Example 3.17. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The orbits of this action are

- conics $\{(x, y) : xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$,
- the punctured x-axis,
- the punctured y-axis,
- the origin.

The origin and the conic orbits are closed whereas the punctured axes both contain the origin in their orbit closures. The dimension of the orbit of the origin is strictly smaller than the dimension of $\mathbb{G}_m$, indicating that its stabiliser has positive dimension.

Example 3.18. Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by scalar multiplication: $t \cdot (a_1, \dots, a_n) = (ta_1, \dots, ta_n)$. In this case, there are two types of orbits:

- punctured lines through the origin,
- the origin.

The origin is the only closed orbit, which has dimension zero. Furthermore, every orbit contains the origin in its closure.

Exercise 3.19. In Examples 3.17 and 3.18, write down the coaction homomorphism explicitly.

Proposition 3.20. *Let G be an affine algebraic group acting on a scheme X. For $x \in X(k)$, we have*

$$\dim(G) = \dim(G_x) + \dim(G \cdot x)$$

Proof. Since the dimension is a topological invariant of a scheme, we can assume G and X are reduced. The orbit $G \cdot x$, which we see as a locally closed subscheme of X according to the previous proposition, is reduced by definition. This implies that the morphism $\sigma_x : G \to G \cdot x$ is flat at every generic point of $G \cdot x$ (every k-scheme is flat over k), hence, by the openess of the flat locus of σ_x (EGA IV$_3$ 11.1.1), there exists a dense open set U such that $\sigma_x^{-1}(U) \to U$ is flat. Using the transitive action of G on $G \cdot x$ (which is well defined because G is reduced), we deduce that σ_x is flat. Moreover, by definition, the fibre of σ_x at x is the stabiliser G_x. We can thus apply the dimension formula for fibres of a flat morphism [14, Proposition III.9.5], which yields

$$\dim(G_x) = \dim(G) - \dim(G \cdot x)$$

as required. □

Proposition 3.21. *Let G be an affine algebraic group acting on a scheme X by a morphism $\sigma : G \times X \to X$. Then the dimension of the stabiliser subgroup (resp. orbit) viewed as a function $X \to \mathbb{N}$ is upper semi-continuous (resp. lower-semi-continuous); that is, for every n, the sets*

$$\{x \in X : \dim G_x \geq n\} \text{ and } \{x \in X : \dim(G \cdot x) \leq n\}$$

are closed in X.

Proof. Consider the graph of the action

$$\Gamma = (\mathrm{pr}_X, \sigma) : G \times X \to X \times X$$

and the fibre product P

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \Delta \\ G \times X & \xrightarrow{\Gamma} & X \times X, \end{array}$$

where $\Delta : X \to X \times X$ is the diagonal morphism; then the k-points of the fibre product P consists of pairs (g, x) such that $g \in G_x$. The function on P which sends $p = (g, x) \in P$ to the dimension of $P_{\varphi(p)} := \varphi^{-1}(\varphi(p))$ is upper semi-continuous (cf. [14] III 12.8 or EGA IV 13.1.3); that is, for all n

$$\{p \in P : \dim P_{\varphi(p)} \geq n\}$$

is closed in P. By restricting to the closed subscheme $X \cong \{(e,x) : x \in X\} \subset P$, we conclude that the dimension of the stabiliser of x is upper semi-continuous; that is,

$$\{x \in X : \dim G_x \geq n\}$$

is closed in X for all n. Using the previous proposition, we deduce the statement for dimensions of orbits. □

Lemma 3.22. *Let G be an affine algebraic group acting on a scheme X over k.*

i) If G is an affine group variety and Y and Z are subschemes of X such that Z is closed, then

$$\{g \in G : gY \subset Z\}$$

is closed.

ii) If X is a variety, then for any subgroup $H \subset G$ the fixed point locus

$$X^H = \{x \in X : H \cdot x = x\}$$

is closed in X.

Proof. Exercise. (Hint: express these subsets as intersections of preimages of closed subschemes under morphisms associated to the action.) □

3.4. **First notions of quotients.** Let G be an affine algebraic group acting on a scheme X over k. In this section and §3.5, we introduce different types of quotients for the action of G on X; the main references for these sections are [4], [25] and [31].

