL_{n+1} is a quotient of SL_{n+1} by a finite subgroup, this implies the stabiliser subgroup of a point in $Y_{d,n}$ corresponding to an irreducible hypersurface is finite dimensional. Since every smooth hypersurface is irreducible, the stabiliser group of a smooth hypersurface is finite. In fact, one can also check that for $d > 2$, the orbit of a smooth hypersurface is closed and so the following result holds.

Proposition 7.10 ([25] §4.3). *For $d > 2$, every degree d smooth hypersurface is stable.*

7.3. **The Hilbert–Mumford criterion for hypersurfaces.** To determine the (semi)stable points for the action of SL_{n+1} on $Y_{d,n}$, we can use the Hilbert–Mumford criterion. Any 1-PS of SL_{n+1} is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & & \\ & t^{r_1} & & \\ & & \ddots & \\ & & & t^{r_n} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^n r_i = 0$ and $r_0 \geq r_1 \geq \cdots \geq r_n$. Then the action of λ is diagonal with respect to the basis of the affine cone over $Y_{d,n}$ given by the monomials

$$x_I = x_0^{i_0} x_1^{i_1} \dots x_n^{i_n},$$

for $I = (i_0, \dots, i_n)$ a tuple of non-negative integers which sum to d. Furthermore, the weight of each monomial x_I for the action of λ is $-\sum_{j=0}^n r_j i_j$, where the negative sign arises as we act by the inverse of $\lambda(t)$.

Let $F = \sum a_I x_I \in k[x_0, \dots, x_n]_d - \{0\}$, where $I = (i_0, \dots, i_n)$ is a tuple of non-negative integers which sum to d and x_I, and let $p_F \in Y_{d,n}$ be the corresponding class. Then

$$\begin{aligned} \mu(p_F, \lambda) &= -\min\{-\sum_{j=0}^n r_j i_j : I = (i_0, \dots, i_n) \text{ and } a_I \neq 0\} \\ &= \max\{\sum_{j=0}^n r_j m_j : I = (i_0, \dots, i_n) \text{ and } a_I \neq 0\}. \end{aligned}$$

For general (d, n), there is not always a clean description of the semistable locus. However for certain small values, we shall see that this has a nice description. In §7.4 below we discuss the case when $n = 1$; in this case, a degree d hypersurface corresponds to d unordered points (counted with multiplicity) on $\mathbb{P}^1$. Then in §7.5 we discuss the case when $(d, n) = (3, 2)$; that is, cubic curves in the projective plane $\mathbb{P}^2$. Both of these classical examples were studied by Hilbert and can also be found in [25] and [31].

7.4. **Binary forms of degree d.** A binary form of degree d is a degree d homogeneous polynomial in two variables x, y. The set of zeros of a binary form F determine d points (counted with multiplicity) in $\mathbb{P}^1$. In this section we study the action of SL_2 on

$$Y_{d,1} = \mathbb{P}(k[x,y]_d) \cong \mathbb{P}^d.$$

Our aim is to describe the (semi)stable locus and the GIT quotient.

One method to determine the semistable and stable locus is to compute the ring of invariants $R(Y_{d,1})^{\mathrm{SL}_2}$ for this action. For $d \leq 6$, the ring of invariants is known due to classical computations in invariant theory going back to Hilbert and later work of Schur. For general values of d, the ring of invariants is still unknown today, which shows how difficult it can be in general to determine the ring of invariants. For $d = 8$, a list of generators of the ring of invariants is given by work of von Gall (1880) and Shioda (1967) [38]. For $d = 9, 10$, generators for the ring of invariants were calculated by Brouwer and Popoviciu in (2010). Instead, we will use the Hilbert–Mumford criterion to obtain a complete description of the semistable locus, which bypasses the need to calculate the ring of invariants.

Remark 7.11. If $d = 1$, then this corresponds to the action of SL_2 on $\mathbb{P}^1$, for which there are no semistable points as the only invariant functions are constant (see also Example 7.7).

