8.9. **Analysis of semistability.** Let SL_N act on $Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ as above. In this section, we will determine the GIT (semi)stable points in Q with respect to the SL_N-linearisation $\mathcal{L}_m$. In fact, we will prove that $Q^{ss}(\mathcal{L}_m) = R^{ss}$ and $Q^s(\mathcal{L}_m) = R^s$.

We will use the Hilbert–Mumford criterion for our stability analysis. Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ denote a closed point in the Quot scheme Q and let $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-parameter subgroup. We recall that the action is given by

$$\lambda(t) \cdot q : \mathcal{O}_X^N \xrightarrow{\lambda^{-1}(t)} \mathcal{O}_X^N \xrightarrow{q} \mathcal{F}.$$

First of all, we would like to calculate the limit as $t \to 0$. For this, we need some notation. The action of λ^{-1} on $V := k^N$ determines a weight space decomposition

$$V = \oplus_{r\in\mathbb{Z}} V_r$$

where $V_r := \{v \in V : \lambda^{-1}(t)v = t^r v\}$ are zero except for finitely many weights r and, as λ is a 1-PS of the special linear group, we have

$$\sum_{r\in\mathbb{Z}} r \dim V_r = 0. \tag{8}$$

There is an induced ascending filtration of $V = k^N$ given by $V^{\leq r} := \oplus_{s\leq r} V_s$ and an induced ascending filtration of $\mathcal{F}$ given by

$$\mathcal{F}^{\leq r} := q(V^{\leq r} \otimes \mathcal{O}_X)$$

and q induces surjections $q_r : V_r \otimes \mathcal{O}_X \to \mathcal{F}_r := \mathcal{F}_{\leq r}/\mathcal{F}_{\leq r-1}$ which fit into a commutative diagram

$$\begin{tikzcd}
V^{\leq r-1} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & V^{\leq r} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & V_r \otimes \mathcal{O}_X \arrow[d, "q_r"] \\
\mathcal{F}^{\leq r-1} \arrow[r] & \mathcal{F}^{\leq r} \arrow[r, two heads] & \mathcal{F}_r.
\end{tikzcd}$$

Lemma 8.51. *Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q and $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-PS as above; then*

$$\lim_{t\to 0} \lambda(t) \cdot q = \bigoplus_{r\in\mathbb{N}} q_r.$$

Proof. As the quot scheme is projective, there is a unique limit. Therefore, it suffices to construct a family of quotient sheaves of $\mathcal{O}_X^N$ over $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$

$$\Phi : \mathcal{O}_{X\times\mathbb{A}^1}^N \twoheadrightarrow \mathcal{E}$$

such that $\Phi_t = \lambda(t) \cdot q$ for all $t \neq 0$ and $\Phi_0 = \oplus_r q_r$.

We will use the equivalence between quasi-coherent sheaves on $\mathbb{A}^1$ and $k[t]$-modules. Consider the $k[t]$-module

$$\mathcal{V} := \bigoplus_r V^{\leq r} \otimes_k t^r k$$

with action given by $t \cdot (v^{\leq r} \otimes t^r) = v^{\leq r} \otimes t^{r+1} \in V^{\leq r+1} \otimes r$, which works as the filtration is increasing. Since the filtration on V is zero for sufficiently small r and stabilises to V for sufficiently large r: there is an integer R such that $V^{\leq r} = 0$ for all $r \leq R$ and so $\mathcal{V} \subset V \otimes_k t^R k[t]$; hence, $\mathcal{V}$ is coherent. The 1-PS λ^{-1} determines a sheaf homomorphism over $\mathbb{A}^1$

$$\gamma : V \otimes_k k[t] \to \mathcal{V} := \bigoplus_r V^{\leq n} \otimes_k t^r k$$

given by $v \otimes t^s = \sum_r v_r \otimes t^s \mapsto \sum_r v_r \otimes t^{r+s}$, where $v_r \in V_r$ and so, as s is non-negative, $v_r \in V^{\leq r+s}$. By construction, $\gamma|_{V_r} = t^r \cdot \mathrm{Id}_{V_r}$. We leave it to the reader to write down an inverse which shows that γ is an isomorphism.

Over $\mathrm{Spec}\, k$, the k-module $k[t]$ determines a quasi-coherent (but not coherent) sheaf, we let $\mathcal{O}_X \otimes_k k[t]$ denote the pullback of this quasi-coherent sheaf to X. Then to describe coherent sheaves on $X \times \mathbb{A}^1$, we will use the equivalence between the category of quasi-coherent sheaves on $X \times \mathbb{A}^1$ and the category of $\mathcal{O}_X \otimes_k k[t]$-modules. Using the filtration $\mathcal{F}^{\leq r}$ we construct a quasi-coherent sheaf $\mathcal{E}$ over $X \times \mathbb{A}^1$ as follows. Let

$$\mathcal{E} := \bigoplus_n \mathcal{F}^{\leq r} \otimes_k t^r k \subset \mathcal{F} \otimes_k t^R k[t]$$

for R as above. The action of t is identical to the action of t on $\mathcal{V}$ given above. Furthermore, we have the above inclusion as the filtration is zero for r sufficiently small and stabilises to $\mathcal{F}$ for r sufficiently large; in particular $\mathcal{E}$ is a coherent sheaf on $X \times \mathbb{A}^1$.

