In preparation for proving that the GIT quotient is a good quotient, we need the following lemma.

Lemma 4.29. *Let G be a geometrically reductive group acting on an affine scheme X. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in \mathcal{O}(X)^G$ which separates these sets i.e.*

$$f(W_1) = 0 \quad \text{and} \quad f(W_2) = 1.$$

Proof. As W_i are disjoint and closed, we have

$$(1) = I(\emptyset) = I(W_1 \cap W_2) = I(W_1) + I(W_2)$$

and so we can write $1 = f_1 + f_2$, where $f_i \in I(W_i)$. Then $f_1(W_1) = 0$ and $f_1(W_2) = 1$. By Lemma 3.8, the function f_1 is contained in a finite dimensional G-invariant linear subspace V of $\mathcal{O}(X)$; therefore, so we can choose a basis $h_1, \dots, h_n$ of V. This basis defines a morphism $h : X \to \mathbb{A}^n$ by

$$h(x) = (h_1(x), \dots, h_n(x)).$$

For each i, the function h_i is a linear combination of translates of f_1, so we have

$$h_i = \sum_{l=1}^{n_i} c_{il}\, g_{il} \cdot f_1$$

for constants c_{il} and group elements g_{il}. Then $h_i(x) = \sum_{l=1}^{n_i} c_{il}\, f_1(g_{il}^{-1} \cdot x)$ and, as W_i are G-invariant subsets and f_1 takes the value 0 (resp. 1) on W_1 (resp. W_2), it follows that $h(W_1) = 0$ and $h(W_2) = v \neq 0$.

As the functions $g \cdot h_i$ also belong to V, we can write them in terms of our given basis as

$$g \cdot h_i = \sum_{j=1}^{n} a_{ij}(g) h_j.$$

This defines a representation $G \to \mathrm{GL}_n$ given by $g \mapsto (a_{ij}(g))$ such that $h : X \to \mathbb{A}^n$ is G-equivariant with respect to the G-action on X and the G-action on $\mathbb{A}^n$ via this representation $G \to \mathrm{GL}_n$. Therefore $v = h(W_2)$ is a non-zero G-invariant point. Since G is geometrically reductive, there is a non-constant homogeneous polynomial $P \in k[x_1, \dots, x_n]^G$ such that $P(v) \neq 0$ and $P(0) = 0$. Then $f = cP \circ h$ is the desired invariant function where $c = 1/P(v)$. □

Theorem 4.30. *Let G be a reductive group acting on an affine scheme X. Then the affine GIT quotient $\varphi : X \to X//G$ is a good quotient and, moreover, $X//G$ is an affine scheme.*

Proof. As G is reductive and so also geometrically reductive, it follows from Nagata's Theorem that the algebra of G-invariant regular functions on X is a finitely generated k-algebra. Hence $Y := X//G = \operatorname{Spec} \mathcal{O}(X)^G$ is an affine scheme of finite type over k. Since the affine GIT quotient is defined by taking the morphism of affine schemes associated to the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$, it is G-invariant and affine: this gives part i) and vi) in the definition of good quotient.

To prove ii), we take $y \in Y(k)$ and want to construct $x \in X(k)$ whose image under $\varphi : X \to Y$ is y. Let $\mathfrak{m}_y$ be the maximal idea in $\mathcal{O}(Y) = \mathcal{O}(X)^G$ of the point y and choose generators $f_1, \dots, f_m$ of $\mathfrak{m}_y$. Since G is reductive, we claim that it follows that

$$\sum_{i=1}^{m} f_i \mathcal{O}(X) \neq \mathcal{O}(X).$$

For a linearly reductive group, this claim follows from Lemma 4.24 as

$$\left(\sum_{i=1}^{m} f_i \mathcal{O}(X) \right) \cap \mathcal{O}(X)^G = \sum_{i=1}^{m} f_i \mathcal{O}(X)^G \neq \mathcal{O}(X)^G.$$

For a proof for geometrically reductive groups, see [31] Lemma 3.4.2. Then, as $\sum_{i=1}^{m} f_i \mathcal{O}(X)$ is not equal to $\mathcal{O}(X)$, it is contained in some maximal idea $\mathfrak{m}_x \subset \mathcal{O}(X)$ corresponding to a closed point $x \in X(k)$. In particular, we have that $f_i(x) = 0$ for $i = 1, \dots, m$ and so $\varphi(x) = y$

as required. Therefore, every closed point is in the image of φ and as the image of φ is a constructible subset by Chevalley's Theorem, we can conclude that φ is surjective.