Henceforth, we assume $d \geq 2$ and use the Hilbert–Mumford criterion for semistability. We fix the maximal torus $T \subset \mathrm{SL}_2$ given by the diagonal matrices

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}.$$

Any primitive 1-PS of G is conjugate to the 1-PS of T given by

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

If $F(x,y) = \sum_i a_i x^{d-i} y^i \in k[x,y]_d - \{0\}$ lies over $p_F \in Y_{d,1}$, then

$$\lambda(t) \cdot F(x,y) = \sum t^{2i-d} a_i x^{d-i} y^i$$

and

$$\mu(p_F, \lambda) = -\min\{2i - d : a_i \neq 0\} = \max\{d - 2i : a_i \neq 0\} = d - 2i_0,$$

where i_0 is the smallest integer for which $a_i \neq 0$. Hence

(1) $\mu(p_F, \lambda) \geq 0$ if and only if $i_0 \leq d/2$ if and only if $[1:0]$ occurs with multiplicity at most $d/2$.
(2) $\mu(p_F, \lambda) > 0$ if and only if $i_0 < d/2$ if and only if $[1:0]$ occurs as a root with multiplicity strictly less than $n/2$.

By the Hilbert–Mumford criterion, $p_F \in Y_{d,1}$ is semistable if and only if $\mu(p_F, \lambda') \geq 0$ for all 1-PSs λ'. For a general 1-PS λ' we can write $\lambda = g^{-1}\lambda' g$, then

$$\mu(p_F, \lambda') = \mu(g \cdot p_F, \lambda).$$

If F has roots $p_1, \dots, p_d \in \mathbb{P}^1$, then $g \cdot F$ has roots $g \cdot p_1, \dots, g \cdot p_d$. As SL_2 acts transitively on $\mathbb{P}^1$, we deduce the following result.

Proposition 7.12. *Let $F \in k[x,y]_d$ lie over $p_F \in Y_{d,1}$; then:*

i) p_F is semistable if and only if all roots of F in $\mathbb{P}^1$ have multiplicity less than or equal to $d/2$.
ii) p_F is stable if and only if all roots of F in $\mathbb{P}^1$ have multiplicity strictly less than $d/2$.

In particular, if d is odd then $Y_{d,1}^{ss} = Y_{d,1}^{s}$ and the GIT quotient is a projective variety which is a geometric quotient of the space of stable degree d hypersurfaces in $\mathbb{P}^1$.

Example 7.13. If $d = 2$, then the semistable locus corresponds to binary forms F with two distinct roots and the stable locus is empty. Given any two distinct points (p_1, p_2) on $\mathbb{P}^1$, there is a mobius transformation taking these points to any other two distinct points (q_1, q_2). However this mobius transformation is far from unique; in fact given points p_3 distinct from (p_1, p_2)

and q_3 distinct from (q_1, q_2), there is a unique mobius transformation taking p_i to q_i. Hence all semistable points have positive dimensional stabilisers and so can never be stable. As the action on the semistable locus is transitive, the GIT quotient is just the point $\mathrm{Spec}\, k$.

Example 7.14. If $d = 3$, then the stable locus (which coincides with the stable locus) consists of forms with 3 distinct roots. We recall that given any 3 distinct points (p_1, p_2, p_3) on $\mathbb{P}^1$, there is a unique mobius transformation taking these points to any other 3 distinct points. Hence the GIT quotient is the projective variety $\mathbb{P}^0 = \mathrm{Spec}\, k$. In fact, the SL_2-invariants have a single generator: the discriminant

$$\Delta\left(\sum a_i x^{d-i} y^i\right) := 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3$$

which is zero if and only if there is a repeated root.

Example 7.15. If $d = 4$, then we are considering binary quartics. In this case the semistable locus is the set of degree 4 binary forms F with at most 2 repeated roots and the stable locus is the set of points in which all 4 roots are distinct. Given 4 distinct ordered points $(p_1, \dots, p_4)$ there is a unique mobius transformation which takes this ordered set of points to $(0, 1, \infty, \lambda)$ where $\lambda \in \mathbb{A}^1 - \{0, 1\}$ is the cross-ratio of these points. However, the points in our case do not have a natural ordering and so there are 6 possible values of the cross-ratio depending on how we choose to order our points:

$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}, \frac{1}{1-\lambda}.$$

The morphism $f : Y_{4,1}^s \to \mathbb{A}^1$ given by

$$\left(\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right)^3$$

is symmetric in the six possible values of the cross-ratio, and so is SL_2-invariant. It is easy to check that f is surjective and in fact an orbit space: for each value of f in $\mathbb{A}^1 - \{0, -27\}$, there are six distinct possible choices for λ as above and so this corresponds to a unique stable orbit. For the values 0 (resp. -27), there are 3 (resp. 2) possible values for λ and these correspond to a unique stable orbit.