The homomorphism q induces a surjective homomorphism of coherent sheaves over $X \times \mathbb{A}^1$

$$q_{\mathbb{A}^1} : \bigoplus_n V^{\leq r} \otimes_k t^r k \to \mathcal{E} := \bigoplus_n \mathcal{F}^{\leq r} \otimes_k t^r k$$

and we define our family of quotient sheaves over $X \times \mathbb{A}^1$ to be $\Phi := q_{\mathbb{A}^1} \circ \pi^*_{\mathbb{A}^1}\gamma$, where $\pi_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection.

If we restrict Φ to $\mathbb{A}^1 - \{0\}$, then this corresponds to inverting the variable t. In this case, we have an commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^N \otimes_k k[t,t^{-1}] \arrow[r, "\pi^*_{\mathbb{A}^1-\{0\}}(\gamma)"] \arrow[d, "\Phi|_{\mathbb{A}^1-\{0\}}"] & \mathcal{O}_X^N \otimes_k k[t,t^{-1}] \arrow[d, "q\otimes \mathrm{id}"] \\
\mathcal{E} \otimes_k k[t,t^{-1}] \arrow[r, "\cong"] & \mathcal{F} \otimes_k k[t,t^{-1}]
\end{tikzcd}$$

where γ gives the action of λ^{-1}; hence $[\Phi_t] = [\lambda(t) \cdot q]$ for all $t \neq 0$. Let $i : 0 \hookrightarrow \mathbb{A}^1$ denote the closed immersion; then the composition i_*i^* kills the action of t. We have

$$i_*i^*(\mathcal{E}) = \mathcal{E}/t \cdot \mathcal{E} = (\bigoplus_{r \geq R} \mathcal{F}^{\leq r} \otimes_k t^r k)/(\bigoplus_{r\geq R} \mathcal{F}^{\leq r} \otimes_k t^{r+1}k) = \bigoplus_r \mathcal{F}_r \otimes_k t^r k,$$

with trivial action by t. Hence, the restriction of $\mathcal{E}$ to the special fibre $0 \in \mathbb{A}^1$ is $\mathcal{E}_0 = \oplus_r \mathcal{F}_r$ and this completes the proof of the lemma. □

Lemma 8.52. *Let $\lambda : \mathbb{G}_m \to \mathrm{SL}_N$ be a 1-PS and $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q. Then using the notation introduced above for the weight decomposition of λ^{-1} acting on $V = k^N$, we have*

$$\mu^{\mathcal{L}_m}(q,\lambda) = -\sum_{r\in\mathbb{Z}} rP(\mathcal{F}_r, m) = \sum_{r\in\mathbb{Z}} \left(P(\mathcal{F}^{\leq r}, m) - \frac{\dim V^{\leq r}}{N} P(\mathcal{F}, m) \right).$$

Proof. By definition, this Hilbert–Mumford weight is negative the weight of the action of $\lambda(\mathbb{G}_m)$ on the fibre of the line bundle L_m over the fixed point $q' := \bigoplus_{r\in\mathbb{N}} q_r = \lim_{t\to 0} \lambda(t) \cdot q$. The fibre over this fixed point is

$$L_{m,q'} = \bigotimes_{r\in\mathbb{Z}} \det H^*(X, \mathcal{F}_r(m)),$$

where $H^*(X, F_r(m))$ denotes the complex defining the cohomology groups $H^i(X, F_r(m))$ for $i = 1, 2$ and the determinant of this complex is the 1-dimensional vector space

$$\bigotimes_{i\geq 0} \det H^i(X, \mathcal{F}_r(m))^{\otimes(-1)^i} = \wedge^{h^0(X,\mathcal{F}_r(m))} H^0(X, \mathcal{F}_r(m)) \otimes \wedge^{h^1(X,\mathcal{F}_r(m))} H^1(X, \mathcal{F}_r(m))^\vee.$$

The virtual dimension of $H^*(X, \mathcal{F}_r(m))$ is given by the alternating sums of the dimensions of the cohomology groups of $\mathcal{F}_r(m)$ and thus is equal to $P(\mathcal{F}_r, m)$. Since λ acts with weight r on $\mathcal{F}_r$, it also acts with weight r on $H^i(X, \mathcal{F}_r(m))$. Therefore, the weight of λ acting on $\det H^*(X, \mathcal{F}_r(m))$ is $rP(\mathcal{F}_r, m)$. The first equality then follows from this and the definition of the Hilbert–Mumford weight.

For the second equality, we recall that as λ is a 1-PS of SL_N, we have a relation (8) and by definition, we have $\dim V_r = \dim V^{\leq r} - \dim V^{\leq r-1}$. Furthermore, as $\mathcal{F}_r := \mathcal{F}^{\leq r}/\mathcal{F}^{\leq r-1}$, we have

$$P(\mathcal{F}_r) = P(\mathcal{F}^{\leq r}) - P(\mathcal{F}^{\leq r-1}).$$

The second equality then follows from these observations. □

Remark 8.53. The second expression for the Hilbert–Mumford weight is of greater use to us, as it is expressed in terms of subsheaves of $\mathcal{F}$. The number of distinct weights for the λ^{-1}-action on $V = k^N$, tells us the number of jumps in the filtration of $\mathcal{F}$.

