8.7. **The Quot scheme.** The Quot scheme is a fine moduli space which generalises the Grassmannian in the sense that it parametrises quotients of a fixed sheaf. In this section, we will define the moduli problem that the Quot scheme represents and give an overview of the construction of the Quot scheme following [33].

Let Y be a projective scheme and $\mathcal{F}$ be a fixed coherent sheaf on X. Then one can consider the moduli problem of classifying quotients of $\mathcal{F}$. More precisely, we consider surjective sheaf homomorphisms $q : \mathcal{F} \to \mathcal{E}$ up to the equivalence relation

$$(q : \mathcal{F} \to \mathcal{E}) \sim (q' : \mathcal{F} \to \mathcal{E}') \iff \ker q = \ker q'.$$

Equivalently, there is a sheaf isomorphism $\phi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F} \arrow[r, "q"] \arrow[d, "\mathrm{Id}"'] & \mathcal{E} \arrow[d, "\phi"] \\ \mathcal{F} \arrow[r, "q'"'] & \mathcal{E}'. \end{tikzcd}$$

This gives the naive moduli problem and the following definition of families gives the extended moduli problem.

Definition 8.37. Let $\mathcal{F}$ be a coherent sheaf over Y. Then for any scheme S, we let $\mathcal{F}_S := \pi_Y^*\mathcal{F}$ denote the pullback of $\mathcal{F}$ to $Y \times S$ via the projection $\pi_Y : Y \times S \to Y$. A *family of quotients of $\mathcal{F}$ over a scheme S* is a surjective $\mathcal{O}_{Y\times S}$-linear homomorphism of sheaves over $Y \times S$

$$q_S : \mathcal{F}_S \to \mathcal{E},$$

such that $\mathcal{E}$ is flat over S. Two families $q_S : \mathcal{F}_S \to \mathcal{E}$ and $q'_S : \mathcal{F}_S \to \mathcal{E}'$ are equivalent if $\ker q_S = \ker q'_S$. It is easy to check that we can pullback families, as flatness is preserved by base change; therefore, we let

$$\mathcal{Q}uot_Y(\mathcal{F}) : \mathrm{Sch} \to \mathrm{Set}$$

denote the associated moduli functor.

Remark 8.38.

(1) With these definitions, it is clear that we can think of the Quot scheme as instead parametrising coherent subsheaves of $\mathcal{F}$ up to equality rather than quotients of $\mathcal{F}$ up to the above equivalence. Indeed this perspective can also be taken with the Grassmannian (and even projective space). For us, the quotient perspective will be the most useful.

(2) For the moduli problem of the Grassmannian, we fix the dimension of the quotient vector spaces. Similarly for the quotient moduli problem, we can fix invariants, as for two quotient sheaves to be equivalent, they must be isomorphic. Thus we can refine the above moduli functor by fixing the invariants of our quotient sheaves.

Definition 8.39. For a coherent sheaf $\mathcal{E}$ over a projective scheme Y equipped with a fixed ample invertible sheaf $\mathcal{L}$, the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{L}$ is a polynomial $P(\mathcal{E}, \mathcal{L}) \in \mathbb{Q}[t]$ such that for $l \in \mathbb{N}$ sufficiently large,

$$P(\mathcal{E}, \mathcal{L}, l) = \chi(\mathcal{E} \otimes \mathcal{L}^{\otimes l}) := \sum_{i\geq 0} (-1)^i \dim H^i(Y, \mathcal{E} \otimes \mathcal{L}^{\otimes l}).$$

Serre's vanishing theorem states that for l sufficiently large (depending on $\mathcal{E}$), all the higher cohomology groups of $\mathcal{E} \otimes \mathcal{L}^{\otimes l}$ vanish (see [14] III Theorem 5.2). Hence, for l sufficiently large, $P(\mathcal{E}, \mathcal{L}, l) = \dim H^0(Y, \mathcal{E} \otimes \mathcal{L}^{\otimes l})$.

The proof that there is such a polynomial is given by reducing to the case of $\mathbb{P}^n$ (as L is ample, we can use a power of L to embed X into a projective space) and then the proof proceeds by induction on the dimension d of the support of the sheaf (where the inductive step is given by restricting to a hypersurface and the base case $d = 0$ is trivial as the Hilbert polynomial is constant); for a proof, see [16] Lemma 1.2.1. However, for a smooth projective curve X, we can explicitly write down the Hilbert polynomial of a locally free sheaf over X using the Riemann–Roch Theorem.