The strictly semistable points have either one or two double roots and so correspond to two orbits. The orbit consisting of one double root is not closed: its closure contains the orbit of points with two double roots (imagine choosing a family of mobius transformations h_t that sends (p, p, q, r) to $(1, 1, 0, t)$, then as $t \to 0$, we see that the point $(1, 1, 0, 0)$ lies in this orbit closure). This suggests that the GIT quotient $Y_{4,1}/\!/\mathrm{SL}_2$ is $\mathbb{P}^1$, the single point compactification of $\mathbb{A}^1$.

In fact, this is true: there are two independent generators for the SL_2-invariants of binary quartics (called the I and J invariants - for example, see [31] Example 4.5 or [4], where they are called S and T) and the good quotient is $\varphi : Y_{4,1}^{ss} \to \mathbb{P}^1$.

7.5. **Plane cubics.** In this section, we study moduli of degree 3 hypersurfaces in $\mathbb{P}^2$; that is, plane cubic curves. We write a degree 3 homogeneous polynomial F in variables x, y, z as

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j.$$

We want to describe all plane cubic curves up to projective equivalence; that is, describe the quotient for the action of SL_3 on $Y_{3,2}$. For simplicity, we assume that the characteristic of k is not equal to 2 or 3.

An important classical result about the intersection of plane curves is Bézout's Theorem, which says for two projective plane curves C_1 and C_2 in $\mathbb{P}^2$ with no common components, the number of points of intersection of C_1 and C_2 counted with multiplicities is equal to the product of the degrees of these curves. The fact that k is algebraically closed is crucial for this result. For a basic introduction to algebraic curves and an elementary proof of Bézout's Theorem, see [19]. In this section, we will use without proof the following easy applications of Bézout's Theorem.

Proposition 7.16. *(1) Any non-singular projective plane curve $C \subset \mathbb{P}^2$ is irreducible.*
(2) Any irreducible projective plane curve $C \subset \mathbb{P}^2$ has at most finitely many singular points.

Furthermore, Bézout's Theorem can be used to obtain a classification of plane curves of low degree.

Lemma 7.17. *Any irreducible plane conic $C \subset \mathbb{P}^2$ is projectively equivalent to the conic defined by $x^2 + yz = 0$, which is isomorphic to $\mathbb{P}^1$.*

Proof. By the above proposition, C has only finitely many singular points, and so we can choose coordinates so that $[0 : 1 : 0] \in C$ is non-singular and the tangent line to the curve at this point is the line $z = 0$. Then we must have that C is the zero locus of a polynomial

$$P(x, y, z) = ayz + bx^2 + cxz + dz^2$$

and, as P is irreducible, we must have $b \neq 0$. Since $\partial P(p)/\partial z \neq 0$, we have $a \neq 0$. Then the change of coordinates $x' := \sqrt{b}x$, $y' := ay + cx + dz$, $z' := z$ transforms the above conic into the desired form.

Finally, for $C : (x^2 + yz = 0)$, we have an isomorphism $f : C \to \mathbb{P}^1$ given by

$$f([x : y : z]) = \begin{cases} [x : y] & \text{if } y \neq 0 \\ [-z : x] & \text{if } z \neq 0. \end{cases}$$

The inverse of f is $f^{-1} : \mathbb{P}^1 \to C$ given by $[u : v] \mapsto [uv : v^2 : -u^2]$. □

This enables us to easily classify all reducible plane cubics up to projective equivalence, as any reducible plane conic is either the union of an irreducible conic with a line or a union of three lines. In fact, one can also prove that two reducible plane cubics are projectively equivalent if and only if they are isomorphic. If the reducible plane cubic curve is a union of a line and a conic, then the line can either meet the conic at two distinct points or a single point (so that the line is tangent to the conic). By the above lemma, the irreducible conic is projectively equivalent to $y^2 + xz = 0$. As the projective automorphism group of this conic acts transitively on the set of tangents lines to this conic and the set of lines meeting the conic at two distinct points, any reducible cubic which is a union of a conic and a line is projectively equivalent to either

- $(xz + y^2)y = 0$, where the line meets the conic in two distinct points, or
- $(xz + y^2)z = 0$, where the line meets the conic tangentially.

