2.3. **Fine moduli spaces.** The ideal situation is when there is a scheme that represents our given moduli functor.

Definition 2.15. Let $\mathcal{M} : \mathrm{Sch} \to \mathrm{Set}$ be a moduli functor; then a scheme M is a fine moduli space for $\mathcal{M}$ if it represents $\mathcal{M}$.

Let's carefully unravel this definition: M is a fine moduli space for $\mathcal{M}$ if there is a natural isomorphism $\eta : \mathcal{M} \to h_M$. Hence, for every scheme S, we have a bijection

$$\eta_S : \mathcal{M}(S) := \{\text{families over } S\}/\sim_S \longleftrightarrow h_M(S) := \{\text{morphisms } S \to M\}.$$

In particular, if $S = \operatorname{Spec} k$, then the k-points of M are in bijection with the set $\mathcal{A}/\sim$. Furthermore, these bijections are compatible with morphisms $T \to S$, in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathcal{M}(S) \arrow[r, "\eta_S"] \arrow[d, "\mathcal{M}(f)"'] & h_M(S) \arrow[d, "h_M(f)"] \\ \mathcal{M}(T) \arrow[r, "\eta_T"'] & h_M(T). \end{tikzcd}$$

The natural isomorphism $\eta : \mathcal{M} \to h_M$ determines an element $\mathcal{U} = \eta_M^{-1}(\mathrm{id}_M) \in \mathcal{M}(M)$; that is, $\mathcal{U}$ is a family over M (up to equivalence).

Definition 2.16. Let M be a fine moduli space for $\mathcal{M}$; then the family $\mathcal{U} \in \mathcal{M}(M)$ corresponding to the identity morphism on M is called the universal family.

This family is called the universal family, as any family $\mathcal{F}$ over a scheme S (up to equivalence) corresponds to a morphism $f : S \to M$ and, moreover, as the families $f^*\mathcal{U}$ and $\mathcal{F}$ correspond to the same morphism $\mathrm{id}_M \circ f = f$, we have

$$f^*\mathcal{U} \sim_S \mathcal{F};$$

that is, any family is equivalent to a family obtained by pulling back the universal family.

Remark 2.17. If a fine moduli space for $\mathcal{M}$ exists, it is unique up to unique isomorphism: that is, if (M, η) and (M', η') are two fine moduli spaces, then they are related by unique isomorphisms $\eta'_M((\eta_M)^{-1}(\mathrm{Id}_M)) : M \to M'$ and $\eta_{M'}((\eta'_{M'})^{-1}(\mathrm{Id}_{M'})) : M' \to M$.

We recall that a presheaf $F : \mathrm{Sch} \to \mathrm{Set}$ is said to be a sheaf in the Zariski topology if for every scheme S and Zariski cover $\{S_i\}$ of S, the natural map

$$\{f \in F(S)\} \longrightarrow \{(f_i \in F(S_i))_i : f_i|_{S_i \cap S_j} = f_j|_{S_j \cap S_i} \text{ for all } i, j\}$$

is a bijection. A presheaf is called a separated presheaf if these natural maps are injective.

Exercise 2.18.

(1) Show that the functor of points of a scheme is a sheaf in the Zariski topology. In particular, deduce that for a presheaf to be representable it must be a sheaf in the Zariski topology.
(2) Consider the moduli functor of vector bundles over a fixed scheme X, where we say two families $\mathcal{E}$ and $\mathcal{F}$ are equivalent if and only if they are isomorphic. Show that the corresponding moduli functor fails to be a separable presheaf (it may be useful to consider the second equivalence relation we introduced for families of vector bundles in Exercise 2.12).

Example 2.19. Let us consider the projective space $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. This variety can be interpreted as a fine moduli space for the moduli problem of lines through the origin in $V := \mathbb{A}^{n+1}$. To define this moduli problem carefully, we need to define a notion of families and equivalences of families. A family of lines through the origin in V over a scheme S is a line bundle $\mathcal{L}$ over S which is a subbundle of the trivial vector bundle $V \times S$ over S (by subbundle we mean that the quotient is also a vector bundle). Then two families are equivalent if and only if they are equal.