If we suppose there are only two weights $r_1 < r_2$ for λ, then we get a filtration of $\mathcal{F}$ by a single subsheaf $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$:

$$0 = \cdots = 0 = \mathcal{F}^{\leq r_1 - 1} \subsetneq \mathcal{F}' := \mathcal{F}^{\leq r_1} = \cdots = \mathcal{F}^{\leq r_2 - 1} \subsetneq \mathcal{F}^{\leq r_2} = \mathcal{F} = \cdots \mathcal{F}.$$

Let $V' := V^{\leq r_1}$; then we have

$$\mu^{\mathcal{L}_m}(q, \lambda) = (r_2 - r_1)\left(P(\mathcal{F}', m) - \frac{\dim V'}{\dim V} P(\mathcal{F}, m) \right),$$

where $r_2 - r_1 > 0$.

Proposition 8.54. *Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q. Then $q \in Q^{(s)s}(\mathcal{L}_m)$ if and only if for all subspaces $0 \neq V' \subsetneq V = K^N$ we have an inequality*

$$\frac{\dim V'}{P(\mathcal{F}', m)} \ (\leq)\ \frac{\dim V}{P(\mathcal{F}, m)} \tag{9}$$

where $\mathcal{F}' := q(V' \otimes \mathcal{O}_X) \subset \mathcal{F}$.

Proof. Suppose the inequality (9) holds for all subspaces V'. We will show q is (semi)stable using the Hilbert–Mumford criterion. For any 1-PS $\lambda : \mathbb{G}_m \to \mathrm{GL}_N$, there are finitely many weights $r_1 < r_2 < \cdots < r_s$ for the λ^{-1}-action on $V = K^N$, which give rise to subspaces $V^{(i)} = V^{\leq r_i} \subset V$ and subsheaves $\mathcal{F}^{(i)} := q(V^{(i)} \otimes \mathcal{O}_X) \subset \mathcal{F}$. Furthermore, we have $\mathcal{F}^{\leq n} = \mathcal{F}^{(i)}$ for $r_i \leq n < r_{i+1}$. Therefore, by Lemma 8.52, we have

$$\mu^{\mathcal{L}_m}(q, \lambda) = \sum_{i=1}^{s-1} (r_{i+1} - r_i) \left(P(\mathcal{F}^{(i)}, m) - \frac{\dim V^{(i)}}{\dim V} P(\mathcal{F}, m) \right) \ (\geq) 0.$$

Conversely, if there exists a subspace $0 \subsetneq V' \subsetneq k^N$ for which the inequality (9) does not hold (or holds with equality respectively), then we can construct a 1-PS λ with two weights $r_1 > r_2$ such that $V^{(1)} = V'$ and $V^{(2)} = V$. Then

$$\mu^{\mathcal{L}_m}(q, \lambda) = (r_2 - r_1)\left(P(\mathcal{F}', m) - \frac{\dim V'}{\dim V} P(\mathcal{F}, m) \right) < 0 \quad (\text{resp. } \mu^{\mathcal{L}_m}(q, \lambda) = 0);$$

that is q is unstable for the SL_N-action with respect to $\mathcal{L}_m$. □

Remark 8.55. For m sufficiently large $P(\mathcal{F}', m)$ and $P(\mathcal{F}, m)$ are both positive; thus, we can multiply by the denominators in the inequality (9) to obtain an equivalent inequality

$$(\dim V' \operatorname{rk} \mathcal{F})m + (\dim V')(\deg \mathcal{F} + \operatorname{rk} \mathcal{F}(1-g)) (\leq) (\dim V \operatorname{rk} \mathcal{F}')m + (\dim V)(\deg \mathcal{F}' + \operatorname{rk} \mathcal{F}'(1-g)).$$

An inequality of polynomials in a variable m holds for all m sufficiently large if and only if there is an inequality of their leading terms. If $\operatorname{rk} \mathcal{F}' \neq 0$, then the leading term of the polynomial $P(\mathcal{F}')$ is $\operatorname{rk} \mathcal{F}'$ and if $\operatorname{rk} \mathcal{F}' = 0$, then the Hilbert polynomial of $\mathcal{F}'$ is constant. Therefore, there exists M (depending on $\mathcal{F}$ and $\mathcal{F}'$) such that for $m \geq M$

$$\operatorname{rk} \mathcal{F}' > 0 \quad \text{and} \quad \frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}} > 0 \quad \iff \quad \frac{\dim V'}{P(\mathcal{F}', m)} (\leq) \frac{\dim V}{P(\mathcal{F}, m)}.$$

In fact, M only depends on $P(\mathcal{F})$ and $P(\mathcal{F}')$. Moreover, as the subspaces $0 \neq V' \subsetneq V = k^N$ form a bounded family (they are parametrised by a product of Grassmannians) and the quotient sheaves $q : \mathcal{O}_X^N \to \mathcal{F}$ form a bounded family (they are parametrised by the Quot scheme Q), the family of sheaves $\mathcal{F}' = q(V' \otimes \mathcal{F})$ are also bounded. Therefore, there are only finitely many possibilities for $P(\mathcal{F}')$. Hence, there exists M such that for $m \geq M$ the following holds: for any $q : \mathcal{O}_X^N \to \mathcal{F}$ in Q and $0 \neq V' \subsetneq V = k^N$, we have

$$\operatorname{rk} \mathcal{F}' > 0 \quad \text{and} \quad \frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}} > 0 \quad \iff \quad \frac{\dim V'}{P(\mathcal{F}', m)} (\leq) \frac{\dim V}{P(\mathcal{F}, m)}$$

where $\mathcal{F}' = q(V' \otimes \mathcal{O}_X)$.