Example 8.40. On a smooth projective genus g curve X, we fix a degree 1 line bundle $\mathcal{L} = \mathcal{O}_X(x) =: \mathcal{O}_X(1)$. For a vector bundle $\mathcal{E}$ over X of rank n and degree d, the twist $\mathcal{E}(m) := \mathcal{E} \otimes \mathcal{O}_X(m)$ has rank n and degree $d + mn$. The Riemann–Roch formula gives

$$\chi(\mathcal{E}(m)) = d + mn + n(1 - g).$$

Thus $\mathcal{E}$ has Hilbert polynomial $P(t) = nt + d + n(1 - g)$ of degree 1 with leading coefficient given by the rank n.

Definition 8.41. For a fixed ample line bundle L on Y, we have a decomposition

$$\mathcal{Q}uot_Y(\mathcal{F}) = \bigsqcup_{P \in \mathbb{Q}[t]} \mathcal{Q}uot_Y^{P,L}(\mathcal{F})$$

into Hilbert polynomials P taken with respect to L.

If $Y = X$ is a curve, then we have a decomposition of the Quot moduli functor by ranks and degrees of the quotient sheaf:

$$\mathcal{Q}uot_X(\mathcal{F}) = \bigsqcup_{(n,d)} \mathcal{Q}uot_X^{n,d}(\mathcal{F}).$$

Example 8.42. The grassmannian moduli functor is a special example of the Quot moduli functor:

$$\mathcal{G}r(d, n) = \mathcal{Q}uot^{n-d}_{\operatorname{Spec} k}(k^n).$$

Theorem 8.43 (Grothendieck)**.** *Let Y be a projective scheme and $\mathcal{L}$ an ample invertible sheaf on Y. Then for any coherent sheaf $\mathcal{F}$ over Y and any polynomial P, the functor $\mathcal{Q}uot_Y^{P,\mathcal{L}}(\mathcal{F})$ is represented by a projective scheme $\mathrm{Quot}_Y^{P,\mathcal{L}}(\mathcal{F})$.*

The idea of the construction is very beautiful but also technical; therefore, we will just give an outline of a proof. We split the proof up into the 4 following steps.

Step 1. Reduce to the case where $Y = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{F}$ is a trivial vector bundle $\mathcal{O}_{\mathbb{P}^n}^N$.

Step 2. For m sufficiently large, construct an injective natural transformation of moduli functors

$$\mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N) \hookrightarrow \mathcal{G}r(V, P(m))$$

to the Grassmannian moduli functor of $P(m)$-dimensional quotients of $V := k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m))$.

Step 3. Prove that $\mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N)$ is represented by a locally closed subscheme of $\mathrm{Gr}(V, P(m))$.

Step 4. Prove that the Quot scheme is proper using the valuative criterion for properness.

Before we explain the proof of each step, we need the following definition.

Definition 8.44. A natural transformation of presheaves $\eta : \mathcal{M}' \to \mathcal{M}$ is a closed (resp. open, resp. locally closed) immersion if η_S is injective for every scheme S and moreover, for any natural transformation $\gamma : h_S \to \mathcal{M}$ from the functor of points of a scheme S, there is a closed (resp. open, resp. locally closed) subscheme $S' \subset S$ such that

$$h_{S'} \cong \mathcal{M}' \times_{\mathcal{M}} h_S$$

where the fibre product is given by

$$(\mathcal{M}' \times_{\mathcal{M}} h_S)(T) = \left\{(f : T \to S \in h_S(T), F \in \mathcal{M}'(T)) : \gamma_T(f) = \eta_T(F) \in \mathcal{M}(T)\right\}.$$

Sketch of Step 1. First, we can assume that L is very ample by replacing L by a sufficiently large positive power of L; this will only change the Hilbert polynomial. Then L defines a projective embedding $i : Y \hookrightarrow \mathbb{P}^n$ such that L is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$. Since i is a closed immersion, $i_*\mathcal{F}$ is coherent and, moreover i_* is exact; therefore, we can push-forward quotient sheaves on Y to $\mathbb{P}^n$. Hence, one obtains a natural transformation

$$\mathcal{Q}uot_Y(\mathcal{F}) \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F}),$$