If the reducible cubic curve is a union of three lines, there are four possibilities: one line occurring with multiplicity three; a union of a double line with another distinct line; a union of three lines meeting in a single intersection point; a union of three lines which meet in three intersection points. Since the group of projective transformations acts transitively on the space of 3 lines, we see that a reducible cubic curve which is a union of three lines is projectively equivalent to either

- $y^3 = 0$ (a triple line), or
- $y^2(y + z) = 0$ (a union of a double line with a distinct line), or
- $yz(y + z) = 0$ (three concurrent lines), or
- $xyz = 0$ (three non-concurrent lines).

The above reducible plane cubics contain a singular point at $[1 : 0 : 0]$. In fact, we can define a notion of multiplicities for singularities to distinguish between different types of singularities. For a plane cubic, all points have multiplicity at most 3.

Definition 7.18. A singular point at p of cubic curved defined by $F(x, y, z) = 0$ is a *triple point* if all second order partial derivatives of F vanish at p; otherwise we say p is a *double point.* A non-singular point is called a single point or point of multiplicity 1.

Example 7.19. The cubics defined by $y^3 = 0$ (a triple line), $y^2(y+z) = 0$ (a union of a double line with a distinct line), $yz(y + z) = 0$ (three concurrent lines) all contain a triple point at $[1 : 0 : 0]$. The cubic defined by $xyz = 0$ (three non-concurrent lines) has three double points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. The cubic defined by $(xz + y^2)y = 0$ (a union of an irreducible conic with a non-tangential line) has two double points: $[1:0:0]$ and $[0:0:1]$. The cubic defined by $(xz+y^2)z=0$ (a union of an irreducible conic with a tangential line) has a single double point at $[1:0:0]$ (with a single tangent direction).

Since tangent lines will play an important role in the classification of semistable plane cubics, we recall their definition. Every non-singular point has a single tangent line, whereas singular points have multiple tangents.

Definition 7.20. Let $p=[p_0:p_1:p_2]$ be a point of a plane algebraic curve $C:(F(x,y,z)=0)$.

(1) If p is a non-singular point, then the *tangent line* to C at p is given by

$$\frac{\partial F(\tilde{p})}{\partial x}x+\frac{\partial F(\tilde{p})}{\partial y}y+\frac{\partial F(\tilde{p})}{\partial z}z=0$$

where $\tilde{p}=(p_0,p_1,p_2)$.

(2) If $p=[p_0:p_1:p_2]$ is a double point of C; then the *tangent lines* to C at p are given by the degree 2 homogeneous polynomial

$$0=(x-p_0,y-p_1,z-p_2)\begin{pmatrix}\frac{\partial^2F(\tilde{p})}{\partial x^2} & \frac{\partial^2F(\tilde{p})}{\partial y\partial x} & \frac{\partial^2F(\tilde{p})}{\partial z\partial x}\\ \frac{\partial^2F(\tilde{p})}{\partial x\partial y} & \frac{\partial^2F(\tilde{p})}{\partial y^2} & \frac{\partial^2F(\tilde{p})}{\partial z\partial y}\\ \frac{\partial^2F(\tilde{p})}{\partial x\partial z} & \frac{\partial^2F(\tilde{p})}{\partial y\partial z} & \frac{\partial^2F(\tilde{p})}{\partial z^2}\end{pmatrix}\begin{pmatrix}x-p_0\\ y-p_1\\ z-p_2\end{pmatrix}.$$

The 3×3 matrix appearing in this expression is called the Hessian of F at $\tilde{p}$ and has rank $0<r<3$ as $\tilde{p}$ is a double point. As the Hessian does not have full rank, the above equation for the tangent lines factorises into a product of two linear polynomials.

For a plane cubic C, there are two types of singular double points:

(1) A *node (or ordinary double point)* is a double point with two distinct tangent lines (which is a self intersection of the curve, so that both branches of the curve have distinct tangent lines at the intersection point).
(2) A *cusp* is a double point with a single tangent line of multiplicity two (which is not a self intersection point of the curve).

