3.6. **Moduli spaces and quotients.** Let us give one result about the construction of moduli spaces using group quotients. For a moduli problem $\mathcal{M}$, a family $\mathcal{F}$ over a scheme S has the *local universal property* if for any other family $\mathcal{G}$ over a scheme T and for any k-point $t \in T$, there exists a neighbourhood U of t in T and a morphism $f : U \to S$ such that $\mathcal{G}|_U \sim_U f^*\mathcal{F}$.

Proposition 3.35. *For a moduli problem $\mathcal{M}$, let $\mathcal{F}$ be a family with the local universal property over a scheme S. Furthermore, suppose that there is an algebraic group G acting on S such that two k-points s, t lie in the same G-orbit if and only if $\mathcal{F}_t \sim \mathcal{F}_s$. Then*

a) any coarse moduli space is a categorical quotient of the G-action on S;
b) a categorical quotient of the G-action on S is a coarse moduli space if and only if it is an orbit space.

Proof. For any scheme M, we claim that there is a bijective correspondence

$$\{\text{natural transformations } \eta : \mathcal{M} \to h_M\} \longleftrightarrow \{G\text{-invariant morphisms } f : S \to M\}$$

given by $\eta \mapsto \eta_S(\mathcal{F})$, which is G-invariant by our assumptions about the G-action on S. The inverse of this correspondence associates to a G-invariant morphism $f : S \to M$ and a family $\mathcal{G}$ over T a morphism $\eta_T(\mathcal{G}) : T \to M$ by using the local universal property of $\mathcal{F}$ over S. More precisely, we can cover T by open subsets U_i such that there is a morphism $h_i : U_i \to S$ and $h_i^*\mathcal{F} \sim_{U_i} \mathcal{G}|_{U_i}$. For $u \in U_i \cap U_j$, we have

$$\mathcal{F}_{h_i(u)} \sim (h_i^*\mathcal{F})_u \sim \mathcal{G}_u \sim (h_j^*\mathcal{F})_u \sim \mathcal{F}_{h_j(u)}$$

and so by assumption $h_i(u)$ and $h_j(u)$ lie in the same G-orbit. Since f is G-invariant, we can glue the compositions $f \circ h_i : U_i \to M$ glue to a morphism $\eta_T(\mathcal{G}) : T \to M$. We leave it to the reader to verify that this determines a natural transformation η (that is, this is functorial with respect to morphisms) and that these correspondences are inverse to each other.

Hence, if $(M, \eta : \mathcal{M} \to h_M)$ is a coarse moduli space, then $\eta_S(\mathcal{F}) : S \to M$ is G-invariant and universal amongst all G-invariant morphisms from S, by the universality of η. This proves statement a). Furthermore, the G-invariant morphism $\eta_S(\mathcal{F}) : S \to M$ is an orbit space if and only if $\eta_{\operatorname{Spec} k}$ is bijective. This proves statement b). □

4. Affine Geometric Invariant Theory

In this section we consider an action of an affine algebraic group G on an affine scheme X of finite type over k and show that this action has a good quotient when G is linearly reductive. The main references for this section are [25] and [31] (for further reading, see also [2], [4] and [32]).

Let X be an affine scheme of finite type over k; then the ring of regular functions $\mathcal{O}(X)$ is a finitely generated k-algebra. Conversely, for any finitely generated k-algebra A, the spectrum of prime ideals $\operatorname{Spec} A$ is an affine scheme of finite type over k.

The action of an affine algebraic group G on an affine scheme X given by a morphism

$$\sigma : G \times X \to X$$

corresponds to a homomorphism of k-algebras $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(X)$, which gives a G-co-module structure on the (typically infinite dimensional) k-vector space $\mathcal{O}(X)$. This co-module structure in turn determines a linear representation $G \to \mathrm{GL}(\mathcal{O}(X))$. Concretely, on the level of k-points, the action of $g \in G(k)$ on $f \in \mathcal{O}(X)$ is given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

The ring of G-invariant regular functions on X is

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : \sigma^*(f) = 1 \otimes f\}.$$

Any G-invariant morphism $\varphi : X \to Z$ of schemes induces a homomorphism $\varphi^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ whose image is contained in the subalgebra of G-invariant regular functions $\mathcal{O}(X)^G$. This leads us to an interesting problem in invariant theory which was first considered by Hilbert.