which is injective as $i^*i_* = \mathrm{Id}$. We claim that this natural transformation is a closed immersion in the sense of the above definition. More precisely, we claim for any scheme S and natural transformation $h_S \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F})$ there exists a closed subscheme $S' \subset S$ with the following property: a morphism $f : T \to S$ determines a family in $\mathcal{Q}uot_Y(\mathcal{F})(T)$ if and only if the morphism f factors via S'. To define the closed subscheme associated to a map $\eta : h_S \to \mathcal{Q}uot_{\mathbb{P}^n}(i_*\mathcal{F})$, we let $(i_*\mathcal{F})_S \twoheadrightarrow \mathcal{E}$ denote the family over S of quotients associated to $\eta_S(\mathrm{id}_S)$ and apply $(\mathrm{id}_S \times i)^*$ to obtain a homomorphism of sheaves over $Y \times S$

$$(\mathcal{F})_S \cong i^*(i_*\mathcal{F})_S \to i^*\mathcal{E};$$

then we take $S' \subset S$ to be the closed subscheme on which this homomorphism is surjective (the fact that this is closed follows from a semi-continuity argument). Hence, we may assume that $(Y, L) = (\mathbb{P}^n, \mathcal{O}(1))$.

We can tensor any quotient sheaf by a power of $\mathcal{O}(1)$ and this induces a natural transformation between

$$\mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F}) \cong \mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F} \otimes \mathcal{O}(r))$$

(under this natural transformation the Hilbert polynomial undergoes a explicit transformation corresponding to this tensorisation). Hence, by replacing $\mathcal{F}$ with $\mathcal{F}(r) := \mathcal{F} \otimes \mathcal{O}(r)$, we can assume without loss of generality that $\mathcal{F}$ has trivial higher cohomology groups and is globally generated; that is, the evaluation map

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus N} \cong H^0(\mathbb{P}^n, \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}$$

is surjective and $N = P(\mathcal{F}, 0)$. By composition, this surjection induces a natural transformation

$$\mathcal{Q}uot_{\mathbb{P}^n}(\mathcal{F}) \to \mathcal{Q}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}^N)$$

which can also be shown to be a closed embedding. In conclusion, we obtain a natural transformation

$$\mathcal{Q}uot_Y^{P',L}(\mathcal{F}) \to \mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N)$$

which is a closed embedding of moduli functors.

Sketch of Step 2. By a result of Mumford and Castelnuovo concerning 'Castelnuovo–Mumford regularity' of subsheaves of $\mathcal{O}_{\mathbb{P}^n}^N$, there exists $M \in \mathbb{N}$ (depending on P, n and N) such that for all $m \geq M$, the following holds for any short exact sequence of sheaves

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^n}^N \to \mathcal{F} \to 0$$

such that $\mathcal{F}$ has Hilbert polynomial P:

(1) the sheaf cohomology groups H^i of $\mathcal{K}(m)$, $\mathcal{F}(m)$ vanish for $i > 0$,
(2) $\mathcal{K}(m)$ and $\mathcal{F}(m)$ are globally generated.

The proof of this result is by induction on n, where one restricts to a hyperplane $H \cong \mathbb{P}^{n-1}$ in $\mathbb{P}^n$ to do the inductive step; for a full proof, see [33] Theorem 2.3. Now if we fix $m \geq M$, we claim there is a natural transformation

$$\eta : \mathcal{Q}uot_{\mathbb{P}^n}^{P,\mathcal{O}(1)}(\mathcal{O}_{\mathbb{P}^n}^N) \hookrightarrow \mathcal{G}r(k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m)), P(m)).$$

First, let us define this for families over $S = \operatorname{Spec} k$: for a quotient $q : \mathcal{O}_{\mathbb{P}^n}^{\oplus N} \twoheadrightarrow \mathcal{F}$ with kernel $\mathcal{K}$, we have an associated long exact sequence in cohomology

$$0 \to H^0(\mathcal{K}(m)) \to H^0(\mathcal{O}_{\mathbb{P}^n}^N(m)) \to H^0(\mathcal{F}(m)) \to H^1(\mathcal{K}(m)) = 0.$$

Hence, we define

$$\eta_{\operatorname{Spec} k}(q : \mathcal{O}_{\mathbb{P}^n}^N \twoheadrightarrow \mathcal{F}) = \left(H^0(q(m)) : W \twoheadrightarrow H^0(\mathcal{F}(m))\right),$$

where $W := H^0(\mathcal{O}_{\mathbb{P}^n}^N(m)) = k^N \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ and we have $\dim H^0(\mathcal{F}(m)) = P(m)$, as all higher cohomology of $\mathcal{F}(m)$ vanishes.