Example 7.21. Let $F_1(x,y,z)=xz^2+y^3+y^2x$ and $F_2(x,y,z)=xz^2+y^3$. The corresponding cubics are irreducible and have a singular point at $p=[1:0:0]$. The point p is a double point which is a node of the first cubic corresponding to $(F_1=0)$ as the tangent lines are given by

$$0=y^2+z^2=(y-\sqrt{-1}z)(y+\sqrt{-1}z).$$

The point p is a double point of the second cubic corresponding to $(F_2=0)$, which is a cusp as the tangent lines are given by

$$0=z^2.$$

Exercise 7.22. Fix a non-zero homogeneous polynomial

$$F(x,y,z)=\sum_{i=0}^{3}\sum_{j=0}^{3-i}a_{ij}x^{3-i-j}y^iz^j$$

of degree 3 and let C be the plane cubic curve defined by $F=0$. For $p=[1:0:0]\in\mathbb{P}^2$, show the following statements hold.

i) $p\in C$ if and only if $a_{00}=0$.
ii) p is a singular point of F if and only if $a_{00}=a_{10}=a_{01}=0$.
iii) p is a triple point of F if and only if $a_{00}=a_{10}=a_{01}=a_{11}=a_{20}=a_{02}=0$.
iv) If $p=[1:0:0]$ is a double point of F, then its tangent lines are defined by

$$a_{20}y^2+a_{11}yz+a_{02}z^2=0.$$

For non-singular plane cubics, we have a classification following Bézout's Theorem in terms of Legendre cubics or Weierstrass cubics. It is important for the following classification, that we remember that the characteristic of k is assumed to be not equal to 2 or 3.

Proposition 7.23. *Let $C \subset \mathbb{P}^2$ be an irreducible plane cubic curve.*

(1) If C is non-singular it is projectively equivalent to a Legendre cubic *of the form*

$$y^2 z = x(x-z)(x-\lambda z)$$

for some $\lambda \in k - \{0,1\}$.

(2) C is projectively equivalent to a Weierstrass cubic *of the form*

$$y^2 z = x^3 + axz^2 + bz^3$$

for scalars a and b.

Proof. i) Let C be a non-singular plane cubic defined by $P(x,y,z) = 0$. The Hessian $\mathcal{H}_P$ of P is the degree 3 polynomial which is the determinant of the 3×3 matrix of second order derivatives of P. By Bézout's theorem, $\mathcal{H}_P$ and P have at least one common solution, which gives a point $p \in C$ known as an inflection point. By a change of coordinates, we can assume $p = [0:1:0]$ and the tangent line T_pC is defined by $z = 0$. Hence P, $\partial P/\partial x$, $\partial P/\partial y$ and $\mathcal{H}_P$ all vanish at p, but $\partial P/\partial z$ is non-zero at p. It follows from the Euler relations that

$$\mathcal{H}_P(p) = -4\left(\frac{\partial P}{\partial z}(p)\right)^2 \frac{\partial^2 P}{\partial x^2}(p)$$

and so also $\partial^2 P/\partial x^2(p) = 0$. Hence, P does not involve the monomials y^3, xy^2 and x^2y. Therefore,

$$P(x,y,z) = Q(x,z) + yz(\alpha x + \beta y + \gamma z)$$

where Q is homogeneous of degree 3 and $\beta \neq 0$. After a change of coordinates in the y variable, we may assume that

$$P(x,y,z) = R(x,z) + y^2 z$$

for R a degree 3 homogeneous polynomial in x and z. Since C is non-singular, z does not divide R; that is, the coefficient of x^3 in R is non-zero. We can factorise this homogeneous polynomial in two variables as:

$$R(x,z) = u(x-az)(x-bz)(x-cz)$$

where $u \neq 0$ and a,b,c are distinct as C is non-singular. Let $\lambda = (b-c)/(b-a)$; then one further change of coordinates reduces the equation to a Legendre cubic. As the characteristic of k is not equal to 3, any Legendre cubic can be transformed into a Weierstrass cubic by a change of coordinates.

ii) It suffices to consider irreducible singular plane conics. By a change of coordinates, we can assume that $[0:0:1]$ is a singular point and the equation of our cubic has the form

$$zQ(x,y) + R(x,y) = 0$$

where Q is homogeneous of degree 2 and R is homogeneous of degree 3. After a linear change of variables in x, y, the degree 2 polynomial Q in two variables is either $Q(x,y) = y^2$ or $Q(x,y) = xy$. T he first case corresponds to a cuspidal cubic and the second case corresponds to a nodal cubic; we merely sketch the argument below and refer to [4] §10.3 for further details, where a classification for fields of characteristic 2 and 3 is also given.