4.1. **Hilbert's 14th problem.** For a rational action of an affine algebraic group G on a finitely generated k-algebra A, Hilbert asked whether the algebra of G-invariants A^G is finitely generated.

The answer to Hilbert's 14th problem is negative in this level of generality: Nagata gave an example of an action of an affine algebraic group (constructed using copies of the additive groups) for which the ring of invariants is not finitely generated (see [27] and [29]). However, for reductive groups (which we introduce below), the answer is positive due to a Theorem of Nagata. The proof of this result is beyond the scope of this course. However, we will prove that for a rational action of a 'linearly reductive' group on an algebra, the subalgebra of invariants is finitely generated, using a Reynolds operator, which essentially mimics Hilbert's 19th century proof that, over the complex numbers, a rational action of the general linear group GL_n on an algebra has a finitely generated invariant subalgebra.

4.2. **Reductive groups.** In this section, we will give the definition of a reductive group, a linearly reductive group and a geometrically reductive group, and explain the relationship between these different notions of reductivity.

Our starting point is the Jordan decomposition for affine algebraic groups over k. We first recall the Jordan decomposition for GL_n: an element $g \in \mathrm{GL}_n(k)$ has a decomposition

$$g = g_{ss}g_u = g_u g_{ss}$$

where g_{ss} is semisimple (or, equivalently, diagonalisable, as k is algebraically closed) and g_u is unipotent (that is, $g - I_n$ is nilpotent).

For any affine algebraic group G, we would like to have an analogous decomposition, and we can hope to make use of the fact that G admits a faithful linear representation $G \hookrightarrow \mathrm{GL}_n$. However, this would require the decomposition to be functorial with respect to closed immersions of groups.

Definition 4.1. Let G be an affine algebraic group over k. An element g is *semisimple* (resp. *unipotent*) if there is a faithful linear representation $\rho : G \hookrightarrow \mathrm{GL}_n$ such that $\rho(g)$ is diagonalisable (resp. unipotent).

Theorem 4.2 (Jordan decomposition, see [23] X Theorem 2.8 and 2.10)**.** *Let G be an affine algebraic group over k. For every $g \in G(k)$, there exists a unique semisimple element g_{ss} and a unique unipotent element g_u such that*

$$g = g_{ss}g_u = g_u g_{ss}.$$

Furthermore, this decomposition is functorial with respect to group homomorphisms. In particular, if $g \in G(k)$ is semisimple (resp. unipotent), then for all linear representations $\rho : G \to \mathrm{GL}_n$, the element $\rho(g)$ is semisimple (resp. unipotent).

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of an affine algebraic group G on a vector space V and let $\rho^* : V \to \mathcal{O}(G) \otimes_k V$ denote the associated co-module. Then a vector subspace $V' \subset V$ is *G-invariant* if $\rho^*(V') \subset \mathcal{O}(G) \otimes_k V'$ and a vector $v \in V$ is *G-invariant* if $\rho^*(v) = 1 \otimes v$. We let V^G denote the subspace of G-invariant vectors.

Definition 4.3. An affine algebraic group G is *unipotent* if every non-trivial linear representation $\rho : G \to \mathrm{GL}(V)$ has a non-zero G-invariant vector.

Proposition 4.4. *For an affine algebraic group G, the following statements are equivalent.*

i) G is unipotent.
ii) For every representation $\rho : G \to \mathrm{GL}(V)$ there is a basis of V such that $\rho(G)$ is contained in the subgroup $\mathbb{U} \subset \mathrm{GL}(V)$ consisting of upper triangular matrices with diagonal entries equal to 1.
iii) G is isomorphic to a subgroup of a standard unipotent group $\mathbb{U}_n \subset \mathrm{GL}_n$ consisting of upper triangular matrices with diagonal entries equal to 1.