Over $\mathbb{P}^n$, we have a tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset V \times \mathbb{P}^n$, whose fibre over $p \in \mathbb{P}^n$ is the corresponding line in V. This provides a tautological family of lines over $\mathbb{P}^n$. The dual of the tautological line bundle is the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, known as the Serre twisting sheaf. The important fact we need about $\mathcal{O}_{\mathbb{P}^n}(1)$ is that it is generated by the global sections $x_0, \dots, x_n$.

Given any morphism of schemes $f : S \to \mathbb{P}^n$, the line bundle $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is generated by the global sections $f^*(x_0), \dots, f^*(x_n)$. Hence, we have a surjection $\mathcal{O}_S^{n+1} \to f^*\mathcal{O}_{\mathbb{P}^n}(1)$. For locally free sheaves, pull back commutes with dualising and so

$$f^*\mathcal{O}_{\mathbb{P}^n}(-1) \cong (f^*\mathcal{O}_{\mathbb{P}^n}(1))^\vee.$$

Dually the above surjection gives an inclusion $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_S^{n+1} = V \times S$ which determines a family of lines in V over S.

Conversely, let $\mathcal{L} \subset V \times S$ be a family of lines through the origin in V over S. Then, dual to this inclusion, we have a surjection $q : V^\vee \times S \to \mathcal{L}^\vee$. The vector bundle $V^\vee \times S$ is generated by the global sections $\sigma_0, \dots, \sigma_n$ corresponding to the dual basis for the standard basis on V. Since q is surjective, the dual line bundle $\mathcal{L}^\vee$ is generated by the global sections $q \circ \sigma_0, \dots, q \circ \sigma_n$. In particular, there is a unique morphism $f : S \to \mathbb{P}^n$ given by

$$s \mapsto [q \circ \sigma_0(s) : \cdots : q \circ \sigma_n(s)]$$

such that $f^*\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{L} \subset V \times S$ (for details, see [14] II Theorem 7.1).

Hence, there is a bijective correspondence between morphisms $S \to \mathbb{P}^n$ and families of lines through the origin in V over S. In particular, $\mathbb{P}^n$ is a fine moduli space and the tautological family is a universal family. The keen reader may note that the above calculations suggests we should rather think of $\mathbb{P}^n$ as the space of 1-dimensional quotient spaces of a $n+1$-dimensional vector space (a convention that many algebraic geometers use).

Exercise 2.20. Consider the moduli problem of d-dimensional linear subspaces in a fixed vector space $V = \mathbb{A}^n$, where a family over S is a rank d vector subbundle $\mathcal{E}$ of $V \times S$ and the equivalence relation is given by equality. We denote the associated moduli functor by $\mathcal{G}r(d, n)$.

We recall that there is a projective variety $\mathrm{Gr}(d, n)$ whose k-points parametrise d-dimensional linear subspaces of k^n, called the *Grassmannian variety.* Let $\mathcal{T} \subset V \times \mathrm{Gr}(d, n)$ be the tautological family over $\mathrm{Gr}(d, n)$ whose fibre over a point in the Grassmannian is the corresponding linear subspace of V. In this exercise, we will show that the Grassmannian variety $\mathrm{Gr}(d, n)$ is a fine moduli space representing $\mathcal{G}r(d, n)$.

Let us determine the natural isomorphism $\eta : \mathcal{G}r(d, n) \to h_{\mathrm{Gr}(d,n)}$. Consider a family $\mathcal{E} \subset V \times S$ over S. As $\mathcal{E}$ is a rank d vector bundle, we can pick an open cover $\{U_\alpha\}$ of S on which $\mathcal{E}$ is trivial, i.e. $\mathcal{E}|_{U_\alpha} \cong U_\alpha \times \mathbb{A}^d$. Then, since we have $U_\alpha \times \mathbb{A}^d \cong \mathcal{E}|_{U_\alpha} \subset V \times S|_{U_\alpha} = \mathbb{A}^n \times U_\alpha$, we obtain a homomorphism $U_\alpha \times \mathbb{A}^d \hookrightarrow U_\alpha \times \mathbb{A}^n$ of trivial vector bundles over U_α. This determines a $n \times d$ matrix with coefficients in $\mathcal{O}(U_\alpha)$ of rank d; that is, a morphism $U_\alpha \to M_{n\times d}^d(k)$, to the variety of $n \times d$ matrices of rank d. By taking the wedge product of the d rows in this matrix, we obtain a morphism $f_\alpha : U_\alpha \to \mathbb{P}(\wedge^d(k^n))$ with image in the Grassmannian $\mathrm{Gr}(d, n)$. Using the fact that the transition functions of $\mathcal{E}$ are linear, verify that these morphisms glue to define a morphism $f = f_{\mathcal{E}} : S \to \mathbb{P}(\wedge^d(k^n))$ such that $f^*\mathcal{T} = \mathcal{E}$. In particular, this procedure determines the natural isomorphism: $\eta_S(\mathcal{E}) = f_{\mathcal{E}}$.

