3. Algebraic group actions and quotients

In this section we consider group actions on algebraic varieties and also describe what type of quotients we would like to have for such group actions.

3.1. Affine Algebraic groups.

An algebraic group (over k) is a group object in the category of schemes (over k). By a Theorem of Chevalley, every algebraic group is an extension of an abelian variety (that is, a smooth connected *projective* algebraic group) by an *affine* algebraic group (whose underlying scheme is affine) [22, Theorem 10.25]. In this course, we only work with affine algebraic groups and cover the results which are most important for our purposes. A good reference for affine algebraic group schemes is the book (in preparation) of Milne [23]. For those who are interested in discovering more about algebraic groups, see [3, 22, 11].

Definition 3.1. An *algebraic group over* k is a scheme G over k with morphisms $e : \mathrm{Spec}\, k \to G$ (identity element), $m : G \times G \to G$ (group law) and $i : G \to G$ (group inversion) such that we have commutative diagrams

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times \mathrm{id}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}\, k \times G \arrow[r, "e\times \mathrm{id}"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "m"] & G \times \mathrm{Spec}\, k \arrow[l, "\mathrm{id}\times e"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "(i,\mathrm{id})"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(\mathrm{id},i)"'] \arrow[d] \\ \mathrm{Spec}\, k \arrow[r, "e"'] & G & \mathrm{Spec}\, k. \arrow[l, "e"] \end{tikzcd}$$

We say G is an *affine algebraic group* if the underlying scheme G is affine. We say G is a *group variety* if the underlying scheme G is a variety (recall in our conventions, varieties are not necessarily irreducible).

A *homomorphism* of algebraic groups G and H is a morphism of schemes $f : G \to H$ such that the following square commutes

$$\begin{tikzcd} G \times G \arrow[r, "m_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "m_H"'] & H. \end{tikzcd}$$

An *algebraic subgroup* of G is a closed subscheme H such that the immersion $H \hookrightarrow G$ is a homomorphism of algebraic groups. We say an algebraic group G' is an *algebraic quotient* of G if there is a homomorphism of algebraic groups $f : G \to G'$ which is flat and surjective.

Remark 3.2.

(1) The functor of points h_G of an algebraic group has a natural factorisation through the category of (abstract) groups, i.e, for every scheme X the operations m, e, i equip $\mathrm{Hom}(X, G)$ with a group structure and with this group structure, every map $h_G(f) : \mathrm{Hom}(X, G) \to \mathrm{Hom}(Y, G)$ for $f : Y \to X$ is a morphism of groups. In fact, one can show using the Yoneda lemma that there is an equivalence of categories between the category of algebraic groups and the category of functors $F : \mathrm{Sch} \to \mathrm{Grp}$ such that the composition $\mathrm{Sch} \stackrel{F}{\to} \mathrm{Grp} \to \mathrm{Set}$ is representable. When restricting to the category of affine k-schemes, this can give a very concrete description of an algebraic group, as we will see in the examples below.

(2) Let $\mathcal{O}(G) := \mathcal{O}_G(G)$ denote the k-algebra of regular functions on G. Then the above morphisms of affine varieties correspond to k-algebra homomorphisms $m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ (comultiplication) and $i^* : \mathcal{O}(G) \to \mathcal{O}(G)$ (coinversion) and the identity element corresponds to $e^* : \mathcal{O}(G) \to k$ (counit). These operations define a *Hopf algebra*

structure on the k-algebra $\mathcal{O}(G)$. Furthermore, there is a bijection between finitely generated Hopf algebras over k and affine algebraic groups (see [23] II Theorem 5.1).

(3) By a Theorem of Cartier, every affine algebraic group over a field k of characteristic zero is smooth (see [23] VI Theorem 9.3). Moreover, in Exercise sheet 3, we see that every algebraic group is separated. Hence, in characteristic zero, the notion of affine algebraic group and affine group variety coincide.

(4) In the definition of homomorphisms, we only require a compatibility with the group law m; it turns out that the compatibility for the identity and group inversion is then automatic. This is well known in the case of homomorphisms of abstract groups, and the algebraic case can then be deduced by applying the Yoneda lemma.

(5) For the definition of a quotient group, the condition that the homomorphism is flat is only needed in positive characteristic, as in characteristic zero this morphism is already smooth (this follows from the Theorem of Cartier mentioned above and the fact that the kernel of a homomorphism of smooth group schemes is smooth; see [22] Proposition 1.48)

Example 3.3. Many of the groups that we are already familiar with are affine algebraic groups.