For $f \in \mathcal{O}(X)^G$, the open sets $U = Y_f$ form a basis of the open subsets of Y. Therefore, to prove iii), it suffices to consider open sets U of the form Y_f for $f \in \mathcal{O}(X)^G$. Let $f \in \mathcal{O}(X)^G$; then $\mathcal{O}_Y(Y_f) = (\mathcal{O}(X)^G)_f$ is the localisation of $\mathcal{O}(X)^G$ with respect to f and

$$\mathcal{O}_X(\varphi^{-1}(Y_f))^G = \mathcal{O}_X(X_f)^G = (\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f = \mathcal{O}_Y(Y_f)$$

as localisation with respect to an invariant function commutes with taking G-invariants. Hence the image of the inclusion homomorphism $\mathcal{O}_Y(Y_f) = (\mathcal{O}(X)^G)_f \to \mathcal{O}_X(\varphi^{-1}(Y_f)) = \mathcal{O}(X)_f$ is $\mathcal{O}_X(\varphi^{-1}(Y_f))^G = (\mathcal{O}(X)_f)^G$ which proves iii).

By Remark 3.28, given the surjectivity of φ, properties iv) and v) are equivalent to v)$'$ and so it suffices to prove v)$'$. By Lemma 4.29, for any two disjoint G-invariant closed subsets W_1 and W_2 in X, there is a function $f \in \mathcal{O}(X)^G$ such that $f(W_1) = 0$ and $f(W_2) = 1$. Since $\mathcal{O}(X)^G = \mathcal{O}(Y)$, we can view f as a regular function on Y with $f(\varphi(W_1)) = 0$ and $f(\varphi(W_2)) = 1$. Hence, it follows that

$$\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \emptyset$$

which finishes the proof. □

Corollary 4.31. *Suppose a reductive group G acts on an affine scheme X and let $\varphi : X \to Y := X//G$ be the affine GIT quotient. Then*

$$\varphi(x) = \varphi(x') \iff \overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset.$$

Furthermore, the preimage of each point $y \in Y$ contains a unique closed orbit. In particular, if the action of G on X is closed, then φ is a geometric quotient.

Proof. As φ is a good quotient, this follows immediately from Corollary 3.32 □

Example 4.32. Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$ as in Example 3.17. In this case $\mathcal{O}(X) = k[x, y]$ and $\mathcal{O}(X)^G = k[xy] \cong k[z]$ so that $Y = \mathbb{A}^1$ and the GIT quotient $\varphi : X \to Y$ is given by $(x, y) \mapsto xy$. The three orbits consisting of the punctured axes and the origin are all identified and so the quotient is not a geometric quotient.

Example 4.33. Consider the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ by $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ as in Example 3.18. Then $\mathcal{O}(X) = k[x_1, \dots, x_n]$ and $\mathcal{O}(X)^G = k$ so that $Y = \operatorname{Spec} k$ is a point and the GIT quotient $\varphi : X \to Y = \operatorname{Spec} k$ is given by the structure morphism. In this case all orbits are identified and so this good quotient is not a geometric quotient.

Remark 4.34. We note that the fact that G is reductive was used several times in the proof, not just to show the ring of invariants is finitely generated. In particular, there are non-reductive group actions which have finitely generated invariant rings but for which other properties listed in the definition of good quotient fail. For example, consider the additive group $\mathbb{G}_a$ acting on $X = \mathbb{A}^4$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}_4$

$$s \mapsto \begin{pmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{pmatrix}.$$

Even though $\mathbb{G}_a$ is non-reductive, the invariant ring is finitely generated: one can prove that

$$k[x_1, x_2, x_3, x_4]^{\mathbb{G}_a} = k[x_2, x_4, x_1x_4 - x_2x_3].$$

However the GIT 'quotient' map $X \to X//\mathbb{G}_a = \mathbb{A}^3$ is not surjective: its image misses the punctured line $\{(0, 0, \lambda) : \lambda \in k^*\} \subset \mathbb{A}^3$. For further differences, see [6].

