8.4. **Vector bundles and locally free sheaves.** We will often use the equivalence between the category of algebraic vector bundles on X and the category of locally free sheaves. We recall that this equivalence is given by associating to an algebraic vector bundle $F \to X$ the sheaf $\mathcal{F}$ of sections of F. Under this equivalence, the trivial line bundle $X \times \mathbb{A}^1$ on X corresponds to the structure sheaf $\mathcal{O}_X$.

We will use the notation $\mathcal{E}$ to mean a sheaf or locally free sheaf and E to mean a vector bundle. We also denote the stalk of $\mathcal{E}$ at x by $\mathcal{E}_x$ and the fibre of E at x by E_x.

For a smooth projective curve X, the local rings $\mathcal{O}_{X,x}$ are DVRs, which are PIDs. Using this one can show the following.

Exercise 8.18. Prove the following statements for a smooth projective curve X.

a) Any torsion free sheaf on X is locally free.
b) A subsheaf of a locally free sheaf over X is locally free.
c) A non-zero homomorphism $f : \mathcal{L} \to \mathcal{E}$ of locally free sheaves over X with $\operatorname{rk} \mathcal{L} = 1$ is injective.

One should be careful when going between vector bundles and locally free sheaves, as this correspondence does not preserve subobjects. More precisely, if $\mathcal{F}$ is a locally free sheaf with associated vector bundle F and $\mathcal{E} \subset \mathcal{F}$ is a subsheaf, then the map on stalks $\mathcal{E}_x \to \mathcal{F}_x$ is injective for all $x \in X$. However, the map on fibres of the associated vector bundles $E_x \to F_x$ is not necessarily injective, as E_x is obtained by tensoring $\mathcal{E}_x$ with the residue field $k(x) \cong k$, which is not exact in general.

Example 8.19. For an effective divisor D, we have that $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ is a locally free subsheaf but this does not induce a vector subbundle of the trivial line bundle, as a line bundle has no non-trivial vector subbundles.

However, if we have a subsheaf $\mathcal{E}$ of a locally free sheaf $\mathcal{F}$ for which the quotient $\mathcal{G} := \mathcal{F}/\mathcal{E}$ is torsion free (and so locally free, as X is a curve), then the associated vector bundle E is a vector subbundle of F, because if we tensor the short exact sequence

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$$

with the residue field, then we get a long exact sequence

$$\cdots \to \operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) \to E_x \to F_x \to G_x \to 0,$$

where $\operatorname{Tor}^1_{\mathcal{O}_{X,x}}(k, \mathcal{G}_x) = 0$ as $\mathcal{G}_x$ is flat.

Definition 8.20. Let $\mathcal{E}$ be a subsheaf of a locally free sheaf $\mathcal{F}$ and let E and F denote the corresponding vector bundles. Then the *vector subbundle of F generically generated by E* is a vector subbundle $\overline{E}$ of F which is the vector bundle associated the locally free sheaf

$$\overline{\mathcal{E}} := \pi^{-1}(\mathcal{T}(\mathcal{F}/\mathcal{E}))$$

where $\pi : \mathcal{F} \to \mathcal{F}/\mathcal{E}$ and $\mathcal{T}(\mathcal{F}/\mathcal{E})$ denotes the torsion subsheaf of $\mathcal{F}/\mathcal{E}$ (i.e. $(\mathcal{F}/\mathcal{E})/\mathcal{T}(\mathcal{F}/\mathcal{E})$ is torsion free).

Indeed, as $\mathcal{F}/\overline{\mathcal{E}}$ is torsion free (and so locally free), the vector bundle homomorphism associated to $\overline{\mathcal{E}} \to \mathcal{F}$ is injective; that is, $\overline{E}$ is a vector subbundle of F. Furthermore, we have that

$$\operatorname{rk} \overline{\mathcal{E}} = \operatorname{rk} \mathcal{E} \qquad \text{and} \qquad \deg \overline{\mathcal{E}} \geq \mathcal{E}.$$

Example 8.21. Let D be an effective divisor and consider the subsheaf $\mathcal{L}' := \mathcal{O}_X(-D)$ of $\mathcal{L} := \mathcal{O}_X$; then the vector subbundle of L generically generated by L' is $\overline{L'} = L$.