(1) The additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$ over k is the algebraic group whose underlying variety is the affine line $\mathbb{A}^1$ over k and whose group structure is given by addition:

$$m^*(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad i^*(t) = -t.$$

Let us indicate how to show these operations satisfy the group axioms. We only prove the associativity, the other axioms being similar and easier. We have to show that

$$(m^* \otimes \mathrm{id}) \circ m^* = (\mathrm{id} \otimes m^*) \circ m^* : k[t] \to k[t] \otimes k[t] \otimes k[t].$$

This is a map of k-algebras, so it is enough to check it for t. We have

$$((m^* \otimes \mathrm{id}) \circ m^*)(t) = (m^* \otimes \mathrm{id})(t \otimes 1 + 1 \otimes t) = t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t$$

and similarly

$$((\mathrm{id} \otimes m^*) \circ m^*)(t) = t \otimes 1 \otimes 1 + 1 \otimes t \otimes 1 + 1 \otimes t \otimes 1$$

which completes the proof. For a k-algebra R, we have $\mathbb{G}_a(R) = (R, +)$; this justifies the name of the 'additive group'.

(2) The multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ over k is the algebraic group whose underlying variety is the $\mathbb{A}^1 - \{0\}$ and whose group action is given by multiplication:

$$m^*(t) = t \otimes t \quad \text{and} \quad i^*(t) = t^{-1}.$$

For a k-algebra R, we have $\mathbb{G}_m(R) = (R^\times, \cdot)$; hence, the name of the 'multiplicative group'.

(3) The general linear group GL_n over k is an open subvariety of $\mathbb{A}^{n^2}$ cut out by the condition that the determinant is non-zero. It is an affine variety with coordinate ring $k[x_{ij} : 1 \leq i, j \leq n]_{\det(x_{ij})}$. The co-group operations are defined by:

$$m^*(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad i^*(x_{ij}) = (x_{ij})^{-1}_{ij}$$

where $(x_{ij})^{-1}_{ij}$ is the regular function on GL_n given by taking the (i,j)-th entry of the inverse of a matrix. For a k-algebra R, the group $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with coefficients in R, with the usual matrix multiplication.

(4) More generally, if V is a finite-dimensional vector space over k, there is an affine algebraic group $\mathrm{GL}(V)$ which is (non-canonically) isomorphic to $\mathrm{GL}_{\dim(V)}$. For a k-algebra R, we have $\mathrm{GL}(V)(R) = \mathrm{Aut}_R(V \otimes_k R)$.

(5) Let G be a finite (abstract) group. Then G can be naturally seen as an algebraic group $\underline{G}_k$ over k as follows. The group operations on G make the group algebra $k[G]$ into a Hopf algebra over k, and $\underline{G}_k := \operatorname{Spec}(k[G])$ is a 0-dimensional variety whose points are naturally identified with elements of G.

(6) Let $n \geq 1$. Put $\mu_n := \operatorname{Spec} k[t, t^{-1}]/(t^n - 1) \subset \mathbb{G}_m$, the subscheme of n-roots of unity. Write I for the ideal $(t^n - 1)$ of $R := k[t, t^{-1}]$. Then

$$m^*(t^n - 1) = t^n \otimes t^n - 1 \otimes 1 = (t^n - 1) \otimes t^n + 1 \otimes (t^n - 1) \in I \otimes R + R \otimes I$$

which implies that μ_n is an algebraic subgroup of $\mathbb{G}_m$. If n is different from char(k), the polynomial $X^n - 1$ is separable and there are n distinct roots in k. Then the choice of a primitive n-th root of unity in k determines an isomorphism $\mu_n \simeq \underline{\mathbb{Z}/n\mathbb{Z}}_k$. If $n = \operatorname{char}(k)$, however, we have $X^n - 1 = (X - 1)^n$ in $k[X]$, which implies that the scheme μ_n is non-reduced (with 1 as only closed point). This is the simplest example of a non-reduced algebraic group.

A linear algebraic group is by definition a subgroup of GL_n which is defined by polynomial equations; for a detailed introduction to linear algebraic groups, see [1, 15, 40]. For instance, the special linear group is a linear algebraic group. In particular, any linear algebraic group is an affine algebraic group. In fact, the converse statement is also true: any affine algebraic group is a linear algebraic group (see Theorem 3.9 below).

