We will now sketch the proof that over the complex numbers every reductive group is linearly reductive.

Proposition 4.18. *Every reductive group G over $\mathbb{C}$ is linearly reductive.*

Proof. We let $K \subset G(\mathbb{C})$ be a maximal compact subgroup.
Step 1. For a compact Lie subgroup K, we claim that every finite dimensional representation of the Lie group K is completely reducible. Let us sketch the proof of this claim. Let V be a finite dimensional representation of K (i.e. there is a morphism $\rho : K \to \mathrm{GL}(V)$ of Lie groups); then analogously to Proposition 4.14 above, it suffices to prove that every K-invariant subspace $W \subset V$ has a K-stable complement. There is a K-invariant Hermitian inner product on V, as we can take any Hermitian inner product h on V and integrate over the compact group K using a Haar measure $d\mu$ on K to obtain a K-invariant Hermitian inner product

$$h^K(v_1, v_2) := \int_K h(k \cdot v_1, k \cdot v_2) d\mu(k).$$

Then, we define the K-stable complement of $W \subset V$ to be the orthogonal complement of $W \subset V$ with respect to this K-invariant Hermitian inner product.
Step 2. For G reductive and a maximal compact subgroup $K \subset G(\mathbb{C})$, the elements of K are Zariski dense in G. We prove this statement in Lemma 4.19 below. The proof works with the Lie algebras $\mathfrak{k}$ and $\mathfrak{g}(\mathbb{C})$, using the fact that the exponential map $\exp : \mathfrak{g}(\mathbb{C}) \to G(\mathbb{C})$ is holomorphic, the fact that $\mathfrak{g}(\mathbb{C}) = \mathfrak{k}_{\mathbb{C}}$ as $G(\mathbb{C})$ is reductive (for a proof see, for example, [34] Theorem 2.7) and the Identity Theorem from complex analysis.
Step 3. For any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we claim that $V^G = V^K$, where K is a maximal compact of G. As $K \subset G$ is a subgroup, we have $V^G \subset V^K$. To prove the reverse inclusion, let $v \in V^K$ and consider the morphism

$$\sigma : G \to V$$

given by $g \mapsto \rho(g) \cdot v$. Then $\sigma^{-1}(v) \subset G$ is Zariski closed. Since $v \in V^K$, we have $K \subset \sigma^{-1}(v)$ and so also $\overline{K} \subset \sigma^{-1}(v)$. However, as $K \subset G$ is Zariski dense, it follows that $G \subset \sigma^{-1}(v)$; that is, $v \in V^G$ as required.
Step 4. The reductive group G is linearly reductive. By Proposition 4.14, it suffices to show for every surjective homomorphism of finite dimensional linear G-representations $\phi : V \to W$, the induced homomorphism ϕ^G on invariant subspaces is also surjective. By Step 3, this is equivalent to showing that ϕ^K is surjective, which follows by Step 1. □

Lemma 4.19. *Over the complex numbers, let G be a reductive group and $K \subset G(\mathbb{C})$ be a maximal compact subgroup. Then the elements of K are Zariski dense in G.*

Proof. If this is not the case, then there exists a function $f \in \mathcal{O}(G)$ which is not identically zero such that $f(K) = 0$. On the level of Lie algebras, as $G(\mathbb{C})$ is a complex reductive group and $K \subset G(\mathbb{C})$ a maximal compact subgroup, we have

$$\mathfrak{g}(\mathbb{C}) = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$$

(see [34] Theorem 2.7). Furthermore, the exponential map $\exp : \mathfrak{g}(\mathbb{C}) \to G(\mathbb{C})$ is holomorphic and maps $\mathfrak{k}$ to K. Therefore, $h := f \circ \exp : \mathfrak{g}(\mathbb{C}) \to \mathbb{C}$ is holomorphic and vanishes on $\mathfrak{k}$. However, if V is a real vector space and $l : V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is holomorphic with $l(V \otimes_{\mathbb{R}} \mathbb{R}) = 0$, then l is identically zero (the proof of this follows from the Identity Theorem in complex analysis when V has dimension 1 and, for higher dimensional V, we can view l as a function in a single variable x_i by fixing all other variables and by applying this argument for each i, we deduce that $l = 0$). In particular, $h : \mathfrak{g}(\mathbb{C}) \to \mathbb{C}$ is identically zero and, as the exponential map is a local homeomorphism, we deduce that f is identically zero which is a contradiction. □

4.3. **Nagata's theorem.** In this section, when we talk of a group G acting on a k-algebra A, we will always mean that the group G acts by k-algebra homomorphisms. We recall that a G-action on a k-algebra A is rational if every element of A is contained in a finite dimensional G-invariant linear subspace of A. In particular, if $A = \mathcal{O}(X)$ and the G-action on A comes from an algebraic action of an affine algebraic group G on X, then this action is rational by Lemma 3.8.