4.6. **Geometric quotients on open subsets.** As we saw above, when a reductive group G acts on an affine scheme X in general a geometric quotient (i.e. orbit space) does not exist because the action is not necessarily closed. For finite groups G, every good quotient is a geometric quotient as the action of a finite group is always closed. In this section, we define an open subset X^s of 'stable' points in X for which there is a geometric quotient.

Definition 4.35. We say $x \in X$ is *stable* if its orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We let X^s denote the set of stable points.

Proposition 4.36. *Suppose a reductive group G acts on an affine scheme X and let $\varphi : X \to Y := X//G$ be the affine GIT quotient. Then $X^s \subset X$ is an open and G-invariant subset, $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$. Moreover, $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. We first show that X^s is open by showing for every $x \in X^s(k)$ there is an open neighbourhood of x in X^s. By Lemma 3.21, the set $X_+ := \{x \in X : \dim G_x > 0\}$ of points with positive dimensional stabilisers is a closed subset of X. If $x \in X^s$, then by Lemma 4.29 there is a function $f \in \mathcal{O}(X)^G$ such that

$$f(X_+) = 0 \quad \text{and} \quad f(G \cdot x) = 1.$$

Then $x \in X_f(k)$ and we claim that $X_f \subset X^s$ so that X_f is an open neighbourhood of x in X^s. Since all points in X_f have stabilisers of dimension zero, it remains to check that their orbits are closed. Suppose $z \in X_f(k)$ has a non-closed orbit so $w \notin G \cdot z$ belongs to the orbit closure of z; then $w \in X_f(k)$ too as f is G-invariant and so w must have stabiliser of dimension zero. However, by Proposition 3.15 the boundary of the orbit $G \cdot z$ is a union of orbits of strictly lower dimension and so the orbit of w must be of dimension strictly less than that of z which contradicts the fact that w has zero dimensional stabiliser. Hence, X^s is an open subset of X, and is covered by sets of the form X_f as above.

Since $\varphi(X_f) = Y_f$ is open in Y and also $X_f = \varphi^{-1}(Y_f)$, it follows that Y^s is open in Y and also $X^s = \varphi^{-1}(\varphi(X^s))$. Then $\varphi : X^s \to Y^s$ is a good quotient by Corollary 3.33. Furthermore, the action of G on X^s is closed and so $\varphi : X^s \to Y^s$ is a geometric quotient by Corollary 3.32. □

Example 4.37. We can now calculate the stable set for the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ as in Examples 3.17 and 4.32. The closed orbits are the conics $\{xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 - \{0\}$ and the origin, but the origin has a positive dimensional stabiliser. Thus

$$X^s = \{(x, y) \in \mathbb{A}^2 : xy \neq 0\} = X_{xy}.$$

In this example, it is clear why we need to insist that $\dim G_x = 0$ in the definition of stability: so that the stable set is open. In fact this requirement should also be clear from the proof of Proposition 4.36.

Example 4.38. We may also consider which points are stable for the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ as in Examples 3.18 and 4.33. The only closed orbit is the origin, whose stabiliser is positive dimensional, and so $X^s = \emptyset$. In particular, this example shows that the stable set may be empty.