The orbit space $X/G = \{G \cdot x : x \in X\}$ for the G-action on X, may not always admit the structure of a scheme. Instead we ask for a universal quotient in the category of schemes (of finite type over k).

Definition 3.23. A *categorical quotient* for the action of G on X is a G-invariant morphism $\varphi : X \to Y$ of schemes which is universal; that is, every other G-invariant morphism $f : X \to Z$ factors uniquely through φ so that there exists a unique morphism $h : Y \to Z$ such that $f = \varphi \circ h$. Furthermore, if the preimage of each k-point in Y is a single orbit, then we say φ is an *orbit space.*

As φ is constant on orbits, it is also constant on orbit closures. Hence, a categorical quotient is an orbit space only if the action of G on X is closed; that is, all the orbits $G \cdot x$ are closed.

Remark 3.24. The categorical quotient has nice functorial properties in the following sense: if $\varphi : X \to Y$ is G-invariant and we have an open cover U_i of Y such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ is a categorical quotient for each i, then φ is a categorical quotient.

Exercise 3.25. Let $\varphi : X \to Y$ be a categorical quotient of a G-action on X.

i) If X is connected, show that Y is connected.
ii) If X is irreducible, show that Y is irreducible.
iii) If X is reduced, show that Y is reduced.

Example 3.26. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ as in Example 3.18. As the origin is in the closure of every single orbit, any G-invariant morphism $\mathbb{A}^n \to Z$ must be a constant morphism. Therefore, we claim that the categorical quotient is the structure map $\varphi : \mathbb{A}^n \to \operatorname{Spec} k$ to the point $\operatorname{Spec} k$. This morphism is clearly G-invariant and any other G-invariant morphism $f : \mathbb{A}^n \to Z$ is a constant morphism to $z \in Z(k)$. Therefore, there is a unique morphism $z : \operatorname{Spec} k \to Z$ such that $f = z \circ \varphi$.

We now see the sort of problems that may occur when we have non-closed orbits. In Example 3.18 our geometric intuition tells us that we would ideally like to remove the origin and then take the quotient of $\mathbb{G}_m$ acting on $\mathbb{A}^n - \{0\}$. In fact, we already know what we want this quotient to be: the projective space $\mathbb{P}^{n-1} = (\mathbb{A}^n - \{0\})/\mathbb{G}_m$ which is an orbit space for this action.

3.5. **Second notions of quotient.** Let G be an affine algebraic group acting on a scheme X over k. The group G acts on the k-algebra $\mathcal{O}(X)$ of regular functions on X by

$$g \cdot f(x) = f(g^{-1} \cdot x)$$

and we denote the subalgebra of invariant functions by

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : g \cdot f = f \text{ for all } g \in G\}.$$

Similarly if $U \subset X$ is a subset which is invariant under the action of G (that is, $g \cdot u \in U$ for all $u \in U$ and $g \in G$), then G acts on $\mathcal{O}_X(U)$ and we write $\mathcal{O}_X(U)^G$ for the subalgebra of invariant functions.

The following notion of a good quotient came out of geometric invariant theory; more precisely, we will later see that GIT quotients are good quotients. However, it is clear that many of the properties of a good quotient and desirable. Furthermore, we will soon see that a good quotient is a categorical quotient.

Definition 3.27. A morphism $\varphi : X \to Y$ is a *good quotient* for the action of G on X if

i) φ is G-invariant.
ii) φ is surjective.
iii) If $U \subset Y$ is an open subset, the morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto the G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.
iv) If $W \subset X$ is a G-invariant closed subset of X, its image $\varphi(W)$ is closed in Y.
v) If W_1 and W_2 are disjoint G-invariant closed subsets, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.
vi) φ is affine (i.e. the preimage of every affine open is affine).

If moreover, the preimage of each point is a single orbit then we say φ is a *geometric quotient.*

Exercise 3.28. Assuming that ii) holds, prove that conditions iv) and v) together are equivalent to:

v)′ If W_1 and W_2 are disjoint G-invariant closed subsets, then the closures of $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

Remark 3.29. In fact, surjectivity is a consequence of iii) and iv): condition iii) shows that φ is dominant (i.e. the image of φ is dense in Y) and condition iv) shows that the image of φ is closed in Y.