Remark 8.56. Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F} \in Q(k)$. Then we note

(1) If $0 \subsetneq V' \subset V = k^N$ and $\mathcal{F}' := q(V' \otimes \mathcal{O}_X)$, then $V' \subset H^0(q)^{-1}(H^0(\mathcal{F}'))$,
(2) If $\mathcal{G} \subset \mathcal{F}$ and $V' = H^0(q)^{-1}(H^0(\mathcal{G}))$, then $q(V' \otimes \mathcal{O}_X) \subset \mathcal{G}$.

Using these two remarks, we obtain a corollary to Proposition 8.54.

Corollary 8.57. *There exists M such that for $m \geq M$ and for a k-point $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ in Q, the following statements are equivalent:*

(1) q is GIT (semi)stable for SL_N-acting on Q with respect to $\mathcal{L}_m$;
(2) for all subsheaves $\mathcal{F}' \subset \mathcal{F}$ with $V' := H^0(q)^{-1}(H^0(\mathcal{F}')) \neq 0$, we have $\operatorname{rk} \mathcal{F}' > 0$ and

$$\frac{\dim V'}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\dim V}{\operatorname{rk} \mathcal{F}}.$$

In the remaining part of this section, we prove some additional results concerning semistability of vector bundles, which we will eventually relate to GIT semistability.

Lemma 8.58. *Let n and d be fixed such that $d > n^2(2g-2)$. Then a locally free sheaf $\mathcal{F}$ of rank n and degree d is (semi)stable if for all $\mathcal{F}' \subset \mathcal{F}$ we have*

$$\frac{h^0(X, \mathcal{F}')}{\operatorname{rk} \mathcal{F}'} (\leq) \frac{\chi(\mathcal{F})}{\operatorname{rk} \mathcal{F}}. \tag{10}$$

Proof. Suppose $\mathcal{F}$ is not semistable; then there exists a subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $\mu(\mathcal{F}') > \mu(\mathcal{F})$. In fact, we can assume $\mathcal{F}'$ is semistable (if not, there is a vector subbundle F'' of F' with larger slope, and so we can replace $\mathcal{F}'$ with $\mathcal{F}''$). Then

$$\deg \mathcal{F}' > \frac{d}{n} \operatorname{rk} \mathcal{F}' > \frac{d}{n} > n(2g-2) > \operatorname{rk} \mathcal{F}'(2g-2).$$

Then it follow from Lemma 8.36 that $H^1(X, \mathcal{F}') = 0$. However, in this case

$$\frac{h^0(X, \mathcal{F}')}{\operatorname{rk} \mathcal{F}'} = \mu(\mathcal{F}') + (1-g) > \mu(\mathcal{F}) + (1-g) = \frac{\chi(\mathcal{F})}{\operatorname{rk} \mathcal{F}}$$

which contradicts (10). Furthermore, if the inequality (10) holds with a strict inequality and $\mathcal{F}$ is not stable, then we can apply the above argument to any subsheaf $\mathcal{F}' \subset \mathcal{F}$ with the same slope as $\mathcal{F}$ and get a contradiction. $\square$

The converse to this lemma also holds for d sufficiently large, as we will demonstrate in Proposition 8.61; however, first we need some preliminary results.

Lemma 8.59. *(Le Potier bounds) For any semi-stable locally free sheaf $\mathcal{F}$ of rank n and slope μ, we have*

$$\frac{h^0(X, \mathcal{F})}{n} \leq [\mu + 1]_+ := \max(\mu + 1, 0)$$

Proof. If $\mu < 0$, then $H^0(X, \mathcal{F}) = 0$. For $\mu \geq 0$, we proceed by induction on the degree d of $\mathcal{F}$. If we assume the lemma holds for all degrees less than d, then we can consider the short exact sequence

$$0 \to \mathcal{F}(-x) \to \mathcal{F} \to F_x \to 0$$

where $x \in X$. By considering the associated long exact sequence, we see that

$$h^0(X, \mathcal{F}) \leq h^0(X, \mathcal{F}(-x)) + n.$$

Since $\mu(\mathcal{F}) = \mu(\mathcal{F}(-x)) + 1$, the result follows by applying the inductive hypothesis to $\mathcal{F}(-x)$. $\square$

We recall that any vector bundle has a unique maximal destabilising sequence of vector subbundles, known as its Harder–Narasimhan filtration (cf. Definition 8.32).

Corollary 8.60. *Let $\mathcal{F}$ be a locally free sheaf of rank n and slope μ with Harder–Narasimhan filtration*

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

i.e. $\mathcal{F}_i = \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ *are semistable and* $\mu_{\max}(\mathcal{F}) = \mu(\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_s) = \mu_{\min}(\mathcal{F})$*; then*

$$\frac{h^0(X,\mathcal{F})}{n} \leq \sum_{i=1}^{s} \frac{\operatorname{rk}\mathcal{F}_i}{n}[\mu(\mathcal{F}_i)+1]_+ \leq \left(1-\frac{1}{n}\right)[\mu+1]_+ + \frac{1}{r}[\mu_{\min}(\mathcal{F})+1]_+.$$

Proposition 8.61. *Let n and d be fixed such that $d > gn^2 + n(2g-2)$. Let $\mathcal{F}$ be a semistable locally free sheaf over X with rank r and degree d. Then for all non-zero subsheaves $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$, we have*

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

and if equality holds, then $h^1(X,\mathcal{F}') = 0$ and $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

Proof. Let $\mu = d/n$ denote the slope and pick a constant C such that $2g-2 < C < \mu - gn$ (this is possible, as $\mu - gn > 2g-2$ by our choice of d). We will prove the following statements for subsheaves $\mathcal{F}' \subset \mathcal{F}$.