For a comprehensive coverage of the Grassmannian moduli functor and its representability, see [8] Section 8. The Grassmannian moduli functor has a natural generalisation to the moduli problem of classifying subsheaves of a fixed sheaf (or equivalently quotient sheaves with a natural notion of equivalence). This functor is representable by a *quot scheme* constructed by Grothendieck [9, 10] (for a survey of the construction, see [33]). Let us mention two special cases of this construction. Firstly, if we take our fixed sheaf to be the structure sheaf of a scheme X, then we are considering ideal sheaves and obtain a Hilbert scheme classifying subschemes of X. Secondly, if we take our fixed sheaf to be a locally free coherent sheaf $\mathcal{E}$ over X and consider quotient line bundles of $\mathcal{E}$, we obtain the projective space bundle $\mathbb{P}(\mathcal{E})$ over X (see [14] II §7).

2.4. **Pathological behaviour.**

Unfortunately, there are many natural moduli problems which do not admit a fine moduli space. In this section, we study some examples and highlight two particular pathologies which prevent a moduli problem from admitting a fine moduli space, namely:

(1) The jump phenomena: moduli may jump in families (in the sense that we can have a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_{s'}$ for all $s, s' \in \mathbb{A}^1 - \{0\}$, but $\mathcal{F}_0 \nsim \mathcal{F}_s$ for $s \in \mathbb{A}^1 - \{0\}$).
(2) The moduli problem may be unbounded (in that there is no family $\mathcal{F}$ over a scheme S which parametrises all objects in the moduli problem).

Example 2.21. We consider the naive moduli problem of classifying endomorphisms of a n-dimensional k-vector space. More precisely $\mathcal{A}$ consists of pairs (V, T), where V is an n-dimensional k-vector space and T is an endomorphism of V. We say $(V, \phi) \sim (V', \phi')$ if there exists an isomorphism $h : V \to V'$ compatible with the endomorphisms i.e. $h \circ \phi = \phi' \circ h$. We extend this to a moduli problem by defining a family over S to be a rank n vector bundle $\mathcal{F}$ over S with an endomorphism $\phi : \mathcal{F} \to \mathcal{F}$. Then we say $(\mathcal{F}, \phi) \sim_S (\mathcal{G}, \phi')$ if there is an isomorphism $h : \mathcal{F} \to \mathcal{G}$ such that $h \circ \phi = \phi' \circ h$. Let $\mathcal{E}nd_n$ be the corresponding moduli functor.

For any $n \geq 2$, we can construct families which exhibit the jump phenomena. For concreteness, let $n = 2$. Then consider the family over $\mathbb{A}^1$ given by $(\mathcal{F} = \mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \phi)$ where for $s \in \mathbb{A}^1$

$$\phi_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For $s, t \neq 0$, these matrices are similar and so $\phi_t \sim \phi_s$. However, $\phi_0 \nsim \phi_1$, as this matrices have distinct Jordan normal forms. Hence, we have produced a family with the jump phenomenon.

Example 2.22. Let us consider the moduli problem of vector bundles over $\mathbb{P}^1$ of rank 2 and degree 0.