The category of locally free sheaves is not an abelian category and also the category of vector bundles is not abelian. Given a homomorphism of locally free sheaves $f : \mathcal{E} \to \mathcal{G}$, the quotient $\mathcal{E}/\ker f$ may not be locally free (and similarly for the image). Similarly, the kernel (and the image) of a morphism of vector bundles may not be a vector bundle; essentially because the rank can jump. Instead, we can define the vector subbundle that is generically generated by the kernel (and the same for the image) sheaf theoretically.

Definition 8.22. Let $f : E \to F$ be a morphism of vector bundles; then we can define

(1) the vector subbundle K of E generically generated by the kernel Ker$f$, which satisfies
$$\operatorname{rk} K = \operatorname{rk} \operatorname{Ker} f \qquad \deg K \geq \deg \operatorname{Ker} f;$$
(2) the vector subbundle I of F generically generated by the image Image$f$, which satisfies
$$\operatorname{rk} I = \operatorname{rk} \operatorname{Image} f \qquad \deg I \geq \deg \operatorname{Image} f.$$

Exercise 8.23. Let $\mathcal{E}$ be a locally free sheaf of rank r over X. In this exercise, we will prove that there exists a short exact sequence of locally free sheaves over X
$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$
such that $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ has rank $r-1$.

a) Show that for any effective divisor D with $r \deg D > h^1(\mathcal{E})$, the vector bundle $\mathcal{E}(D)$ admits a section by considering the long exact sequence in cohomology associated to the short exact sequence
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0$$
tensored by $\mathcal{E}$ (here k_D denotes the skyscraper sheaf with support D). Deduce that $\mathcal{E}$ has an invertible subsheaf.
b) For an invertible sheaf $\mathcal{L}$ with $\deg \mathcal{L} > 2g-2$, prove that $h^1(\mathcal{L}) = 0$ using Serre duality.
c) Show that the degree of an invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$ is bounded above, using the Riemann–Roch formula for invertible sheaves and part b).
d) Let $\mathcal{L}$ to be an invertible subsheaf of $\mathcal{E}$ of maximal degree; then verify that the quotient $\mathcal{F}$ of $\mathcal{L} \subset \mathcal{E}$ is locally free.

Exercise 8.24. In this exercise, we will prove for locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over X that
$$\deg(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rk} \mathcal{E} \deg \mathcal{F} + \operatorname{rk} \mathcal{F} \deg \mathcal{E}$$
by induction on the rank of $\mathcal{E}$.

a) Prove the base case where $\mathcal{E} = \mathcal{O}_X(D)$ by splitting into two cases. If D is effective, use the short exact sequence
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D \to 0$$
and the Riemann–Roch Theorem to prove the result. If D is not effective, write D as $D_1 - D_2$ for effective divisors D_i and modify $\mathcal{F}$ by twisting by a line bundle.
b) For the inductive step, use Exercise 8.23.

Example 8.25. Let $\mathcal{E}$ be a locally free sheaf of rank r and degree d over a genus g smooth projective curve X; then for any line bundle $\mathcal{L}$, we have that
$$\chi(\mathcal{E} \otimes \mathcal{L}^{\otimes m}) = d + rm \deg \mathcal{L} + r(1-g)$$
is a degree 1 polynomial in m.

8.5. **Semistability.** In order to construct moduli spaces of algebraic vector bundles over a smooth projective curve, Mumford introduced a notion of semistability for algebraic vector bundles. One advantage to restricting to semistable bundles of fixed rank and degree is that the moduli problem is then bounded (without adding the semistability hypothesis, the moduli problem is unbounded; see Example 2.22). A second advantage, which explains the term semistable, is that the notion of semistability for vector bundles corresponds to the notion of semistability coming from an associated GIT problem (which we will describe later on).

Definition 8.26. The slope of a non-zero vector bundle E on X is the ratio
$$\mu(E) := \frac{\deg E}{\operatorname{rk} E}.$$

Remark 8.27. Since the degree and rank are both additive on short exact sequences of vector bundles
$$0 \to E \to F \to G \to 0,$$
it follows that

(1) If two out of the three bundles have the same slope μ, the third also has slope μ;
(2) $\mu(E) < \mu(F)$ (resp. $\mu(E) > \mu(F)$) if and only if $\mu(F) < \mu(G)$ (resp. $\mu(F) > \mu(G)$).

Definition 8.28. A vector bundle E is stable (resp. semistable) if every proper non-zero vector subbundle $S \subset E$ satisfies

$$\mu(S) < \mu(E) \qquad \text{(resp. } \mu(S) \leq \mu(E) \text{ for semistability).}$$

A vector bundle E is polystable if it is a direct sum of stable bundles of the same slope.