An affine algebraic group G over k determines a group-valued functor on the category of finitely generated k-algebras given by $R \mapsto G(R)$. Similarly, for a vector space V over k, we have a group valued functor $\mathrm{GL}(V)$ given by $R \mapsto \mathrm{Aut}_R(V \otimes_k R)$, the group of R-linear automorphisms. If V is finite dimensional, then $\mathrm{GL}(V)$ is an affine algebraic group.

Definition 3.4. A *linear representation* of an algebraic group G on a vector space V over k is a homomorphism of group valued functors $\rho : G \to \mathrm{GL}(V)$. If V is finite dimensional, this is equivalent to a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$, which we call a *finite dimensional linear representation* of G.

If G is affine, we can describe a linear representation $\rho : G \to \mathrm{GL}(V)$ more concretely in terms of its associated *co-module* as follows. The natural inclusion $\mathrm{GL}(V) \to \mathrm{End}(V)$ and $\rho : G \to \mathrm{GL}(V)$ determine a functor $G \to \mathrm{End}(V)$, such that the universal element in $G(\mathcal{O}(G))$ given by the identity morphism corresponds to an $\mathcal{O}(G)$-linear endomorphism of $V \otimes_k \mathcal{O}(G)$, which by the universality of the tensor product is uniquely determined by its restriction to a k-linear homomorphism $\rho^* : V \to V \otimes_k \mathcal{O}(G)$; this is the associated co-module. If V is finite dimensional, we can even more concretely describe the associated co-module by considering the group homomorphism $G \to \mathrm{End}(V)$ and its corresponding homomorphism of k-algebras $\mathcal{O}(V \otimes_k V^*) \to \mathcal{O}(G)$, which is determined by a k-linear homomorphism $V \otimes_k V^* \to \mathcal{O}(G)$ or equivalently by the co-module $\rho^* : V \to V \otimes_k \mathcal{O}(G)$. In particular, a linear representation of an affine algebraic group G on a vector space V is equivalent to a co-module structure on V (for the full definition of a co-module structure, see [23] Chapter 4).

3.2. **Group actions.**

Definition 3.5. An (algebraic) action of an affine algebraic group G on a scheme X is a morphism of schemes $\sigma : G \times X \to X$ such that the following diagrams commute

$$\begin{tikzcd} \operatorname{Spec} k \times X \arrow[r, "e \times \mathrm{id}_X"] \arrow[dr, "\cong"'] & G \times X \arrow[d, "\sigma"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id}_G \times \sigma"] \arrow[d, "m_G \times \mathrm{id}_X"'] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

Suppose we have actions $\sigma_X : G \times X \to X$ and $\sigma_Y : G \times Y \to Y$ of an affine algebraic group G on schemes X and Y. Then a morphism $f : X \to Y$ is *G-equivariant* if the following diagram commutes

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\sigma_X"'] & G \times Y \arrow[d, "\sigma_Y"'] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

If Y is given the trivial action $\sigma_Y = \pi_Y : G \times Y \to Y$, then we refer to a G-equivariant morphism $f : X \to Y$ as a *G-invariant morphism*.

Remark 3.6. If X is an affine scheme over k and $\mathcal{O}(X)$ denotes its algebra of regular functions, then an action of G on X gives rise to a coaction homomorphism of k-algebras:

$$\begin{array}{rccc} \sigma^* : & \mathcal{O}(X) & \to & \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X) \\ & f & \mapsto & \sum h_i \otimes f_i. \end{array}$$

This gives rise to a homomorphism $G \to \mathrm{Aut}(\mathcal{O}(X))$ where the k-algebra automorphism of $\mathcal{O}(X)$ corresponding to $g \in G$ is given by

$$f \mapsto \sum h_i(g) f_i \in \mathcal{O}(X)$$

for $f \in \mathcal{O}(X)$ with $\sigma^*(f) = \sum h_i \otimes f_i$.

Definition 3.7. An action of an affine algebraic group G on a k-vector space V (resp. k-algebra A) is given by, for each k-algebra R, an action of $G(R)$ on $V \otimes_k R$ (resp. on $A \otimes_k R$)

$$\sigma_R : G(R) \times (V \otimes_k R) \to V \otimes_k R \quad (\text{resp. } \sigma_R : G(R) \times (A \otimes_k R) \to A \otimes_k R)$$

such that $\sigma_R(g, -)$ is a morphism of R-modules (resp. R-algebras) and these actions are functorial in R. We say that an action of G on a k-algebra A is *rational* if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Lemma 3.8. *Let G be an affine algebraic group acting on an affine scheme X. Then any $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$. Furthermore, for any finite dimensional vector subspace W of $\mathcal{O}(G)$, there is a finite dimensional G-invariant vector subspace V of $\mathcal{O}(X)$ containing W.*