To define η_S for a family of quotients $q_S : \mathcal{O}_{\mathbb{P}^n \times S}^N \to \mathcal{E}$ over an arbitrary scheme S, we essentially do the above process in a family. More precisely, we let $\pi_S : X \times S \to S$ be the projection and push forward $q_S(m)$ by π_S to S to obtain a surjective homomorphism of sheaves over S

$$\mathcal{O}_S \otimes W \cong \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) \to \pi_{S*}(\mathcal{E}(m)).$$

By our assumptions on m and the semi-continuity theorem, $\pi_{S*}(\mathcal{E}(m))$ is locally free of rank $P(m)$ (as the higher rank direct images of $\mathcal{E}(m)$ vanish so the claim follows by EGA III 7.9.9). Hence, we have a family of $P(m)$-dimensional quotients of W over S, which defines the desired S-point in the Grassmannian.

Let $\mathrm{Gr} = \mathrm{Gr}(W, P(m))$. We claim this natural transformation η is an injection. Let us explain how to reconstruct q_S from the morphism $f_{q_s} : S \to \mathrm{Gr}$ corresponding to the surjection

$$\pi_{S*}(q_s(m)) : \mathcal{O}_S \otimes W \to \pi_{S*}(\mathcal{E}(m)).$$

Over the Grassmannian, we have a universal inclusion (and a corresponding surjection)

$$\mathcal{K}_{\mathrm{Gr}} \hookrightarrow \mathcal{O}_{\mathrm{Gr}} \otimes W,$$

whose pullback to S via the morphism $S \to \mathrm{Gr}$ is the homomorphism

$$\pi_S^* \pi_{S*}(\mathcal{K}_S(m)) \to V \otimes \mathcal{O}_S = \pi_S^* \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)),$$

where $\mathcal{K}_S := \ker q_S$. We claim that the homomorphism

$$f : \pi_S^* \pi_{S*}(\mathcal{K}_S(m)) \to \pi_S^* \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) \to \mathcal{O}_{\mathbb{P}^n \times S}^N(m)$$

has cokernel $q_S(m)$. To prove the claim, consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_S^* \pi_{S*}(\mathcal{K}_S(m)) \arrow[r] \arrow[d] & \pi_S^* \pi_{S*}(\mathcal{O}_{\mathbb{P}^n \times S}^N(m)) \arrow[r] \arrow[d] & \pi_S^* \pi_{S*}(\mathcal{E}(m)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}_S(m) \arrow[r] & \mathcal{O}_{\mathbb{P}^n \times S}^N(m) \arrow[r] & \mathcal{E}(m) \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact and whose columns are surjective by our assumption on m. Finally, we can recover q_S from $q_S(m)$ by twisting by $\mathcal{O}(-m)$.

Sketch of Step 3. For any morphism $f : T \to \mathrm{Gr} = \mathrm{Gr}(W, P(m))$, we let $\mathcal{K}_{T,f}$ denote the pullback of the universal subsheaf $\mathcal{K}_{\mathrm{Gr}}$ on the Grassmannian to T via f. Then consider the induced composition

$$h_{T,f} : \pi_T^* \mathcal{K}_{T,f} \to \pi_T^*(W \otimes \mathcal{O}_T) \cong \pi_T^* \pi_{T*}(\mathcal{O}_{\mathbb{P}^n \times T}^N(m)) \to \mathcal{O}_{T \times \mathbb{P}^n}^N(m)$$

where $\pi_T : \mathbb{P}^n \times T \to T$ denotes the projection. Let $q_{T,f}(m) : \mathcal{O}_{\mathbb{P}^n \times T}^N(m) \to \mathcal{F}_{T,f}(m)$ denote the cokernel of $h_{T,f}$; then $\mathcal{F}_{T,f}$ is a coherent sheaf over $\mathbb{P}^n \times T$.

We claim that the natural transformation defined in Step 2 is a locally closed immersion. To prove the claim, we need to show for any morphism $h : S \to \mathrm{Gr}$, there is a unique locally closed subscheme $S' \hookrightarrow S$ with the property that a morphism $f : T \to S$ factors via $S' \hookrightarrow S$ if and only if the sheaf $\mathcal{F}_{T,h\circ f}$ is flat over T and has Hilbert polynomial P at each $t \in T$. This locally closed subscheme $S' \subset S$ is constructed as the stratum with Hilbert polynomial P in the flattening stratification for the sheaf $\mathcal{F}_{T,f'}$ over $T \times \mathbb{P}^n$. For details, see [33] Theorem 4.3.