Proof. i) $\iff$ ii): If $e_1, \dots, e_n$ is a basis of V such that $\rho(G) \subset \mathbb{U}$, then e_1 is fixed by ρ. Conversely if $\rho : G \to \mathrm{GL}(V)$ is a representation of a unipotent group G, then we can proceed by induction on the dimension of V. As U is unipotent, the linear subspace of G-fixed points V^G is non-zero; let $e_1, \cdots e_m$ be a basis of V^G. Then there is a basis $\overline{e}_{m+1}, \dots, \overline{e}_n$ of V/V^G such that the induced representation has image in the upper triangular matrices with diagonal entries equal to 1. By choosing lifts $e_{m+i} \in V$ of $\overline{e}_{m+i}$, we get the desired basis of V.

ii) $\iff$ iii): As every affine algebraic group G has a faithful representation $\rho : G \to \mathrm{GL}_n$, we see that ii) implies iii). Conversely, any subgroup of $\mathbb{U}_n$ is unipotent (see [23] XV Theorem 2.4). □

Remark 4.5. If G is a unipotent affine algebraic group, then every $g \in G(k)$ is unipotent. The converse is true if in addition G is smooth (for example, see [23] XV Corollary 2.6 or SGA3 XVII Corollary 3.8).

Example 4.6.

(1) The additive group $\mathbb{G}_a$ is unipotent, as we have an embedding $\mathbb{G}_a \hookrightarrow \mathbb{U}_2$ given by
$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

(2) In characteristic p, there is a finite subgroup $\alpha_p \subset \mathbb{G}_a$ where we define the functor of points of α_p by associating to a k-algebra R,
$$\alpha_p(R) := \{c \in \mathbb{G}_a(R) : c^p = 0\}.$$
This is represented by the scheme $\operatorname{Spec} k[t]/(t^p)$ and so α_p is a unipotent group which is not smooth.

Definition 4.7. An algebraic subgroup H of an affine algebraic group G is *normal* if the conjugation action $H \times G \to G$ given by $(h, g) \mapsto ghg^{-1}$ factors through $H \hookrightarrow G$.

Definition 4.8. An affine algebraic group G over k is *reductive* if it is smooth and every smooth unipotent normal algebraic subgroup of G is trivial.

Remark 4.9. In fact, one can define reductivity by saying that the unipotent radical of G (which is the maximal connected unipotent normal algebraic subgroup of G) is trivial; however, to define the unipotent radical carefully, we would need to prove that, for a group G, the subgroup generated by two smooth algebraic subgroups of G is also algebraic (see [22] Proposition 2.24).

Exercise 4.10. Show that the general linear group GL_n and the special linear group SL_n are reductive. [Hint: if we have a non-trivial smooth connected unipotent normal algebraic subgroup $U \subset \mathrm{GL}_n$, then there exists $g \in U(k) \subset \mathrm{GL}_n(k)$ whose Jordan normal form has a $r \times r$ Jordan block for $r > 1$ (as g is unipotent). Using normality of U, find another element $g' \in U(k)$ such that the product gg' is not unipotent.]

Definition 4.11. An affine algebraic group G is

(1) *linearly reductive* if every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ is completely reducible; that is the representation decomposes as a direct sum of irreducibles.
(2) *geometrically reductive* if, for every finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G-invariant point $v \in V$, there is a G-invariant non-constant homogeneous polynomial $f \in \mathcal{O}(V)$ such that $f(v) \neq 0$.

Example 4.12. Any algebraic torus $(\mathbb{G}_m)^r$ is linearly reductive by Proposition 3.12.

Exercise 4.13. Show directly that the additive group $\mathbb{G}_a$ is not geometrically reductive. [Hint: there is a representation $\rho : \mathbb{G}_a \to \mathrm{GL}_2$ and a G-invariant point $v \in \mathbb{A}^2$ such that every non-constant G-invariant homogeneous polynomial in two variables vanishes at v].