Proposition 3.30. *Let G be an affine algebraic group acting on a scheme X and suppose we have a morphism $\varphi : X \to Y$ satisfying properties i), iii), iv) and v) in the definition of good quotient. Then φ is a categorical quotient. In particular, any good quotient is a categorical quotient.*

Proof. Property i) of the definition of a good quotient states that φ is G-invariant and so we need only prove that it is universal with respect to all G-invariant morphisms from X. Let $f : X \to Z$ be a G-invariant morphism; then we will construct a unique morphism $h : Y \to Z$ such that $f = h \circ \varphi$ by taking a finite affine open cover U_i of Z (we can take the cover to be finite as Z is of finite type over k), then using this cover to define a cover of Y by open subsets V_i, and finally by locally defining morphisms $h_i : V_i \to U_i$ which glue to give h.

Since $W_i := X - f^{-1}(U_i)$ is G-invariant and closed in X, its image $\varphi(W_i) \subset Y$ is closed by iv). Let $V_i := Y - \varphi(W_i)$ be the open complement; then by construction, we have an inclusion $\varphi^{-1}(V_i) \subset f^{-1}(U_i)$. As U_i cover Z, the intersection $\cap_i W_i$ is empty. We claim by property v) of the good quotient φ, we have $\cap_i \varphi(W_i) = \emptyset$; that is, V_i are an open cover of Y. To see this, suppose for a contradiction that the intersection $\cap_i \varphi(W_i)$ is non-empty; then as we are working with finite type schemes, this intersection has a closed point, which is a k-point as k is algebraically closed. Let W be a closed G-orbit in the preimage of the k-point $p \in \cap_i \varphi(W_i)$. Then by property v), we must have $W \cap W_i \neq \emptyset$ for each i, since $\varphi(W) \cap \varphi(W_i) \neq \emptyset$. Since W is a single G-orbit and each W_i is G-invariant, we must have $W \subset W_i$ and thus $W \subset \cap_i W_i$, which gives a contradiction.

Since f is G-invariant the homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ has image in $\mathcal{O}_X(f^{-1}(U_i))^G$. Therefore, there is a unique morphism h_i^* which makes the following square commute

$$\begin{array}{ccc} \mathcal{O}_Z(U_i) & \xrightarrow{h_i^*} & \mathcal{O}_Y(V_i) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \cong\ \varphi^*} \\ \mathcal{O}_X(f^{-1}(U_i))^G & \longrightarrow & \mathcal{O}_X(\varphi^{-1}(V_i))^G \end{array}$$

where the isomorphism on the right hand side of this square is given by property iii) of the good quotient φ. Since U_i is affine, the k-algebra homomorphism $\mathcal{O}_Z(U_i) \to \mathcal{O}_Y(V_i)$ corresponds to a morphism $h_i : V_i \to U_i$ (see [14] I Proposition 3.5). By construction

$$f|_{\varphi^{-1}(V_i)} = h_i \circ \varphi|_{\varphi^{-1}(V_i)} : \varphi^{-1}(V_i) \to U_i$$

and $h_i = h_j$ on $V_i \cap V_j$; therefore, we can glue the morphisms h_i to obtain a morphism $h : Y \to Z$ such that $f = h \circ \varphi$. Since the morphisms h_i are unique, it follows that h is also unique. □

Example 3.31. We consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ as in Example 3.17. As the origin is in the closure of the punctured axes $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$, all three orbits will be identified by the categorical quotient. The smooth conic orbits $\{(x, y) : xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$ are closed. These conic orbits are parametrised by $\mathbb{A}^1 - \{0\}$ and the remaining three orbits will all be identified in the categorical quotient. Therefore, we may naturally expect that $\varphi : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto xy$ is a categorical quotient. In fact, we will prove that this is a good quotient and therefore also a categorical quotient. This morphism is clearly G-invariant and surjective, which shows parts i) and ii).