(1) If $\mu_{\min}(\mathcal{F}') \leq C$, then
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} < \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}.$$
(2) If $\mu_{\min}(\mathcal{F}') > C$, then $h^1(X,\mathcal{F}) = 0$ and
$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$
and if equality holds, then $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

We can apply Corollary 8.60 to a subsheaf $\mathcal{F}' \subset \mathcal{F}$ to obtain the bound

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \left(1-\frac{1}{n}\right)[\mu+1]_+ + \frac{1}{n}[\mu_{\min}(\mathcal{F}')+1]_+.$$

If $\mu_{\min}(\mathcal{F}') \leq C$, then

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} \leq \left(1-\frac{1}{n}\right)(\mu+1) + \frac{1}{n}(C+1) < \mu + 1 + g = \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

by our choice of C, which proves (1).

For (2), suppose $\mu_{\min}(\mathcal{F}') > C$; then we claim that $H^1(X,\mathcal{F}') = 0$. To prove the claim, it suffices to show that $H^1(X,\mathcal{F}'_i) = 0$, where $\mathcal{F}'_i$ are the semistable subquotients appearing in the Harder–Narasimhan filtration of $\mathcal{F}'$. For each $\mathcal{F}'_i$, we have

$$\mu(\mathcal{F}'_i) \geq \mu_{\min}(\mathcal{F}') > C > 2g-2.$$

Hence, $\deg \mathcal{F}'_i > \operatorname{rk}\mathcal{F}'_i(2g-2)$ and, as $\mathcal{F}'_i$ is semistable, we conclude that $H^1(X,\mathcal{F}'_i) = 0$ by Lemma 8.36. Then by semistability of $\mathcal{F}$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$; hence

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} = \mu(\mathcal{F}') + 1 - g \leq \mu(\mathcal{F}) + 1 - g = \frac{\chi(\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

with equality only if $\mu(\mathcal{F}') = \mu(\mathcal{F})$. □

Remark 8.62. Proposition 8.61 and Lemma 8.58 together say, for sufficiently large degree d, that (semi)stability of a locally free sheaf $\mathcal{F}$ over X is equivalent to

$$\frac{h^0(X,\mathcal{F}')}{\operatorname{rk}\mathcal{F}'} (\leq) \frac{h^0(X,\mathcal{F})}{\operatorname{rk}\mathcal{F}}$$

for all non-zero proper subsheaves $\mathcal{F}' \subset \mathcal{F}$. This result was first proved by Le Potier for curves (see [35] Propositions 7.1.1 and 7.1.3) and was later generalised to higher dimensions by Simpson [39].

We recall that we defined open subschemes $R^{(s)s} \subset Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}^N)$ whose k-points are quotient sheaves $q : \mathcal{O}_X^N \to \mathcal{F}$ such that $\mathcal{F}$ is a locally free (semi)stable sheaf and $H^0(q)$ is an isomorphism. The following theorem shows that GIT semistability for SL_N acting on Q coincides with vector bundle semistability (provided d and m are sufficiently large).

Theorem 8.63. *Let n and d be fixed such that $d > \max(n^2(2g-2), gn^2 + n(2g-2))$. Then there exists a natural number $M > 0$ such that for all $m \geq M$, we have*

$$Q^{ss}(\mathcal{L}_m) = R^{ss} \qquad \text{and} \qquad Q^s(\mathcal{L}_m) = R^s.$$

Proof. We pick M as required by Corollary 8.57. Since these subschemes are all open subschemes of Q, it suffices to check these equalities of schemes on k-points.

First, let $q : \mathcal{O}_X^N \to \mathcal{F}$ be a k-point in R^{ss}; that is, $\mathcal{F}$ is a semistable locally free sheaf and $H^0(q) : V \to H^0(X, \mathcal{F})$ is an isomorphism. We will show that q is GIT semistable using Corollary 8.57. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf with $\mathrm{rk}\,\mathcal{F}' > 0$ and let $V' := H^0(q)^{-1}(H^0(X, \mathcal{F}'))$. As $H^0(q)$ is a isomorphism, we have $\dim V' = h^0(X, \mathcal{F}')$. By Proposition 8.61, we have either

(1) $h^0(X, \mathcal{F}') < \mathrm{rk}\,\mathcal{F}'\chi(\mathcal{F})/\mathrm{rk}\,\mathcal{F}$, or
(2) $h^1(X, \mathcal{F}') = 0$ and $\mu(\mathcal{F}') = \mu(\mathcal{F})$.

In the first case,

$$\frac{\dim V'}{\mathrm{rk}\,\mathcal{F}'} = \frac{h^0(X, \mathcal{F}')}{\mathrm{rk}\,\mathcal{F}'} < \frac{\chi(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} = \frac{\dim V}{\mathrm{rk}\,\mathcal{F}}$$

and in the second case, $\dim V' = h^0(X, \mathcal{F}') = P(\mathcal{F}')$, and we have

$$\frac{\dim V'}{\mathrm{rk}\,\mathcal{F}'} = \frac{\chi(\mathcal{F}')}{\mathrm{rk}\,\mathcal{F}'} = \frac{\chi(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} = \frac{\dim V}{\mathrm{rk}\,\mathcal{F}}.$$

Hence $q \in \overline{Q}^{ss}(\mathcal{L}_m)$ by Corollary 8.57. In fact, this argument shows that if, moreover, $\mathcal{F}$ is a stable locally free sheaf, then $q \in \overline{Q}^{s}(\mathcal{L}_m)$, because, in this case, condition (2) is not possible and so we always have a strict inequality. Hence, we have inclusions $R^{(s)s}(k) \subset Q^{(s)s}(\mathcal{L}_m)(k)$.