Remark 8.29. If we fix a rank n and degree d such that n and d are coprime, then the notion of semistability for vector bundles with invariants (n, d) coincides with the notion of stability.

Lemma 8.30. *Let L be a line bundle and E a vector bundle over X; then*

i) L is stable.
ii) If E is stable (resp. semistable), then $E \otimes L$ is stable (resp. semistable).

Proof. Exercise. □

Lemma 8.31. *Let $f : E \to F$ be a non-zero homomorphism of vector bundles over X; then*

i) If E and F are semistable, $\mu(E) \leq \mu(F)$.
ii) If E and F are stable of the same slope, then f is an isomorphism.
iii) Every stable vector bundle E is simple i.e. $\mathrm{End}\, E = k$.

Proof. Exercise. □

If E is a vector bundle which is not semistable, then there exists a subbundle $E' \subset E$ with larger slope that E, by taking the sum of all vector subbundles of E with maximal slope, one obtains a unique maximal destabilising vector subbundle of E, which is semistable by construction. By iterating this process, one obtains a unique maximal destabilising filtration of E known as the Harder–Narasimhan filtration of E [13].

Definition 8.32. Let E be a vector bundle; then E has a Harder–Narasimhan filtration

$$0 = E^{(0)} \subsetneq E^{(1)} \subsetneq \cdots \subsetneq E^{(s)} = E$$

where $E_i := \mathcal{E}^{(i)}/E^{(i-1)}$ are semistable with slopes

$$\mu(E_1) > \mu(E_2) > \cdots > \mu(E_s).$$

As we have already mentioned, the moduli problem of vector bundles on X with fixed rank n and degree d is unbounded. Therefore, we restrict to the moduli functors $\mathcal{M}^{(s)s}(n, d)$ of (semi)stable locally free sheaves. Let us refine our notion of families to families of semistable vector bundles.

Definition 8.33. A family over a scheme S of (semi)stable vector bundles on X with invariants (n, d) is a coherent sheaf $\mathcal{E}$ over $X \times S$ which is flat over S and such that for each $s \in S$, the sheaf $\mathcal{E}_s$ is a (semi)stable vector bundle on X with invariants (n, d).

We say two families $\mathcal{E}$ and $\mathcal{F}$ over S are equivalent if there exists an invertible sheaf $\mathcal{L}$ over S and an isomorphism

$$\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}$$

where $\pi_S : X \times S \to S$ denotes the projection.

Lemma 8.34. *If there exists a semistable vector bundle over X with invariants (n, d) which is not polystable, then the moduli problem of semistable vector bundles $\mathcal{M}^{ss}(n, d)$ does not admit a coarse moduli space.*

Proof. If there exists a semistable sheaf $\mathcal{F}$ on X which is not polystable, then there is a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable vector bundles with the same slope as $\mathcal{F}$. The above short exact sequence corresponds to a non-zero point in $\mathrm{Ext}(\mathcal{F}'', \mathcal{F}')$ and if we take the affine line through this extension class then we obtain a family of extensions over $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$. More precisely, let $\mathcal{E}$ be the coherent sheaf over $X \times \mathbb{A}^1$ given by this extension class. Then $\mathcal{E}$ is flat over $\mathbb{A}^1$ and

$$\mathcal{E}_t \cong \mathcal{F} \quad \text{for } t \neq 0, \quad \text{and} \quad \mathcal{E}_0 \cong \mathcal{F}'' \oplus \mathcal{F}'.$$

Since $\mathcal{E}$ is a family of semistable locally free sheaves with the fixed invariants (n, d) which exhibits the jump phenomenon, there is no coarse moduli space by Lemma 2.27. □

However, this cannot happen if the notions of semistability and stability coincide, which happens when n and d are coprime.

8.6. **Boundedness of semistable vector bundles.** To construct a moduli space of vector bundles on X using GIT, we would like to find a scheme R that parametrises a family $\mathcal{F}$ of semistable vector bundles on X of fixed rank n and degree d such that any vector bundle of the given invariants is isomorphic to $\mathcal{F}_p$ for some $p \in R$. In this section, we prove an important boundedness result for the family of semistable vector bundles on X of fixed rank and degree which will enable us to construct such a scheme. In fact, we will show that we can construct a scheme R which parametrises a family with the local universal property.