Let $\mathrm{Quot}_{\mathbb{P}^n}^P(\mathcal{O}_{\mathbb{P}^n}^N)$ be the locally closed subscheme of Gr associated to the identity morphism on Gr (which corresponds to the universal family on the Grassmannian); then it follows from the above arguments that this scheme represents the functor $\mathcal{Q}uot_{\mathbb{P}^n}^P(\mathcal{O}_{\mathbb{P}^n}^N)$.

The Grassmannian $\mathrm{Gr} = \mathrm{Gr}(W, P(m))$ has its Pücker embedding into projective space

$$\mathrm{Gr}(W, P(m)) \hookrightarrow \mathbb{P}(\wedge^{P(m)} W^\vee).$$

Therefore, we have a locally closed embedding

$$\mathrm{Quot}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}^{\oplus N}, P) \hookrightarrow \mathbb{P}(\wedge^{P(m)} W^\vee). \tag{6}$$

In particular, the Quot scheme is quasi-projective; hence, separated and of finite type.

Sketch of Step 4. We will prove the valuative criterion for the Quot scheme using its moduli functor. The Quot scheme $\mathrm{Quot}^P_{\mathbb{P}^n}(\mathcal{O}^{\oplus N}_{\mathbb{P}^n})$ is proper over $\operatorname{Spec} k$ if and only if for every discrete valuation ring R over k with quotient field K, the restriction map

$$\mathcal{Q}uot^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})(\operatorname{Spec} R) \to \mathcal{Q}uot^P_{\mathbb{P}^n}(\mathcal{O}^N_{\mathbb{P}^n})(\operatorname{Spec} K)$$

is bijective. Since the Quot scheme is separated, we already know that this map is injective. Let $j : \mathbb{P}^n_K \hookrightarrow \mathbb{P}^n_R$ denote the open immersion. Any quotient sheaf $q_K : \mathcal{O}^N_{\mathbb{P}^n_K} \to \mathcal{F}_K$ can be lifted to a quotient sheaf $q_R : \mathcal{O}^N_{\mathbb{P}^n_R} \to \mathcal{F}_R$ where $\mathcal{F}_R$ is the image of the homomorphism

$$q_R : \mathcal{O}^{\oplus N}_{\mathbb{P}^n_R} \to j_*(\mathcal{O}^{\oplus N}_{\mathbb{P}^n_K}) \to j_*\mathcal{F}_K.$$

The sheaf $\mathcal{F}_R$ is torsion free as it as a subsheaf of $j_*\mathcal{F}_K$, which is torsion free as j^* is exact, $j^*j_*\mathcal{F}_K = \mathcal{F}_K$ and $\mathcal{F}_K$ is torsion free (as it is flat over K). Hence, $\mathcal{F}_R$ is flat over R, as over a DVR flat is equivalent to torsion free and so this gives a well defined R-valued point of the Quot scheme. It remains to check that the image of q_R under the restriction map is q_K. As j^* is left exact, the map $j^*\mathcal{F}_R \to j^*j_*\mathcal{F}_K = \mathcal{F}_K$ is injective and the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}^{\oplus N}_{\mathbb{P}^n_K} \arrow[r, two heads, "j^*q_R"] \arrow[dr, "q_K"'] & j^*\mathcal{F}_R \arrow[d] \\
& \mathcal{F}_K
\end{tikzcd}$$

implies that the vertical homomorphism must also be surjective; thus $j^*\mathcal{F}_R \cong j^*j_*\mathcal{F}_K = \mathcal{F}_K$ as required. Hence, the Quot scheme is proper over $\operatorname{Spec} k$.

Since $\mathrm{Quot}_{\mathbb{P}^n}(\mathcal{O}^{\oplus N}_{\mathbb{P}^n}; P)$ is proper over $\operatorname{Spec} k$, the embedding (6) is a closed embedding.

Remark 8.45.

(1) As the Quot scheme $Q := \mathrm{Quot}^{P,\mathcal{L}}_Y(\mathcal{F})$ is a fine moduli space, the identity morphism on Q corresponds to a universal quotient homomorphism

$$\pi_Y^*\mathcal{F} \twoheadrightarrow \mathcal{U}$$

over $Q \times Y$, where $\pi_Y : Q \times Y \to Y$ denotes the projection to Y.

(2) The Quot scheme can also be defined in the relative setting, where we replace our field k by a general base scheme S and look at quotients of a fixed coherent sheaf on a scheme $X \to S$; the construction in the relative case is carried out in [33].