For iii), let $U \subset \mathbb{A}^1$ be an open subset and consider the morphism

$$\varphi^* : \mathcal{O}_{\mathbb{A}^1}(U) \to \mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U)).$$

For $U = \mathbb{A}^1$, we have $\varphi : \mathbb{C}[z] \to \mathbb{C}[x, y]$ given by $z \mapsto xy$. We claim that this is an isomorphism on the ring of $\mathbb{G}_m$-invariant functions. The action of $t \in \mathbb{G}_m$ on $\mathcal{O}(\mathbb{A}^2) = \mathbb{C}[x, y]$ is given by

$$t \cdot \left(\sum_{i,j} a_{ij} x^i y^j \right) = \sum_{i,j} a_{ij} t^{j-i} x^i y^j.$$

Therefore, the invariant subalgebra is

$$\mathbb{C}[x, y]^{\mathbb{C}^*} = \left\{ \sum_{ij} a_{ij} x^i y^j : a_{ij} = 0 \ \forall i \neq j \right\} = \mathbb{C}[xy]$$

as required. Now suppose we have an open subset $U \subsetneq \mathbb{A}^1$; then $U = \mathbb{A}^1 - \{a_1, \dots, a_n\}$ and $\mathcal{O}_{\mathbb{A}^1}(U) = \mathbb{C}[z]_{(f)}$ where $f(z) = (z - a_1) \cdots (z - a_n) \in \mathbb{C}[z]$. Then $\varphi^{-1}(U)$ is the non-vanishing locus of $F(x, y) := f(xy) \in \mathbb{C}[x, y]$ and $\mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U)) = \mathbb{C}[x, y]_F$. In particular, we can directly verify that

$$\mathcal{O}_{\mathbb{A}^2}(\varphi^{-1}(U))^{\mathbb{G}_m} = (\mathbb{C}[x, y]_F)^{\mathbb{G}_m} = \left(\mathbb{C}[x, y]^{\mathbb{G}_m} \right)_F = \mathbb{C}[xy]_F \cong \mathbb{C}[z]_f = \mathcal{O}_{\mathbb{A}^1}(U).$$

For v)′, we note that any G-invariant closed subvariety in $\mathbb{A}^2$ is either a finite union of orbit closures or the entire space $\mathbb{A}^2$. Therefore, we can assume that the disjoint G-invariant closed subsets W_1 and W_2 are both a finite union of orbit closures and even just that $W_i = \overline{G \cdot p_i}$ are disjoint for $i = 1, 2$. Since we have already determined the orbit closures, we see that there are two cases two consider: either p_1 and p_2 both do not lie on the axes in $\mathbb{A}^2$ (and so their orbits correspond to disjoint conics $\{(x, y) : xy = \alpha_i\}$ and $\varphi(W_1) = \alpha_1 \neq \alpha_2 = \varphi(W_2)$) or one of the points, say p_1 lies on an axis, so that $\varphi(W_1) = 0$, and the second point p_2 cannot also lie on an axis as we assumed the closures of the orbits were disjoint, so $\varphi(W_2) \neq 0$.

Trivially vi) holds, as any morphism of affine schemes is affine.

Finally, we note that φ is not a geometric quotient, as $\varphi^{-1}(0)$ is a union of 3 orbits.

Corollary 3.32. *Let G be an affine algebraic group acting on a scheme X and let $\varphi : X \to Y$ be a good quotient; then:*

a) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$ *if and only if* $\varphi(x_1) = \varphi(x_2)$.

b) For each $y \in Y$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed (i.e. all orbits are closed), then φ is a geometric quotient.

Proof. a). As φ is constant on orbit closures, it follows that $\varphi(x_1) = \varphi(x_2)$ if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi$. By property v) of the good quotient φ, we get the converse. For b), suppose we have two distinct closed orbits W_1 and W_2 in $\varphi^{-1}(y)$, then the fact that their images under φ are both equal to y contradicts property v) of the good quotient φ. □

Corollary 3.33. *If $\varphi : X \to Y$ is a good (resp. geometric) quotient, then for every open $U \subset Y$ the restriction $\varphi| : \varphi^{-1}(U) \to U$ is also a good (resp. geometric) quotient of G acting on $\varphi^{-1}(U)$.*

Proof. Exercise. □

Remark 3.34. The definition of good and geometric quotients are local in the target; thus if $\varphi : X \to Y$ is G-invariant and we have a cover of Y by open sets U_i such that $\varphi| : \varphi^{-1}(U_i) \to U_i$ are all good (respectively geometric) quotients, then so is $\varphi : X \to Y$. We leave the proof as an exercise.