Theorem 4.20 (Nagata). *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. Then the G-invariant subalgebra A^G is finitely generated.*

As every reductive group is geometrically reductive, we can use Nagata's theorem for reductive groups. In the following section, we will prove this result for linearly reductive groups using Reynolds operators (so in characteristic zero this also proves Nagata's theorem). Nagata also gave a counterexample of a non-reductive group action for which the ring of invariants is not finitely generated (see [27] and [29]).

4.4. **Reynolds operators.** Given a linearly reductive group G, for any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we can write $V = V^G \oplus W$ where W is the direct sum of all non-trivial irreducible subrepresentations. This gives a canonical G-complement W to V^G and a unique projection $p_V : V \to V^G$. This projection motivates the following definition.

Definition 4.21. For a group G acting on a k-algebra A, a linear map $R_A : A \to A^G$ is called a *Reynolds operator* if it is a projection onto A^G and, for $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$.

Lemma 4.22. *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A; then there exists a Reynolds operator $R_A : A \to A^G$.*

Proof. Since A is finitely generated, it has a countable basis. Therefore, we can write A as an increasing union of finite dimensional G-invariant vector subspaces $A_n \subset A$ using the fact that the action is rational. More precisely, if we label our basis elements $a_1, a_2, \dots,$ then we iteratively construct the subsets A_n by letting A_n be the finite dimensional G-invariant subspace containing $a_1, \dots, a_n$ and a basses of A_{n-1} and $a_j \cdot A_{n-1}$ for $j = 1, \dots, n$. Then $A = \bigcup_{n \geq 1} A_n$. Since G is linearly reductive and each A_n is a finite dimensional G-representation, we can write

$$A_n = A_n^G \oplus A_n'$$

where A_n' is the direct sum of all non-trivial irreducible G-subrepresentations of A_n. We let $R_n : A_n \to A_n^G$ be the canonical projection onto the direct factor A_n^G.

For $m > n$, we have a commutative square

$$\begin{tikzcd} A_n \arrow[r, "R_n"'] \arrow[d, hook] & A_n^G \arrow[d, hook] \\ A_m \arrow[r, "R_m"'] & A_m^G, \end{tikzcd}$$

as we have $A_n' \subset A_m'$ and $A_n^G \subset A_m^G$. Hence, we have a linear map $R_A : A \to A^G$ given by the compatible projections $R_n : A_n \to A_n^G$ for each n.

It remains to check that for $a \in A^G$ and $b \in A$, we have $R_A(ab) = aR_A(b)$. Pick n such that $a, b \in A_n$ and pick $m \geq n$ such that $a(A_n) \subset A_m$. Then consider the homomorphism of G-representations given by left multiplication by a

$$l_a : A_n \to A_m.$$

We can write $A_n = A_n^G \oplus A_n'$, where $A_n' = W_1 \oplus \cdots \oplus W_r$ is a direct sum of non-trivial irreducible subrepresentations $W_i \subset A_n$. Since G acts by algebra homomorphisms and $a \in A^G$, we have $l_a(A_n^G) \subset A_m^G$. By Schur's Lemma, the image of each irreducible W_i under l_a is either zero or isomorphic to W_i. Therefore, we have $l_a(W_i) \subset A_m'$ and so $l_a(A_n') \subset A_m'$. In particular, if we write $b = b^G + b'$ for $b^G \in A_n^G$ and $b' \in A_n'$, then $ab = l_a(b) = l_a(b^G) + l_a(b')$, where $l_a(b^G) = ab^G \in A_m^G$ and $l_a(b') = ab' \in A_m'$. Hence,

$$R_A(ab) = ab^G = aR_A(b)$$

as required. □

In fact, the arguments used in the final part of this proof, give the following result.

Corollary 4.23. *Let A and B be k-algebras with a rational action of a linearly reductive group G, which have Reynolds operators $R_A : A \to A^G$ and $R_B : B \to B^G$. Then any G-equivariant homomorphism $h : A \to B$ of these k-algebras commutes with the Reynolds operators: $R_B \circ h = h \circ R_A$.*

Lemma 4.24. *Let A be a k-algebra with a rational G-action and suppose that A has a Reynolds operator $R_A : A \to A^G$. Then for any ideal $I \subset A^G$, we have $IA \cap A^G = I$. More generally, if $\{I_j\}_{j\in J}$ are a set of ideals in A^G, then we have*

$$(\sum_{j\in J} I_j A) \cap A^G = \sum_{j \in J} I_j.$$

In particular, if A is noetherian, then so is A^G.