Proof. Let $\sigma : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ denote the coaction homomorphism. Then we can write $\sigma^*(f) = \sum_{i=1}^n h_i \otimes f_i$, for $h_i \in \mathcal{O}(G)$ and $f_i \in \mathcal{O}(X)$. Then $g \cdot f = \sum_i h_i(g) f_i$ and so the vector space spanned by $f_1, \dots, f_n$ is a G-invariant subspace containing f. The second statement follows by applying the same argument to a given basis of W. □

In particular, the action of G on the k-algebra $\mathcal{O}(X)$ is *rational* (that is, every $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant linear subspace of $\mathcal{O}(X)$).

One of the most natural actions is the action of G on itself by left (or right) multiplication. This induces a rational action $\sigma : G \to \mathrm{Aut}(\mathcal{O}(G))$.

Theorem 3.9. *Any affine algebraic group G over k is a linear algebraic group.*

Proof. As G is an affine scheme (of finite type over k), the ring of regular functions $\mathcal{O}(G)$ is a finitely generated k-algebra. Therefore the vector space W spanned by a choice of generators for $\mathcal{O}(G)$ as a k-algebra is finite dimensional. By Lemma 3.8, there is a finite dimensional subspace V of $\mathcal{O}(G)$ which is preserved by the G-action and contains W.

Let $m^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ denote the comultiplication; then for a basis $f_1, \dots, f_n$ of V, we have $m^*(f_i) \in \mathcal{O}(G) \otimes V$, hence we can write

$$m^*(f_i) = \sum_{j=1}^n a_{ij} \otimes f_j$$

for functions $a_{ij} \in \mathcal{O}(G)$. In terms of the action $\sigma : G \to \mathrm{Aut}(\mathcal{O}(G))$, we have that $\sigma(g, f_i) = \sum_j a_{ij}(g) f_j$. This defines a k-algebra homomorphism

$$\rho^* : \mathcal{O}(\mathrm{Mat}_{n \times n}) \to \mathcal{O}(G) \qquad x_{ij} \mapsto a_{ij}.$$

To show that the corresponding morphism of affine schemes $\rho : G \to \mathrm{Mat}_{n \times n}$ is a closed embedding, we need to show ρ^* is surjective. Note that V is contained in the image of ρ^* as

$$f_i = (\mathrm{Id}_{\mathcal{O}(G)} \otimes e^*) m^*(f_i) = (\mathrm{Id}_{\mathcal{O}(G)} \otimes e^*) \sum_{j=1}^n a_{ij} \otimes f_j = \sum_{j=1}^n e^*(f_j) a_{ij}.$$

Since V generates $\mathcal{O}(G)$ as a k-algebra, it follows that ρ^* is surjective. Hence ρ is a closed immersion.

Finally, we claim that $\rho : G \to \mathrm{Mat}_{n\times n}$ is a homomorphism of semigroups (recall that a semigroup is a group without inversion, such as matrices under multiplication) i.e. we want to show on the level of k-algebras that we have a commutative square

$$\begin{tikzcd} \mathcal{O}(\mathrm{Mat}_{n\times n}) \arrow[r, "m^*_{\mathrm{Mat}}"] \arrow[d, "\rho^*\otimes\rho^*"'] & \mathcal{O}(\mathrm{Mat}_{n\times n}) \otimes \mathcal{O}(\mathrm{Mat}_{n\times n}) \arrow[d, "\rho^*"] \\ \mathcal{O}(G) \arrow[r, "m^*_G"'] & \mathcal{O}(G) \otimes \mathcal{O}(G); \end{tikzcd}$$

that is, we want to show for the generators $x_{ij} \in \mathcal{O}(\mathrm{Mat}_{n\times n})$, we have

$$m^*_G(a_{ij}) = m^*_G(\rho^*(x_{ij})) = (\rho^* \otimes \rho^*)(m^*_{\mathrm{Mat}}(x_{ij})) = (\rho^* \otimes \rho^*)\left(\sum_k x_{ik} \otimes x_{kj}\right) = \sum_k a_{ik} \otimes a_{kj}.$$

To prove this, we consider the associativity identity $m_G \circ (\mathrm{id} \times m_G) = m_G \circ (m_G \times \mathrm{id})$ and apply this on the k-algebra level to $f_i \in \mathcal{O}(G)$ to obtain

$$\sum_{k,j} a_{ik} \otimes a_{kj} \otimes f_j = \sum_j m^*_G(a_{ij}) \otimes f_j$$

as desired. Furthermore, as G is a group rather than just a semigroup, we can conclude that the image of ρ is contained in the group GL_n of invertible elements in the semigroup $\mathrm{Mat}_{n\times n}$. □

Tori are a basic class of algebraic group which are used extensively to study the structure of more complicated algebraic groups (generalising the use of diagonal matrices to study matrix groups through eigenvalues and the Jordan normal form).