The Hilbert schemes are special examples of Quot schemes, which also play an important role in the construction of many moduli spaces.

Definition 8.46. A *Hilbert scheme* is a Quot scheme of the form $\mathrm{Quot}^P_Y(\mathcal{O}_Y)$ and represents the moduli functor that sends a scheme S to the set of closed subschemes $Z \subset Y \times S$ that are proper and flat over S with the given Hilbert polynomial.

Exercise 8.47. For a natural number $d \geq 1$, prove that the Hilbert scheme

$$\mathrm{Quot}^d_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1})$$

is isomorphic to $\mathbb{P}^d$ by showing they both have the same functor of points in the following way.

a) Show that any family $Z \subset \mathbb{P}^1$ over $\operatorname{Spec} k$ in this Hilbert scheme is a degree d hypersurface in $\mathbb{P}^1$.

b) Let S be a scheme and $\pi_S : \mathbb{P}^1_S := \mathbb{P}^1 \times S \to S$ denote the projection. Show that any family $Z \subset \mathbb{P}^1_S$ over S in this Hilbert scheme is a Cartier divisor in $\mathbb{P}^1_S$ and so there is a line subbundle of $\pi_{S*}(\mathcal{O}_{\mathbb{P}^1_S}(d))$ which determines a morphism $f_Z : S \to \mathbb{P}^d$. In particular, this gives a natural transformation

$$\mathcal{Q}uot^d_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) \to h_{\mathbb{P}^d}.$$

c) Construct the inverse to the above natural transformation using the tautological family of degree d hypersurfaces in $\mathbb{P}^1$ parametrised by $\mathbb{P}^d$.

8.8. **GIT set up for construction of the moduli space.** Throughout this section, we fix a connected smooth projective curve X and we assume the genus of X is greater than or equal to 2 to avoid special cases in low genus. We fix a rank n and a degree $d > n(2g-1)$ (recall that tensoring with a line bundle does not alter semistability and so we can pick the degree to be arbitrarily large; in fact, eventually we will choose d to be even larger). It follows from Lemma 8.36 that any locally free semistable sheaf $\mathcal{E}$ of rank r and degree d is globally generated with $H^1(X,\mathcal{E}) = 0$. By the Riemann–Roch Theorem,

$$\dim H^0(X,\mathcal{E}) = d + n(1-g) =: N.$$

If we choose an identification $H^0(X,\mathcal{E}) \cong k^N$, then the evaluation map

$$H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \to \mathcal{E},$$

which is surjective as $\mathcal{E}$ is globally generated, determines a quotient sheaf $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{E}$.

Let $Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ be the Quot scheme of quotient sheaves of the trivial rank N vector bundle $\mathcal{O}_X^N$ of rank n and degree d. Let $R^{(s)s} \subset Q$ denote the open subscheme consisting of quotients $q : \mathcal{O}_X^N \to \mathcal{F}$ such that $\mathcal{F}$ is a (semi)stable locally free sheaf and $H^0(q)$ is an isomorphism. For a proof that these conditions are open see [16] Proposition 2.3.1.

The Quot scheme Q parametrises a universal quotient

$$q_Q : \mathcal{O}_{Q\times X}^N \twoheadrightarrow \mathcal{U}$$

and we let $q^{(s)s} : \mathcal{O}^N_{R^{(s)s}\times X} \to \mathcal{U}^{(s)s} := \mathcal{U}|_{R^{(s)s}}$ denote the restriction to $R^{(s)s}$.

Lemma 8.48. *The universal quotient sheaf $\mathcal{U}^{(s)s}$ over $R^{(s)s} \times X$ is a family over $R^{(s)s}$ of (semi)stable locally free sheaves over X with invariants (n,d) with the local universal property.*

Proof. Let $\mathcal{F}$ be a family over a scheme S of (semi)stable locally free sheaves over X with fixed invariants (n,d). Then for each $s \in S$, the locally free semistable sheaf $\mathcal{F}_s$ is globally generated with vanishing first cohomology by our assumption on d. Therefore, by the semi-continuity Theorem, $\pi_{S*}\mathcal{F}$ is a locally free sheaf over S of rank $N = d+n(1-g)$. For each $s \in S$, we need to show there is an open neighbourhood $U \subset S$ of $s \in S$ and a morphism $f : S \to R^{(s)s}$ such that $\mathcal{F}|_U \sim f^*\mathcal{U}^{(s)s}$. Pick an open neighbourhood $U \ni s$ on which $\pi_{S*}\mathcal{F}$ is trivial; that is, we have an isomorphism