Proof. Clearly, $I \subset IA \cap A^G$. Conversely, let $x \in IA \cap A^G$; then we can write $x = \sum_{l=1}^n i_l x_l$ for $i_l \in I$ and $x_l \in A$. As R_A is a Reynolds operator,

$$x = R_A(x) = R_A\left(\sum_{l=1}^n i_l x_l\right) = \sum_{l=1}^n i_l R_A(x_l) \in I.$$

Now suppose that A is Noetherian and consider a chain $I_1 \subset I_2 \subset \cdots$ of ascending ideals in A^G. Then $I_1 A \subset I_2 A \subset \cdots$ is a chain of ascending ideals in A and so must stabilise as A is Noetherian. However, as $I_n = I_n A \cap A^G$, it follows that the chain of ideals I_n must also stabilise. □

Theorem 4.25 (Hilbert, Mumford)**.** *Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A. Then A^G is finitely generated.*

Proof. Let us first reduce to the case where A is a polynomial algebra and the G-action is linear. Let V be a finite dimensional G-invariant vector subspace of A containing a set of generators for A as a k-algebra; the existence of V is guaranteed as our action is rational. As V contains generators for A as an algebra, we have a G-equivariant surjection of k-algebras

$$\mathcal{O}(V^\vee) = \mathrm{Sym}^*(V) \to A.$$

Since G is linearly reductive, both algebras admit a Reynolds operator by Lemma 4.22 and, moreover, these Reynolds operators commute with this surjection by Corollary 4.23. Therefore, we have a surjection $(\mathrm{Sym}^*(V))^G \to A^G$ and so to prove A^G is finitely generated, it suffices to assume that A is a polynomial algebra with a linear G-action.

Let $A = \mathrm{Sym}^*(V)$ where V is a finite dimensional G-representation. Then A is naturally a graded k-algebra, where the grading is by homogeneous degree $A = \oplus_n A_n = \oplus_{n\geq 0} \mathrm{Sym}^n V$. As the G-action on A is linear, the invariant subalgebra A^G is also graded $A^G = \oplus_n A_n^G$. By Hilbert's basis theorem, A is Noetherian and so by Lemma 4.24, the invariant ring A^G is also Noetherian. Hence, the ideal $A_+^G = \oplus_{n>0} A_n^G \subset A^G$ is finitely generated. We then use the following technical but not difficult result: for a graded k-algebra $B = \oplus_{n\geq 0} B_n$ and $b_1, \dots, b_m \in B$ homogeneous elements of positive degree, the following statements are equivalent:

1. B is generated by $b_1, \dots, b_m$ as a B_0-algebra; that is, $B = B_0[b_1, \dots b_m]$;
2. $B_+ := \oplus_{n>0} B_n$ is generated by $b_1, \dots, b_m$ as an ideal; that is $B_+ = Bb_1 + \cdots + Bb_m$.

By applying this to A^G and the finitely generated ideal $A_+^G = \oplus_{n>0} A_n^G$, we deduce that A^G is a finitely generated k-algebra. □

Nagata gave an example of an action of a product of additive groups $\mathbb{G}_a^r$ on an affine space $\mathbb{A}^n$ such that the algebra of invariants fails to be finitely generated; see [27] and [29]. From this example, one can produce an affine scheme X with a $\mathbb{G}_a$-action such that $\mathcal{O}(X)^{\mathbb{G}_a}$ is not finitely generated. More generally, a theorem of Popov states that for any non-reductive group G there is an affine scheme X such that $\mathcal{O}(X)^G$ is not finitely generated. Let us quickly outline the proof following [25] Theorem A.1.0. As G is non-reductive, we can pick a surjective homomorphism from the unipotent radical $R_u(G)$ of G onto $\mathbb{G}_a$, which defines an action of $R_u(G)$ on X such that the algebra of invariants is not finitely generated. Then we can take the Borel construction associated to $R_u(G) \subset G$

$$Y := G \times^{R_u(G)} X := (G \times X)/R_u(G)$$

which is locally trivial over $G/R_u(G)$ with fibre X and there is a natural G-action on Y where

$$\mathcal{O}(Y)^G \cong \mathcal{O}(X)^{R_u(G)}$$

is not finitely generated. In fact Y is affine (and so $\mathcal{O}(Y)$ is finitely generated) as $G \to G/R_u(G)$ has a local section by a result of Rosenlicht and so the fibre bundle $Y \to G/R_u(G)$ also has a local section.

Theorem 4.26 (Popov). *An affine algebraic group G over k is reductive if and only if for every rational G-action on a finitely generated k-algebra A, the subalgebra A^G of G-invariants is finitely generated.*

4.5. **Construction of the affine GIT quotient.** Let G be a reductive group acting on an affine scheme X. We have seen that this induces an action of G on the coordinate ring $\mathcal{O}(X)$, which is a finitely generated k-algebra. By Nagata's Theorem, the subalgebra of invariants $\mathcal{O}(X)^G$ is finitely generated.

Definition 4.27. The *affine GIT quotient* is the morphism $\varphi : X \to X//G := \operatorname{Spec} \mathcal{O}(X)^G$ of affine schemes associated to the inclusion $\varphi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

Remark 4.28. The double slash notation $X//G$ used for the GIT quotient is a reminder that this quotient is not necessarily an orbit space and so it may identify some orbits. In nice cases, the GIT quotient is also a geometric quotient and in this case we shall often write X/G instead to emphasise the fact that it is an orbit space.

We will soon prove that the reductive GIT quotient is a good quotient.