We claim there is no family $\mathcal{F}$ over a scheme S with the property that for any rank 2 degree 0 vector bundle $\mathcal{E}$ on $\mathbb{P}^1$, there is a k-point $s \in S$ such that $\mathcal{F}|_s \cong \mathcal{E}$. Suppose such a family $\mathcal{F}$ over a scheme S exists. For each $n \in \mathbb{N}$, we have a rank 2 degree 0 vector bundle $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ (in fact, by Grothendieck's Theorem classifying vector bundles on $\mathbb{P}^1$, every rank 2 degree 0 vector bundle on $\mathbb{P}^1$ has this form). Furthermore, we have

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = \dim_k(k[x_0, x_1]_n \oplus k[x_0, x_1]_{-n}) = \begin{cases} 2 & \text{if } n = 0, \\ n+1 & \text{if } n \geq 1. \end{cases}$$

Consider the subschemes $S_n := \{s \in S : \dim H^0(\mathbb{P}^1, \mathcal{F}_s) \geq n\}$ of S, which are closed by the semi-continuity theorem (see [14] III Theorem 12.8). Then we obtain a decreasing chain of closed subschemes

$$S = S_2 \supsetneqq S_3 \supsetneqq S_4 \supsetneqq \dots.$$

each of which is distinct as $\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \in S_{n+1} - S_{n+2}$. The existence of this chain contradicts the fact that S is Noetherian (recall that for us scheme means scheme of finite type over k). In particular, the moduli problem of vector bundles of rank 2 and degree 0 is unbounded.

In fact, we also see the jump phenomena: there is a family $\mathcal{F}$ of rank 2 degree 0 vector bundles over $\mathbb{A}^1 = \operatorname{Spec} k[s]$ such that

$$\mathcal{F}_s = \begin{cases} \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} & s \neq 0 \\ \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & s = 0. \end{cases}$$

To construct this family, we note that

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong k$$

by Serre duality. Therefore, there is a family of extensions $\mathcal{F}$ over $\mathbb{A}^1$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$ with the desired property.

In both cases there is no fine moduli space for this problem. To solve these types of phenomena, one usually restricts to a nicer class of objects (we will return to this idea later on).

Example 2.23. We can see more directly that there is no fine moduli space for $\mathcal{E}nd_n$. Suppose M is a fine moduli space. Then we have a bijection between morphisms $S \to M$ and families over S up to equivalence. Choose any $n \times n$ matrix T, which determines a point $m \in M$. Then for $S = \mathbb{P}^1$ we have that the trivial families $(\mathcal{O}_{\mathbb{P}^1}^n, T)$ and $(\mathcal{O}_{\mathbb{P}^1}^n \otimes \mathcal{O}_{\mathbb{P}^1}(1), T \otimes \mathrm{Id}_{\mathcal{O}_{\mathbb{P}^1}(1)})$ are non-equivalent families which determine the same morphism $\mathbb{P}^1 \to M$, namely the constant morphism to the point m.

2.5. **Coarse moduli spaces.** As demonstrated by the above examples, not every moduli functor has a fine moduli space. By only asking for a natural transformation $\mathcal{M} \to h_M$ which is universal and a bijection over $\operatorname{Spec} k$ (so that the k-points of M are in bijection with the equivalence classes $\mathcal{A}/\sim$), we obtain a weaker notion of a coarse moduli space.

Definition 2.24. A coarse moduli space for a moduli functor $\mathcal{M}$ is a scheme M and a natural transformation of functors $\eta : \mathcal{M} \to h_M$ such that

(a) $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is bijective.
(b) For any scheme N and natural transformation $\nu : \mathcal{M} \to h_N$, there exists a unique morphism of schemes $f : M \to N$ such that $\nu = h_f \circ \eta$, where $h_f : h_M \to h_N$ is the corresponding natural transformation of presheaves.

Remark 2.25. A coarse moduli space for $\mathcal{M}$ is unique up to unique isomorphism: if (M, η) and (M', η') are coarse moduli spaces for $\mathcal{M}$, then by Property (b) there exists unique morphisms $f : M \to M'$ and $f' : M' \to M$ such that h_f and $h_{f'}$ fit into two commutative triangles:

$$\begin{tikzcd} h_M \arrow[d, "h_f"'] & \mathcal{M} \arrow[l, "\eta"'] \arrow[r, "\eta'"] \arrow[dl, "\eta'"] \arrow[dr, "\eta"'] & h_{M'} \arrow[d, "h_{f'}"] \\ h_{M'} & & h_M. \end{tikzcd}$$

Since $\eta = h_{f'} \circ h_f \circ \eta$ and $\eta = h_{\mathrm{id}_M} \circ \eta$, by uniqueness in (b) and the Yoneda Lemma, we have $f' \circ f = \mathrm{id}_M$ and similarly $f \circ f' = \mathrm{id}_{M'}$.