Definition 3.10. Let G be an affine algebraic group scheme over k.

(1) G is an (algebraic) torus if $G \cong \mathbb{G}_m^n$ for some $n > 0$.
(2) A torus of G is a subgroup scheme of G which is a torus.
(3) A maximal torus of G is a torus $T \subset G$ which is not contained in any other torus.

For a torus T, we have commutative groups

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \quad X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$$

called the *character group* and *cocharacter group* respectively, where the morphisms are homomorphisms of linear algebraic groups. Let us compute $X^*(\mathbb{G}_m)$.

Lemma 3.11. *The map*

$$\begin{array}{rcl} \theta : \mathbb{Z} & \to & X^*(\mathbb{G}_m) \\ n & \mapsto & (t \mapsto t^n) \end{array}$$

is an isomorphism of groups.

Proof. Let us first show that this is well defined. Write m^* for the comultiplication on $\mathcal{O}(\mathbb{G}_m)$. Then $m^*(t^n) = (t\otimes t)^n = t^n \otimes t^n$ shows that $\theta(n) : \mathbb{G}_m \to \mathbb{G}_m$ is a morphism of algebraic groups. Since $t^a t^b = t^{a+b}$, θ itself is a morphism of groups. It is clearly injective, so it remains to show surjectivity.

Let ϕ be an endomorphism of $\mathbb{G}_m$. Write $\phi^*(t) \in k[t, t^{-1}]$ as $\sum_{|i|<m} a_i t^i$. We have $m^*(\phi^*(t)) = \phi^*(t) \otimes \phi^*(t)$, which translates into

$$\sum_i a_i t^i \otimes t^i = \sum_{i,j} a_i a_j t^i \otimes t^j.$$

From this, we deduce that at most one a_i is non-zero, say a_n. Looking at the compatibility of ϕ with the unit, we see that necessarily $a_n = 1$. This shows that $\phi = \theta(n)$, completing the proof. □

For a general torus T, we deduce from the Lemma that the (co)character groups are finite free $\mathbb{Z}$-modules of rank $\dim T$. There is a perfect pairing between these lattices given by composition

$$< \, , \, >: X^*(T) \times X_*(T) \to \mathbb{Z}$$

where $< \chi, \lambda >:= \chi \circ \lambda$.

An important fact about tori is that their linear representations are completely reducible. We will often use this result to diagonalise a torus action (i.e. choose a basis of eigenvectors for the T-action so that the action is diagonal with respect to this basis).

Proposition 3.12. *For a finite dimensional linear representation of a torus $\rho : T \to \mathrm{GL}(V)$, there is a weight space decomposition*

$$V \cong \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{v \in V : t \cdot v = \chi(t) v \; \forall t \in T\}$ are called the weight spaces *and $\{\chi : V_\chi \neq 0\}$ are called the* weights *of the action.*

Proof. To keep the notation simple, we give the proof for $T \cong \mathbb{G}_m$, where $X^*(T) \cong \mathbb{Z}$; the general case can be obtained either by adapting the proof (with further notation) or by induction on the dimension of T. The representation ρ has an associated co-module

$$\rho^* : V \to V \otimes_k \mathcal{O}(\mathbb{G}_m) \cong V \otimes k[t, t^{-1}].$$

and the diagram

$$\begin{tikzcd} V \arrow[r, "\rho"'] \arrow[d, "\rho"'] & V \otimes k[t,t^{-1}] \arrow[d, "\mathrm{id} \otimes m^*"] \\ V \otimes k[t,t^{-1}] \arrow[r, "\rho \otimes \mathrm{id}"'] & V \otimes k[t,t^{-1}] \otimes k[t,t^{-1}] \end{tikzcd}$$

commutes. From this, it follows easily that, for each integer m, the space

$$V_m = \{v \in V : \rho^*(v) = v \otimes t^m\}$$

is a subrepresentation of V.