Suppose that $q : \mathcal{O}_X^N \to \mathcal{F}$ is a k-point in $Q^{(s)s}(\mathcal{L}_m)$; then for every subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $V' := H^0(q)^{-1}(H^0(X, \mathcal{F}'))$ is non-zero, we have $\mathrm{rk}\,\mathcal{F}' > 0$ and an inequality

$$\frac{\dim V'}{\mathrm{rk}\,\mathcal{F}'} (\leq) \frac{\dim V}{\mathrm{rk}\,\mathcal{F}}$$

by Corollary 8.57.

We first observe that $H^0(q) : V \to H^0(X, \mathcal{F})$ is injective, as otherwise let K be the kernel, then $\mathcal{F}' = q(K \otimes \mathcal{O}_X) = 0$ has rank equal to zero, and so contradicts GIT semistability of q. In fact, we claim that GIT semistability also implies $H^1(X, \mathcal{F}) = 0$; thus, $\dim H^0(X, \mathcal{F}) = \chi(\mathcal{F}) = N = \dim V$ and so the injective map $H^0(q)$ is an isomorphism. If $H^1(X, \mathcal{F}) \neq 0$, then by Serre duality, there is a non-zero homomorphism $\mathcal{F} \to \omega_X$ whose image $\mathcal{F}'' \subset \omega_X$ is an invertible sheaf. We can equivalently phrase the GIT (semi)stability of q in terms of quotient sheaves $\mathcal{F} \twoheadrightarrow \mathcal{F}''$ as giving an inequality

$$\frac{\dim V}{n} \leq \frac{\dim V''}{\mathrm{rk}\,\mathcal{F}''}$$

where V'' denotes the image of the composition $V \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'')$. We note that $\dim V'' \leq g$, as $V'' \subset H^0(X, \mathcal{F}'') \subset H^0(X, \omega_X)$. Therefore, GIT semistability would imply

$$\frac{d}{n} + (1-g) \leq g,$$

which contradicts our choice of d. Thus $H^0(q)$ is an isomorphism.

We next claim that $\mathcal{F}$ is locally free. Since we are working over a curve, the claim is equivalent to showing that $\mathcal{F}$ is torsion free. If $\mathcal{F}' \subset \mathcal{F}$ is a torsion subsheaf (i.e. $\mathrm{rk}\,\mathcal{F}' = 0$), then $H^0(X, \mathcal{F}') \neq 0$, as every torsion sheaf has a section, and so this would contradict GIT semistability.

Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf and $V' := H^0(q)^{-1}(H^0(X, \mathcal{F}'))$; then by GIT (semi)stability

$$\frac{h^0(X, \mathcal{F}')}{\mathrm{rk}\,\mathcal{F}'} = \frac{\dim V'}{\mathrm{rk}\,\mathcal{F}'} (\leq) \frac{\dim V}{\mathrm{rk}\,\mathcal{F}} = \frac{\chi(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}}.$$

Hence, $\mathcal{F}$ is (semi)stable by Lemma 8.58. Since also $H^0(q)$ is an isomorphism, we have shown that $q \in R^{(s)s}$. This completes the proof of the opposite inclusion $Q^{(s)s}(\mathcal{L}_m)(k) \subset R^{(s)s}(k)$. □

8.10. **Construction of the moduli space.** Let X be a connected smooth projective curve of genus $g \geq 2$. We fix a rank n and a degree d. In this section, we will give the construction of the moduli space of stable vector bundles on X.

We defined open subschemes $R^{(s)s} \subset Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}^N)$ (where $N := d + n(1-g)$) whose k-points are quotients $q : \mathcal{O}_X^N \to \mathcal{F}$ such that $\mathcal{F}$ is (semi)stable and $H^0(q)$ is an isomorphism.

The construction of the moduli space of stable vector bundles is originally due to Seshadri [37]; however, we have not followed his construction (Seshadri uses a different linearisation which embeds the Quot scheme in a product of Grassmannians). Instead, we are following the construction due to Le Potier [35] and Simpson [39], which generalises more naturally to higher dimensions; see Remark 8.70 for some comments on the additional complications for higher dimensional base schemes.

Theorem 8.64. *There is a coarse moduli space $M^s(n,d)$ for moduli of stable vector bundles of rank n and degree d over X that has a natural projective completion $M^{ss}(n,d)$ whose k-points parametrise polystable vector bundles of rank n and degree d.*

Proof. We first construct these spaces for large d and then, by tensoring with invertible sheaves of negative degree, we obtain the moduli spaces for smaller degree d. Hence, we may assume that $d > \max(n^2(2g-2), gn^2 + n(2g-2))$ We linearise the SL_N-action on Q in the invertible sheaf $\mathcal{L}_m$, where m is taken sufficiently large as required for the statement of Theorem 8.63. Then $Q^{(s)s}(\mathcal{L}_m) = R^{(s)s}$ and there is a projective GIT quotient

$$\pi : R^{ss} = Q^{ss}(\mathcal{L}_m) \to Q/\!/_{\mathcal{L}_m}\mathrm{SL}_N =: M^{ss}(n,d)$$

which is a categorical quotient of the SL_N-action on R^{ss} and π restrict to a geometric quotient

$$\pi^s : R^s = Q^s(\mathcal{L}_m) \to Q^s(\mathcal{L}_m)/\mathrm{SL}_N =: M^s(n,d).$$

Furthermore, $R^{(s)s}$ parametrises a family $\mathcal{U}^{(s)s}$ of (semi)stable vector bundles over X of rank n and degree d which has the local universal property and such that two k-points in $R^{(s)s}$ lie in the same orbit if and only if the corresponding vector bundles parametrised by these points are isomorphic; see Lemmas 8.48 and 8.49. By Proposition 3.35, a coarse moduli space is a categorical quotient of the SL_N-action on $R^{(s)s}$ if and only if it is an orbit space. Therefore, as π^s is a categorical quotient which is also an orbit space, $M^s(n,d)$ is a coarse moduli space for stable vector bundles on X of rank n and degree d.