Example 4.39. Consider $G = \mathrm{GL}_2$ acting on the space $M_{2\times 2}$ of 2×2 matrices with k-coefficients by conjugation. The characteristic polynomial of a matrix A is given by

$$\mathrm{char}_A(t) = \det(xI - A) = x^2 + c_1(A)x + c_2(A)$$

where $c_1(A) = -\mathrm{Tr}(A)$ and $c_2(A) = \det(A)$ and is well defined on the conjugacy class of a matrix. The Jordan canonical form of a matrix is obtained by conjugation and so lies in the same orbit of the matrix. The theory of Jordan canonical forms says there are three types of orbits:

- matrices with characteristic polynomial with distinct roots α, β. These matrices are diagonalisable with Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$
These orbits are closed and have dimension 2. The stabiliser of the above matrix is the subgroup of diagonal matrices which is 2 dimensional.
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is equal to the characteristic polynomial. These matrices are not diagonalisable and their Jordan canonical form is
$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$
These orbits are also 2 dimensional but are not closed: for example
$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$
- matrices with characteristic polynomial with repeated root α for which the minimum polynomial is $x - \alpha$. These matrices have Jordan canonical form
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$
Since scalar multiples of the identity commute with everything, their stabilisers are equal to the full group GL_2 and their orbits are simply a point, which is closed and zero dimensional.

We note that every orbit of the second type contains an orbit of the third type and so will be identified in the quotient. Clearly there are two G-invariant functions on $M_{2\times 2}$: the trace and determinant, and so
$$k[\mathrm{tr}, \det] \subset \mathcal{O}(M_{2\times 2})^{\mathrm{GL}_2}.$$
We claim that these are the only G-invariant functions on $M_{2\times 2}$. To see this we note that from the above discussion about Jordan normal forms and orbit closures, a G-invariant function on $M_{2\times 2}$ is completely determined by its values on the diagonal matrices $D_2 \subset M_{2\times 2}$. Hence the ring of GL_2-invariants on $M_{2\times 2}$ is contained in the ring $\mathcal{O}(D_2) \cong k[x_{11}, x_{22}]$. In fact, using the GL_2-action we can permute the diagonal entries; therefore,
$$\mathcal{O}(M_{2\times 2})^{\mathrm{GL}_2} \subset k[x_{11}, x_{22}]^{S_2} = k[x_{11} + x_{22}, x_{11}x_{22}],$$
as the symmetric polynomials are generated by the elementary symmetric polynomials. These elementary symmetric polynomials correspond to the trace and determinant respectively, and we see there are no additional invariants. Hence
$$k[\mathrm{tr}, \det] = \mathcal{O}(M_{2\times 2})^{\mathrm{GL}_2}$$
and the affine GIT quotient is given by
$$\varphi = (\mathrm{tr}, \det) : M_{2\times 2} \to \mathbb{A}^2.$$
The subgroup $\mathbb{G}_m I_2$ fixes every point and so there are no stable points for this action.

Example 4.40. More generally, we can consider $G = \mathrm{GL}_n$ acting on $M_{n\times n}$ by conjugation. If A is an $n \times n$ matrix, then the coefficients of its characteristic polynomial
$$\mathrm{char}_A(t) = \det(tI - A) = t^n + c_1(A)t^{n-1} + \cdots + c_n(A)$$
are all G-invariant functions. As in Example 4.39 above, we can use the theory of Jordan normal forms as above to describe the different orbits and the closed orbits correspond to diagonalisable matrices. By a similar argument to above, we have
$$k[c_1, \dots, c_n] \subset \mathcal{O}(M_{n\times n})^{\mathrm{GL}_n} \subset \mathcal{O}(D_n)^{S_n} \cong k[x_{11}, \dots, x_{nn}]^{S_n} = k[\sigma_1, \dots, \sigma_n]$$

where σ_i is the ith elementary symmetric polynomial in the x_js. Hence, we conclude these are all equalities and the affine GIT quotient is given by

$$\begin{aligned} \varphi : M_{n\times n} &\to \mathbb{A}^n \\ A &\mapsto (c_1(A), \dots, c_n(A)). \end{aligned}$$

Again as every orbit contains a copy of $\mathbb{G}_m$ in its stabiliser subgroup, there are no stable points.

Remark 4.41. In situations where there is a non-finite subgroup $H \subset G$ which is contained in the stabiliser subgroup of every point for a given action of G on X, the stable set is automatically empty. Hence, for the purposes of GIT, it is better to work with the induced action of the group G/H. In the above example, this would be equivalent to considering the action of the special linear group on the space of $n \times n$ matrices by conjugation.