For $v \in V$, we have $\rho^*(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m$ where $f_m : V \to V$ is a linear map, and because of the compatibility with the identity element, we find that

$$v = \sum_{m \in \mathbb{Z}} f_m(v).$$

If $\rho^*(v) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m$, then we claim that $f_m(v) \in V_m$. From the diagram above

$$\sum_{m \in \mathbb{Z}} \rho^*(f_m(v)) \otimes t^m = (\rho^* \otimes \mathrm{Id}_{k[t,t^{-1}]})(\rho^*(v)) = (\mathrm{Id}_V \otimes m^*)(\rho^*(v)) = \sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m$$

and as $\{t^m\}_{m \in \mathbb{Z}}$ are linearly independent in $k[t, t^{-1}]$, the claim follows.

Let us show that in fact, the f_m form a collection of orthogonal projectors onto the subspaces V_m. Using the commutative diagram again, we get

$$\sum_{m \in \mathbb{Z}} f_m(v) \otimes t^m \otimes t^m = \sum_{m,n \in \mathbb{Z}} f_m(f_n(v)) \otimes t^m \otimes t^n,$$

which again by linear independence of the $\{t^m\}$ shows that $f_m \circ f_n$ vanishes if $m \neq n$ and is equal to f_n otherwise; this proves that they are orthogonal idempotents. Hence, the V_m are linearly independent and this completes the proof. □

This result can be phrased as follows: there is an equivalence between the category of linear representations of T and $X^*(T)$-graded k-vector spaces. We note that there are only finitely many weights of the T-action, for reasons of dimension.

3.3. **Orbits and stabilisers.**

Definition 3.13. Let G be an affine algebraic group acting on a scheme X by $\sigma : G \times X \to X$ and let x be a k-point of X.

i) The orbit $G \cdot x$ of x to be the (set-theoretic) image of the morphism $\sigma_x = \sigma(-, x) : G(k) \to X(k)$ given by $g \mapsto g \cdot x$.
ii) The stabiliser G_x of x to be the fibre product of $\sigma_x : G \to X$ and $x : \operatorname{Spec} k \to X$.

The stabiliser G_x of x is a closed subscheme of G (as it is the preimage of a closed subscheme of X under $\sigma_x : G \to X$). Furthermore, it is a subgroup of G.

Exercise 3.14. Using the same notation as above, consider the presheaf on Sch whose S-points are the set

$$\{g \in h_G(S) : g \cdot (x_S) = x_S\}$$

where $x_S : S \to X$ is the composition $S \to \operatorname{Spec} k \to X$ of the structure morphism of S with the inclusion of the point x. Describe the presheaf structure and show that this functor is representable by the stabiliser G_x.

The situation for orbits is clarified by the following result.

Proposition 3.15. *Let G be an affine algebraic group acting on a scheme X. The orbits of closed points are locally closed subsets of X, hence can be identified with the corresponding reduced locally closed subschemes.*

Moreover, the boundary of an orbit $\overline{G \cdot x} - G \cdot x$ is a union of orbits of strictly smaller dimension. In particular, each orbit closure contains a closed orbit (of minimal dimension).

Proof. Let $x \in X(k)$. The orbit $G \cdot x$ is the set-theoretic image of the morphism σ_x, hence by a theorem of Chevalley ([14] II Exercise 3.19), it is constructible, i.e., there exists a dense open subset U of $\overline{G \cdot x}$ with $U \subset G \cdot x \subset \overline{G \cdot x}$. Because G acts transitively on $G \cdot x$ through σ_x, this implies that every point of $G \cdot x$ is contained in a translate of U. This shows that $G \cdot x$ is open in $\overline{G \cdot x}$, which precisely means that $G \cdot x$ is locally closed. With the corresponding reduced scheme structure of $G \cdot x$, there is an action of G_{red} on $G \cdot x$ which is transitive on k-points. In particular, it makes sense to talk about its dimension (which is the same at every point because of the transitive action of G_{red}).

The boundary of an orbit $G \cdot x$ is invariant under the action of G and so is a union of G-orbits. Since $G \cdot x$ is locally closed, the boundary $\overline{G \cdot x} - G \cdot x$, being the complement of a dense open set, is closed and of strictly lower dimension than $G \cdot x$. This implies that orbits of minimum dimension are closed and so each orbit closure contains a closed orbit. □

Definition 3.16. An action of an affine algebraic group G on a scheme X is *closed* if all G-orbits in X are closed.