Consider the first case: $Q(x,y) = y^2$. Then our conic has the form

$$y^2 z + ax^3 + bx^2 y + cxy^2 + dy^3 = 0$$

where $a \neq 0$, as the conic is irreducible. By a linear change in the z-coordinate, we can assume $c = d = 0$ and by scaling x, we may assume $a = 1$. A final change of coordinates which fixes the singular point $[0:0:1]$ and moves the unique non-singular inflection point to $[0:1:0]$, with tangent line $z = 0$, reduces the equation to $zy^2 = x^3$, which is the Weierstrass cusp.

Consider the second case: $Q(x,y) = xy$. Then our conic has the form

$$xyz + ax^3 + bx^2 y + cxy^2 + dy^3 = 0.$$

By the change of coordinates in z, we can assume $b = c = 0$. Since C is irreducible, both a and d must be non-zero and so we can scale them to both be 1. After one more change of coordinates, we obtain a nodal Weierstrass form: $y^2 z = x^2(x+y)$. □

Remark 7.24. The constant λ occurring in the Legendre cubic is not unique: it depends on which two roots of the cubic equation are sent to 0 and 1. Hence, there are 6 possible choices of λ for each non-singular cubic: λ, $1-\lambda$, $1/\lambda$, $1/(1-\lambda)$, $\lambda/(\lambda-1)$ and $(\lambda-1)/\lambda$. Similarly, in the Weierstrass cubic, the constants a and b are not unique: as a change of coordinates $y' = \eta^3 y$ and $x' = \eta^2 x$ gives a new Weierstrass cubic with $a' = \eta^4 a$ and $b' = \eta^6 b$.

Weierstrass cubics arise in the study of elliptic curves, which are classified up to isomorphism using the j-invariant. *Elliptic curves* are the non-singular Weierstrass cubics (those for which $4a^3+27b^2 \neq 0$). Two elliptic curves are isomorphic if and only if they have the same *j-invariant*, where

$$j = 1738\frac{4a^3}{4a^3+27b^2}.$$

In terms of the Legendre cubic, we can write the j-invariant in terms of λ as

$$j = \frac{256(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

For further details on elliptic curves and the j-invariant, see [14] IV §4.

This classification of plane cubics does not tell us anything about which ones are (semi)stable. We will use the Hilbert–Mumford criterion to give a complete description of the (semi)stable locus. Any 1-PS of SL_3 is conjugate to a 1-PS of the form

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & \\ & t^{r_1} & \\ & & t^{r_2} \end{pmatrix}$$

where r_i are integers such that $\sum_{i=0}^2 r_i = 0$ and $r_0 \geq r_1 \geq r_2$. It is easy to calculate that

$$\mu(F,\lambda) = -\min\{-(3-i-j)r_0 - ir_1 - jr_2 : a_{ij} \neq 0\} = \max\{(3-i-j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.$$

Lemma 7.25. *A plane cubic curve C is semistable if and only if it has no triple point and no double point with a unique tangent. A plane cubic curve C is stable if and only if it is non-singular.*

Proof. Let C be defined by the vanishing of the non-zero degree 3 homogeneous polynomial

$$F(x,y,z) = \sum_{i=0}^{3}\sum_{j=0}^{3-i} a_{ij}x^{3-i-j}y^i z^j.$$

If F (or really the class of F in $Y_{3,n}$) is not semistable, then by the Hilbert–Mumford criterion there is a 1-PS λ of SL_3 such that $\mu(F,\lambda) < 0$. For some $g \in \mathrm{SL}_3$, the 1-PS $\lambda' := g\lambda g^{-1}$ is of the form $\lambda'(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ for integers $r_0 \geq r_1 \geq r_2$ which satisfy $\sum r_i = 0$. Then

$$\mu(g\cdot F, \lambda') = \mu(F,\lambda) < 0.$$

Let us write $F' := g \cdot F = \sum_{i,j} a'_{ij}x^{3-i-j}y^i z^j$; then

$$\lambda'(t)\cdot F'(x,y,z) = \sum_{i,j} t^{-r_0(3-i-j)-r_1 i - r_2 j} a'_{ij} x^{3-i-j}y^i z^j.$$

Since $\mu(F',\lambda') < 0$, we conclude that

$$-\min\{-r_0(3-i-j) - r_1 i - r_2 j : a'_{ij} \neq 0\} = \max\{r_0(3-i-j) + r_1 i + r_2 j : a'_{ij} \neq 0\} < 0;$$

that is, all weights of F' must be positive. The inequalities $r_0 \geq r_1 \geq r_2$ imply that the monomials with non-positive weights are: x^3, x^2y (which have strictly negative weights), and xy^2, x^2z, and xyz. Hence, $\mu(F,\lambda) < 0$ implies $a'_{00} = a'_{10} = a'_{20} = a'_{11} = a'_{01} = 0$ and so $p = [1:0:0]$ is a singular point of F' by Exercise 7.22. Then $g^{-1}\cdot p$ is a singular point of $F = g^{-1}\cdot F'$. Moreover, if $a'_{02} = 0$ also then $[1:0:0]$ is a triple point of F' and if $a_{02} \neq 0$ then $[1:0:0]$ is a double point with a single tangent.