First, we note that we can assume, without loss of generality, that the degree of our vector bundle is sufficiently large: for, if we take a line bundle $\mathcal{L}$ of degree e, then tensoring with $\mathcal{L}$ preserves (semi)stability and so induces an isomorphism between the moduli functor of (semi)stable vector bundles with rank and degree (n, d) and those with rank and degree $(n, d + ne)$

$$- \otimes \mathcal{L} : \mathcal{M}^{ss}(n, d) \cong \mathcal{M}^{ss}(n, d + ne).$$

Hence, we can assume that $d > n(2g - 1)$ where g is the genus of X. This assumption will be used to prove the boundedness result for semistable vector bundles. However, first we need to recall the definition of a sheaf being generated by global sections.

Definition 8.35. A sheaf $\mathcal{F}$ is generated by its global sections if the natural evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

is a surjection.

Lemma 8.36. *Let $\mathcal{F}$ be a locally free sheaf over X of rank n and degree $d > n(2g - 1)$. If the associated vector bundle F is semistable, then the following statements hold:*

i) $H^1(X, \mathcal{F}) = 0$;
ii) $\mathcal{F}$ is generated by its global sections.

Proof. For i), we argue by contradiction using Serre duality: if $H^1(X, \mathcal{F}) \neq 0$, then dually there would be a non-zero homomorphism $f : \mathcal{F} \to \omega_X$. We let $K \subset F$ be the vector subbundle generically generated by the kernel of f which is a vector subbundle of rank $n - 1$ with

$$\deg K \geq \deg \ker f \geq \deg \mathcal{F} - \deg \omega_X = d - (2g - 2).$$

In this case, by semistability of F, we have

$$\frac{d - (2g - 2)}{n - 1} \leq \mu(K) \leq \mu(F) = \frac{d}{n};$$

this gives $d \leq n(2g - 2)$, which contradicts our assumption on the degree of F.

For ii), we let F_x denote the fibre of the vector bundle at a point $x \in X$. If we consider the fibre F_x as a torsion sheaf over X, then we have a short exact sequence

$$0 \to \mathcal{F}(-x) := \mathcal{O}_X(-x) \otimes \mathcal{F} \to \mathcal{F} \to F_x = \mathcal{F} \otimes k_x \to 0$$

which gives rise to an associated long exact sequence in cohomology

$$0 \to H^0(X, \mathcal{F}(-x)) \to H^0(X, \mathcal{F}) \to H^0(F_x) \to H^1(X, \mathcal{F}(-x)) \to \cdots.$$

Then we need to prove that, for each $x \in X$, the map $H^0(X, \mathcal{F}) \to H^0(X, F_x) = F_x$ is surjective. We prove this map is surjective by showing that $H^1(X, \mathcal{F}(-x)) = 0$ using the same argument as in part i) above, where we use the fact that twisting by a line bundle does not change semistability: $\mathcal{F}(-x) := \mathcal{O}_X(-x) \otimes \mathcal{F}$ is semistable with degree $d - n > n(2g - 2)$ and thus $H^1(X, \mathcal{O}_X(-x) \otimes \mathcal{F}) = 0$. □

As mentioned above, these two properties are important for showing boundedness. In fact, we will see that a strictly larger family of vector bundles of fixed rank and degree are bounded; namely those that are generated by their global sections and have vanishing 1st cohomology. Given a locally free sheaf $\mathcal{F}$ of rank n and degree d that is generated by its global sections, we can consider the evaluation map

$$\mathrm{ev}_{\mathcal{F}} : H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F}$$

which is, by assumption, surjective. If also $H^1(X, \mathcal{F}) = 0$, then by the Riemann–Roch formula

$$\chi(\mathcal{F}) = d + n(1-g) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}) = \dim H^0(X, \mathcal{F});$$

that is, the dimension of the 0th cohomology is fixed and equal to $N := d + n(1-g)$. Therefore, we can choose an isomorphism $H^0(X, \mathcal{F}) \cong k^N$ and combine this with the evaluation map for $\mathcal{F}$, to produce a surjection

$$q_{\mathcal{F}} : \mathcal{O}_X^N = k^N \otimes \mathcal{O}_X \to \mathcal{F}$$

from a fixed trivial vector bundle. Such surjective homomorphisms from a fixed coherent sheaf are parametrised by a projective scheme known as a Quot scheme, which generalises the Grassmannians.