$$\Phi : \mathcal{O}_U^N \cong (\pi_{S*}\mathcal{F})|_U.$$

Then the surjective homomorphism of sheaves over $U \times X$

$$q_U : \mathcal{O}^N_{U\times X} \xrightarrow{\pi_U^*\Phi} \pi_U^*\pi_{U*}(\mathcal{F}|_U) \longrightarrow \mathcal{F}|_U$$

determines a morphism $f : U \to Q$ to the quot scheme such that there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}^N_{U\times X} \arrow[r, "q_u"] \arrow[d, "\mathrm{Id}"'] & \mathcal{F}|_U \arrow[d, "\cong"] \\ \mathcal{O}^N_{U\times X} \arrow[r, "f^*q_Q"] & f^*\mathcal{U} \end{tikzcd}$$

In particular $\mathcal{F}|_U \cong f^*\mathcal{U}$ and, as $\mathcal{F}$ is a family of (semi)stable vector bundles, the morphism $f : U \to Q$ factors via $R^{(s)s}$. □

These families $\mathcal{U}^{(s)s}$ over $R^{(s)s}$ are not universal families as the morphisms described above are not unique: if we take $S = \mathrm{Spec}\, k$ and $\mathcal{E}$ to be a (semi)stable locally free sheaf, then different choices of isomorphism $H^0(X,\mathcal{E}) \cong k^N$ give rise to different points in $R^{(s)s}$.

Any two choices of the above isomorphism are related by an element in the general linear group GL_N and so it is natural to mod out by the action of this group.

Lemma 8.49. *There is an action of GL_N on $Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ such that the orbits in $R^{(s)s}$ are in bijective correspondence with the isomorphism classes of (semi)stable locally free sheaves on X with invariants (n,d).*

Proof. We claim there is a (left) action

$$\sigma : \mathrm{GL}_N \times Q \to Q$$

which on k-points is given by

$$g \cdot (\mathcal{O}_X^N \xrightarrow{q} \mathcal{E}) = (\mathcal{O}_X^N \xrightarrow{g^{-1}} \mathcal{O}_X^N \xrightarrow{q} \mathcal{E}).$$

To construct the action morphism it suffices to give a family over $\mathrm{GL}_N \times Q$ of quotients of $\mathcal{O}_X^N$ with invariants (n, d). The inverse map on the group $i^{-1} : \mathrm{GL}_N \to \mathrm{GL}_N$ determines a universal inversion which is a sheaf isomorphism

$$\tau : k^N \otimes \mathcal{O}_{\mathrm{GL}_N} \to k^N \otimes \mathcal{O}_{\mathrm{GL}_N}. \tag{7}$$

Let $q_Q : k^N \otimes \mathcal{O}_{Q\times X} \to \mathcal{U}$ denote the universal quotient homomorphism on $Q \times X$. Then the action $\sigma : \mathrm{GL}_N \times Q \to Q$ is determined by the following family of quotient maps over $\mathrm{GL}_N \times Q$

$$k^N \otimes \mathcal{O}_{\mathrm{GL}_N \times Q\times X} \xrightarrow{p_{\mathrm{GL}_N}^* \tau} k^N \otimes \mathcal{O}_{\mathrm{GL}_N \times Q\times X} \xrightarrow{(p_{Q\times X})^* q_Q} p_{Q\times X}^* \mathcal{U}$$

where $p_{\mathrm{GL}_N} : \mathrm{GL}_N \times Q \times X \to \mathrm{GL}_N$ and $p_{Q\times X} : \mathrm{GL}_N \times Q \times X \to Q \times X$ denote the projection morphisms.

From the definition of $R^{(s)s}$, we see these subschemes are preserved by the action. Consider quotient sheaves $q_{\mathcal{E}} : \mathcal{O}_X^N \twoheadrightarrow \mathcal{E}$ and $q_{\mathcal{E}} : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ in $R^{(s)s}$. If $g \cdot q_E \sim q_F$, then there is an isomorphism $\mathcal{E} \cong \mathcal{F}$ which fits into a commutative square, and so $\mathcal{E}$ and $\mathcal{F}$ are isomorphic. Conversely, if $\mathcal{E} \cong \mathcal{F}$, then there is an induced isomorphism $\phi : H^0(\mathcal{E}) \cong H^0(\mathcal{F})$. The composition

$$k^N \xrightarrow{H^0(q_{\mathcal{E}})} H^0(\mathcal{E}) \xrightarrow{\phi} H^0(\mathcal{F}) \xrightarrow{H^0(q_{\mathcal{F}})^{-1}} k^N$$

is an isomorphism which determines a point $g \in \mathrm{GL}_N$ such that $g \cdot q_E \sim q_F$. $\square$