Suppose that $F = \sum a_{ij}x_0^{3-i-j}x_1^i x_2^j$ has a double point with a unique tangent or triple point, then we can assume without loss of generality (by using the action of SL_3) that this point is $p = [1:0:0]$ and that $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} = 0$. Then if $\lambda(t) = \mathrm{diag}(t^3, t^{-1}, t^{-2})$, we see $\mu(F, \lambda) < 0$. Therefore F is semistable if and only if it has no triple point or double point with a unique tangent.

For the second statement, if p is a singular point of C defined by $F = 0$, then using the SL_3-action, we can assume $p = [1 : 0 : 0]$ and so $a_{00} = a_{10} = a_{01} = 0$. For $\lambda(t) = \mathrm{diag}(t^2, t^{-1}, t^{-1})$, we see $\mu(F, \lambda) \leq 0$ by direct calculation; that is, F is not stable.

It remains to show that if F is not stable then F is not smooth. Without loss of generality, using the Hilbert–Mumford criterion and the action of SL_3 we can assume that $\mu(F, \lambda) \leq 0$ for $\lambda(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, t^{r_2})$ where $r_0 \geq r_1 \geq r_2$ and $\sum r_i = 0$. In this case, we must have $a_{00} = a_{10} = 0$, as x^3 and x^2y have strictly negative weights. If also $a_{01} = 0$, then $p = [1 : 0 : 0]$ is a singular point as required. If $a_{01} \neq 0$, then

$$0 \geq \mu(F, \lambda) \geq (2r_0 + r_2). \tag{1}$$

The inequalities between the r_i imply that we must have equality in (1) and so $r_1 = r_0$ and $r_2 = -2r_0$. Then

$$\mu(F, \lambda) = \max\{(3 - 3j)r_0 : a_{ij} = 0\} \leq 0$$

and $r_0 > 0$; thus $a_{20} = a_{30} = 0$. In this case, F is reducible, as z divides F, and any reducible plane cubic has a singular point. □

There are three strictly semistable orbits:

(1) nodal irreducible cubics,
(2) cubics which are a union of a conic and a non-tangential line, and
(3) cubics which are the union of three non-concurrent lines.

The lowest dimensional strictly semistable orbit, which is the orbit of three non-concurrent lines (this has a two dimensional stabiliser group and so the orbit has dimension $6 = \dim \mathrm{SL}_3 - 2$), is closed in the semistable locus. One can show that the closure of the orbit of nodal irreducible cubics (which is 8 dimensional) contains both other strictly semistable orbits. In particular, the compactification of the geometric quotient $Y_{3,2}^s \to Y_{3,2}^s/\mathrm{SL}_3$ of smooth cubics is given by adding a single point corresponding to these three strictly semistable orbits.

The geometric quotient of the stable locus classifies isomorphism classes of non-singular plane cubics, and so via the theory of elliptic curves and the j-invariant, is isomorphic to $\mathbb{A}^1$ (see [14] IV Theorem 4.1). Hence its compactification, which is a good quotient of $Y_{3,2}^{ss}$, is $\mathbb{P}^1$.

The ring of invariants $R(Y_{3,2})^{\mathrm{SL}_2}$ is known to be freely generated by two invariants S and T by a classical result of Aaronhold (1850). In terms of the Weierstrass normal form, we have

$$S = \frac{a}{27} \quad T = \frac{4b}{27},$$

which both vanish on the cuspidal Weierstrass cubic (where $a = b = 0$), and $S \neq 0$ for the nodal Weierstrass cubic, which is strictly semi-stable.

Finally, we list the unstable orbits: cuspidal cubics, cubics which are the union of a conic and a tangent line, cubics which are the union of three lines with a common intersection, cubics which are the union of a double line with a distinct line and cubics which are given by a triple line.