Since the k-points of the GIT quotient parametrise closed orbits, to complete the proof it remains to show that the orbit of $q : \mathcal{O}_X^N \to \mathcal{F}$ in R^{ss} is closed if and only if $\mathcal{F}$ is polystable. If $\mathcal{F}$ is not polystable, then there is a non-split short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are semistable with the same slope as $\mathcal{F}$. In this case, we can find a 1-PS λ such that $\lim_{t\to 0} \lambda(t) \cdot [q] = [\mathcal{O}_X^N \to \mathcal{F}'' \oplus \mathcal{F}']$, which shows that the orbit is not closed. In fact, by repeating this argument one case show that a quotient homomorphism for a semistable sheaf contains a quotient homomorphism for a polystable sheaf in its orbit closure. More precisely, one can define a Jordan–Holder filtration of $\mathcal{F}$ by stable vector bundles of the same slope as $\mathcal{F}$:

$$0 \subsetneq \mathcal{F}_{(1)} \subsetneq \mathcal{F}_{(2)} \subsetneq \cdots \subsetneq \mathcal{F}_{(s)} = \mathcal{F}$$

and then pick out a 1-PS λ which inducing this filtration so that the limit as $t \to 0$ is the associated graded object $\mathrm{gr}_{JH}(\mathcal{F}) := \oplus_i \mathcal{F}_{(i)}/\mathcal{F}_{(i-1)}$. We note that unlike the Harder–Narasimhan filtration, the Jordan–Holder filtration is not unique but the associated graded object is unique. Now suppose that $\mathcal{F}$ is polystable so we have $\mathcal{F} = \oplus \mathcal{F}_i^{\oplus n_i}$ for non-isomorphic stable vector bundles $\mathcal{F}_i$; then we want to show the orbit of q is closed: i.e. for every point $q' : \mathcal{O}_X^N \to \mathcal{F}'$ in the closure of the orbit of q, we have an isomorphism $\mathcal{F} \cong \mathcal{F}'$. Using Theorem 6.13, we can produce a 1-PS λ such that $\lim_{t\to 0}\lambda(t)\cdot q = q'$. This corresponds to a family $\mathcal{E}$ over $\mathbb{A}^1$ of semistable vector bundles such that

$$\mathcal{E}_t \cong \mathcal{F} \quad \text{for } t \neq 0, \quad \text{and} \quad \mathcal{E}_0 = \mathcal{F}'.$$

Since the stable bundles $\mathcal{F}_i$ are simple and any non-zero homomorphism between stable vector bundles of the same slope is an isomorphism, we see that $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}) = n_i$. As $\mathcal{E}$ is flat over $\mathbb{A}^1$, this dimension function is upper semi-continuous; hence $\dim \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') =: n_i' \geq n_i$. As $\mathcal{F}_i$ is stable, the evaluation map $e_i : \mathcal{F}_i \otimes \mathrm{Hom}(\mathcal{F}_i, \mathcal{F}') \to \mathcal{F}'$ must be injective. Moreover sum $\sum \mathcal{F}_i^{n_i'} \subset \mathcal{F}'$ is a direct sum as $\mathcal{F}_i \not\cong \mathcal{F}_j$ by assumption. By comparing the ranks, we must have $n_i = n_i'$ for all i and $\mathcal{F}' \cong \oplus \mathcal{F}_i^{\oplus n_i} = \mathcal{F}$. □

Proposition 8.65. *The moduli space $M^s(n,d)$ of stable vector bundles is a smooth quasi-projective variety of dimension $n^2(g-1)+1$.*

Proof. We claim that the open subscheme $R^s \subset Q$ is smooth and has dimension $n^2(g-1)+N^2$. To prove this claim, we use the following results concerning the local smoothness and Zariski tangent spaces of the quot scheme: for a k-point $q : \mathcal{O}_X^N \to \mathcal{F}$ of Q, we have

1. $T_q Q \cong \mathrm{Hom}(\mathcal{K}, \mathcal{F})$, where $\mathcal{K} = \ker q$.
2. If $\mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$, then Q is smooth in a neighbourhood of q.