Remark 8.50. In particular, any coarse moduli space for (semi)stable vector bundles is constructed as a categorical quotient of the GL_N-action on $R^{(s)s}$. Furthermore, if there is an orbit space quotient of the GL_N-action on $R^{(s)s}$, then this is a coarse moduli space. In fact, as the diagonal $\mathbb{G}_m \subset \mathrm{GL}_N$ acts trivially on the Quot scheme, we do not lose anything by instead working with the SL_N-action.

Finally, we would like to linearise the action to construct a categorical quotient via GIT. There is a natural family of invertible sheaves on the Quot scheme arising from Grothendieck's embedding of the Quot scheme into the Grassmannians: for sufficiently large m, we have a closed immersion

$$Q = \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N) \hookrightarrow \mathrm{Gr}(H^0(\mathcal{O}_X^N(m)), M) \hookrightarrow \mathbb{P} := \mathbb{P}(\wedge^M H^0(\mathcal{O}_X^N(m))^\vee)$$

where $M = mr+d+r(1-g)$. We let $\mathcal{L}_m$ denote the pull back of $\mathcal{O}_{\mathbb{P}}(1)$ to the Quot scheme via this closed immersion. There is a natural linear action of SL_N on $H^0(\mathcal{O}_X^N(m)) = (k^N \otimes H^0(\mathcal{O}_X(m)))$, which induces a linear action of SL_N on $\mathbb{P}(\wedge^M H^0(\mathcal{O}_X^N(m))^\vee)$; hence, $\mathcal{L}_m$ admits a linearisation of the SL_N-action.

We can define the linearisation $\mathcal{L}_m$ using the universal quotient sheaf $\mathcal{U}$ on $Q \times X$: we have

$$\mathcal{L}_m = \det(\pi_{Q*}(\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(m)))$$

where $\pi_X : Q \times X \to X$ and $\pi_Q : Q \times X \to Q$ are the projection morphisms. Furthermore, the universal quotient sheaf $\mathcal{U}$ admits a SL_N-linearisation: we have equivalent families of quotients sheaves over $\mathrm{SL}_N \times Q$ given by

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q\times X} \xrightarrow{(\sigma\times \mathrm{id}_X)^* q_Q} (\sigma \times \mathrm{id}_X)^* \mathcal{U}$$

and

$$k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q\times X} \xrightarrow{p_{\mathrm{SL}_N}^* \tau} k^N \otimes \mathcal{O}_{\mathrm{SL}_N \times Q\times X} \xrightarrow{p_{Q\times X}^* q_Q} p_{Q\times X}^* \mathcal{U},$$

where $q_Q : k^N \otimes \mathcal{O}_{Q \times X} \to \mathcal{U}$ denotes the universal quotient homomorphism, $\sigma : \mathrm{SL}_N \times Q \to Q$ denotes the group action, $p_{Q\times X}$ and p_{SL_N} denote the projections from $\mathrm{SL}_N \times Q \times X$ to the relevant factor and τ is the isomorphism given in (7). Hence, there is an isomorphism

$$\Phi : (\sigma \times \mathrm{id}_X)^*\mathcal{U} \to (p_{Q\times X})^*\mathcal{U}$$

satisfying the cocycle condition, which gives a linearisation of the SL_N-action on $\mathcal{U}$. For m sufficiently large, $\mathcal{L}_m$ is ample and admits an SL_N-linearisation, as the construction of $\mathcal{L}_m$ commutes with base change for m sufficiently large. Hence, at $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q, the fibre of the of the associated line bundle L_m is naturally isomorphic to an alternating tensor product of exterior powers of the cohomology groups of $\mathcal{F}(m)$:

$$L_{m,q} \cong \det H^*(X, \mathcal{F}(m)) = \bigotimes_{i\geq 0} \det H^i(X, \mathcal{F}(m))^{\otimes(-1)^i}.$$

In fact, by the Castelnuovo–Mumford regularity result explained in the second step of the construction of the quot scheme, for m sufficiently large, we have $H^i(X, \mathcal{F}(m)) = 0$ for all $i > 0$ for all points $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q. Therefore, for m sufficiently large, the fibre at q is

$$L_{m,q} \cong \det H^0(X, \mathcal{F}(m)).$$