Proposition 2.26. *Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if*

(1) there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$,
(2) for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G} \iff \eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Proof. Exercise. □

Lemma 2.27. *Let $\mathcal{M}$ be a moduli problem and suppose there exists a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_1$ for all $s \neq 0$ and $\mathcal{F}_0 \nsim \mathcal{F}_1$. Then for any scheme M and natural transformation $\eta : \mathcal{M} \to h_M$, we have that $\eta_{\mathbb{A}^1}(\mathcal{F}) : \mathbb{A}^1 \to M$ is constant. In particular, there is no coarse moduli space for this moduli problem.*

Proof. Suppose we have a natural transformation $\eta : \mathcal{M} \to h_M$; then η sends the family $\mathcal{F}$ over $\mathbb{A}^1$ to a morphism $f : \mathbb{A}^1 \to M$. For any $s : \operatorname{Spec} k \to \mathbb{A}^1$, we have that $f \circ s = \eta_{\operatorname{Spec} k}(\mathcal{F}_s)$ and, for $s \neq 0$, $\mathcal{F}_s = \mathcal{F}_1 \in \mathcal{M}(\operatorname{Spec} k)$, so that $f|_{\mathbb{A}^1 - \{0\}}$ is a constant map. Let $m : \operatorname{Spec} k \to M$ be the point corresponding to the equivalence class for $\mathcal{F}_1$ under η. Since the k-valued points of M are closed (recall M is a scheme of finite type over an algebraically closed field), their preimages under morphisms must also be closed. Then, as $\mathbb{A}^1 - \{0\} \subset f^{-1}(m)$, the closure $\mathbb{A}^1$ of $\mathbb{A}^1 - \{0\}$ must also be contained in $f^{-1}(m)$; that is, f is the constant map to the k-valued point m of M. In particular, the map $\eta_{\operatorname{Spec} k} : \mathcal{M}(\operatorname{Spec} k) \to h_M(\operatorname{Spec} k)$ is not a bijection, as $\mathcal{F}_0 \neq \mathcal{F}_1$ in $\mathcal{M}(\operatorname{Spec} k)$, but these non-equivalent objects correspond to the same k-point m in M. □

In particular, the moduli problems of Examples 2.22 and 2.21 do not even admit coarse moduli spaces.

2.6. **The construction of moduli spaces.** The construction of many moduli spaces follows the same general pattern.

(1) Fix any discrete invariants for our objects - here the invariants should be invariant under the given equivalence relation (for example, for isomorphism classes of vector bundles on a curve, one may fix the rank and degree).
(2) Restrict to a reasonable class of objects which are bounded (otherwise, we can't find a coarse moduli space). Usually one restricts to a class of *stable* objects which are better behaved and bounded.
(3) Find a family $\mathcal{F}$ over a scheme P with the *local universal property* (i.e. locally any other family is equivalent to a pullback of this family - see below). We call P a *parameter space*, as the k-points of P surject onto $\mathcal{A}/\sim$; however, this is typically not a bijection.
(4) Find a group G acting on P such that p and q lie in the same G-orbit in P if and only if $\mathcal{F}_p \sim \mathcal{F}_q$. Then we have a bijection $P(k)/G \cong \mathcal{A}/\sim$.
(5) Typically this group action is algebraic (see Section 3) and by taking a quotient, we should obtain our moduli space. The quotient should be taken in the category of schemes (in terminology to come, it should be a *categorical quotient*) and this is done using Mumford's *Geometric Invariant Theory*.

Definition 2.28. For a moduli problem $\mathcal{M}$, a family $\mathcal{F}$ over a scheme S has the local universal property if for any other family $\mathcal{G}$ over a scheme T and for any k-point $t \in T$, there exists a neighbourhood U of t in T and a morphism $f : U \to S$ such that $\mathcal{G}|_U \sim_U f^*\mathcal{F}$.

In particular, we do not require the morphism f to be unique. We note that, for such a family to exist, we need our moduli problem to be bounded.