Proposition 4.14. *For an affine algebraic group G, the following statements are equivalent.*

i) G is linearly reductive.

ii) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, any G-invariant subspace $V' \subset V$ admits a G-stable complement (i.e. there is a subrepresentation $V'' \subset V$ such that $V = V' \oplus V''$).

iii) For any surjection of finite dimensional G-representations $\phi : V \to W$, the induced map on G-invariants $\phi^G : V^G \to W^G$ is surjective.

iv) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and every non-zero G-invariant point $v \in V$, there is a G-invariant linear form $f : V \to k$ such that $f(v) \neq 0$.

v) For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ and any surjective G-invariant linear form $f : V \to k$, there is $v \in V^G$ such that $f(v) \neq 0$.

Proof. The equivalence i) $\iff$ ii) is clear, as we are working with finite dimensional representations.

ii) $\implies$ iii): Let $f : V \to W$ be a surjection of finite dimensional G-representations and $V' := \ker(f) \subset V$. Then, by assumption, V' has a G-stable complement $V'' \cong W$. Since both V' and V'' are G-invariant, $V^G = (V')^G \oplus (V'')^G$ and so $f^G : V^G \to (V'')^G \cong W^G$ is surjective.

iii) $\implies$ ii): Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional linear representation and $V' \subset V$ a G-invariant subspace. Then we have a surjection

$$\phi : \mathrm{Hom}(V, V') \to \mathrm{Hom}(V', V')$$

of finite dimensional G-representations and so by iii) the identity map id'_V lifts to G-equivariant morphism $f : V \to V'$ splitting the inclusion $V' \subset V$. More precisely, V' has G-stable complement $V'' := \ker f$.

iv) $\iff$ v): We can identify V^G with the space of G-invariant linear forms $V^\vee \to k$

$$V^G = \mathrm{Hom}_G(k, V) = \mathrm{Hom}_G(V^\vee, k).$$

iii) $\implies$ iv): Let V be a finite dimensional linear G-representation and $v \in V^G$ be a non-zero G-invariant vector. Then v determines a G-invariant linear form $\phi : V^\vee \to k$. By letting G act trivially on k, we can view ϕ as a surjection of G-representations and so by iii), the fixed point $1 \in k = k^G$ has a lift $f \in (V^\vee)^G = \mathrm{Hom}_G(V, k)$ such that $f(v) = 1$.

iv) $\implies$ iii): Let $\phi : V \to W$ be a finite dimensional G-representation. Then we want to prove that ϕ^G is surjective: i.e. lift any non-zero $w \in W^G$ to a point $v \in V^G$. By iv), there exists a G-invariant form $f : W \to k$ such that $f(w) \neq 0$. Then $f \circ \phi : V \to k$ is a G-invariant surjective form on V and so by v) $\iff$ iv), there exists $v \in V^G$ such that $(f \circ \phi)(v) \neq 0$. By suitably rescaling $v \in V^G$ so that $(f \circ \phi)(v) = f(w)$, we get the desired lift. $\square$

Exercise 4.15. Prove that any finite group of order not divisible by the characteristic of k is linearly reductive. [Hint: consider averaging over the group.]

We summarise the main results relating the different notions of reductivity in the following theorem, whose proof is beyond the scope of this course.

Theorem 4.16. *(Weyl, Nagata, Mumford, Haboush)*

i) Every linearly reductive group is geometrically reductive.

ii) In characteristic zero, every reductive group is linearly reductive.

iii) A smooth affine algebraic group is reductive if and only if it is geometrically reductive.

In particular, for smooth affine algebraic group schemes, we have

$$\textit{linearly reductive} \implies \textit{geometrically reductive} \iff \textit{reductive}$$

and all three notions coincide in characteristic zero.

Statement i) follows immediately from the definition of geometrically reductive and Proposition 4.14. There are several proofs of Statement ii); the earliest goes back to Weyl, where he first reduces to $k = \mathbb{C}$, and then uses the representation theory of compact Lie groups (this argument is known an Weyl's unitary trick; see Proposition 4.18). An alternative approach is to use Lie algebras (for example, see the proof that SL_n is linearly reductive in characteristic zero in [24] Theorem 4.43). Statement iii) was conjectured by Mumford after Nagata proved that every geometrically reductive group is reductive [29], and the converse statement was proved by Haboush [12].

Remark 4.17. In positive characteristic, the groups GL_n, SL_n and PGL_n are not linearly reductive for $n > 1$; see [28].