For a proof of these results, see [16] Propositions 2.2.7 and 2.2.8; in fact, the description of the tangent spaces should remind you of the description of the tangent spaces to the Grassmannian. To prove the claim, for $q \in R^s$, we apply $\mathrm{Hom}(-, \mathcal{F})$ to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^N \to \mathcal{F} \to 0$$

to obtain a long exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{K}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{K}, \mathcal{F}) \to 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}_X^N, \mathcal{F}) = H^1(X, \mathcal{F})^N = 0$ (by our assumption on the degree of d), we see that Q is smooth in a neighbourhood of every point $q \in Q$. To calculate the dimension, we consider the following long exact sequence for $q \in R^s$:

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}_X^N, \mathcal{F}) \to \mathrm{Hom}(\mathcal{K}, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0$$

where $\hom(\mathcal{F}, \mathcal{F}) = 1$ as every stable bundle is simple, and $\hom(\mathcal{O}_X^N, \mathcal{F}) = N^2$ as our assumption on d implies $H^1(X, \mathcal{F}) = 0$, and $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) = H^1(\mathcal{F}^\vee \otimes \mathcal{F}) = n^2(g-1)+1$ by the Riemann–Roch formula. Hence,

$$\dim R^{ss} = \dim T_q Q = \dim \mathrm{Hom}(\mathcal{K}, \mathcal{F}) = n^2(g-1) + 1 + N^2 - 1 = n^2(g-1) + N^2.$$

Since SL_N acts with only a finite global stabiliser on the smooth quasi-projective variety R^s and the quotient $R^s \to M^s(n,d)$ is geometric, it follows from a deep result concerning étale slices of GIT quotients known as Luna's slice theorem [21], that $M^s(n,d)$ is smooth. Furthermore, we have

$$\dim M^s(n,d) = \dim R^s - \dim \mathrm{SL}_N = n^2(g-1) + 1$$

which completes the proof. □

Remark 8.66. In fact, using deformation theory of vector bundles, one can identify the Zariski tangent space to $M^s(n,d)$ at the isomorphism class $[E]$ of a stable vector bundle E as follows

$$T_{[E]} M^s(n,d) \cong \mathrm{Ext}^1(E, E).$$

The obstruction to $M^s(n,d)$ being smooth is controlled by $\mathrm{Ext}^2(E,E)$, which vanishes as we are working over a curve. The same description holds in higher dimensions, except now this second Ext group could be non-zero and so in general the moduli space is not smooth; see [16] Corollary 4.52.

If the degree and rank are coprime, the notions of semistability and stability coincide; hence, in the coprime case, the moduli space of stable vector bundles of rank r and degree d on X is a smooth projective variety.

Finally, we ask whether this coarse moduli space is ever a fine moduli space. In fact, we see why it is necessary to allow a more general notion of equivalence of families of vector bundles with a twist by a line bundle:

Remark 8.67. Two families $\mathcal{E}$ and $\mathcal{F}$ parametrised by S determine the same morphism to $M^s(n,d)$ if $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^*\mathcal{L}$ for a line bundle $\mathcal{L}$ on S where $\pi_S : S \times X \to S$ is the projection map and, in fact, this is an if and only if statement by [31] Lemma 5.10.

It is a result of Mumford and Newstead, for $n = 2$ [26], and Tjurin [43] in general that the moduli space of stable vector bundles is a fine moduli space for coprime rank and degree.

Theorem 8.68. *If $(n,d) = 1$, then $M^s(n,d) = M^{ss}(n,d)$ is a fine moduli space.*

The idea of the proof is to construct a universal family over this moduli space by descending the universal family $\mathcal{U}$ over $R^s \times X$ to the GIT quotient. For more details, we recommend the exposition given by Newstead [31], Theorem 5.12.

Remark 8.69. If $(n,d) \neq 1$, then Ramanan observes that a fine moduli space for stable sheaves does not exist [36].

Remark 8.70. In this remark, we briefly explain some of the additional complications that arise when studying moduli of vector bundles over a higher dimensional projective base Y.

(1) Instead of fixing just the rank and degree, one must fix higher Chern classes (or the Hilbert polynomial) of the sheaves.
(2) In higher dimensions, torsion free and locally free not longer agree; therefore, rather than working with locally free sheaves, we must enlarge our category to torsion free sheaves in order to get a projective completion of the moduli space of stable sheaves.
(3) As we have seen for curves, slope (semi)stability is equivalent to an inequality of reduced Hilbert polynomials, known as Gieseker (semi)stability
$$\mu(\mathcal{E}') \leq \mu(\mathcal{E}) \iff \frac{P(\mathcal{E}')}{\mathrm{rk}\mathcal{E}'} \leq \frac{P(\mathcal{E})}{\mathrm{rk}\mathcal{E}}.$$
However, in higher dimensions, slope (semi)stability and Gieseker (semi)stability do not coincide: we have
$$\text{slope stable} \implies \text{Gieseker stable} \implies \text{Gieseker semistable} \implies \text{slope semistable}.$$
In higher dimensions, one constructs a moduli space for Gieseker stable torsion free sheaves (or for Gieseker semistable pure sheaves).
(4) Since the Hilbert polynomial is taken with respect to a choice of ample line bundle on Y, the notion of Gieseker (semi)stability also depends on this choice. Over a curve, the Hilbert polynomial of a vector bundle only depends on the degree of the ample line bundle we take and consequently all ample line bundles determine the same notion of semistability. In particular, one can study how the moduli space changes as we vary this ample line bundle on Y.
(5) The Quot scheme is longer smooth, due to the existence of some non-vanishing second Ext groups. In particular, the moduli space of stable torsion free sheaves is no longer smooth in general.
(6) To construct the moduli spaces in higher dimensions, we do not take a GIT quotient of the whole Quot scheme, but rather the closure of R^{ss} in Q. The reason for this, is that there may be semistable points in the quot scheme which are not torsion free (or pure) sheaves;
(7) In higher dimensions, the Le Potier bounds become more difficult to prove; although there are essentially analogous statements.

For the interested reader, we recommend the excellent book of Huybrechts and Lehn [16].