5. Projective GIT quotients

In this section we extend the theory of affine GIT developed in the previous section to construct GIT quotients for reductive group actions on projective schemes. The idea is that we would like construct our GIT quotient by gluing affine GIT quotients. In order to do this we would like to cover our scheme X by affine open subsets which are invariant under the group action and glue the affine GIT quotients of these affine open subsets of X. However, it may not be possible to cover all of X by such compatible open invariant affine subsets.

For a projective scheme X with an action of a reductive group G, there is not a canonical way to produce an open subset of X which is covered by open invariant affine subsets. Instead, this will depend on a choice of an equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ by a linear representation $G \to \mathrm{GL}_{n+1}$. We recall that a projective embedding of X corresponds to a choice of a (very) ample line bundle $\mathcal{L}$ on X. We will shortly see that equivariant projective embeddings are given by an ample *linearisation* of the G-action on X, which is a lift of the G-action to a ample line bundle on X such that the projection to X is equivariant and the action on the fibres is linear.

In this section, we will show for a reductive group G acting on a projective scheme X and a choice of ample linearisation of the action, there is a good quotient of an open subset of *semistable* points in X. Furthermore, this quotient is itself projective and restricts to a geometric quotient on an open subset of *stable* points. The main reference for the construction of the projective GIT quotient is Mumford's book [25] and other excellent references are [4, 24, 31, 32, 42].

5.1. Construction of the projective GIT quotient.

Definition 5.1. Let X be a projective scheme with an action of an affine algebraic group G. A linear G-equivariant projective embedding of X is a group homomorphism $G \to \mathrm{GL}_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$. We will often simply say that the G-action on $X \hookrightarrow \mathbb{P}^n$ is linear to mean that we have a linear G-equivariant projective embedding of X as above.

Suppose we have a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$. Since the projective embedding $X \subset \mathbb{P}^n$ is G-equivariant, there is an induced action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over $X \subset \mathbb{P}^n$. More precisely, we have

$$\mathcal{O}(\mathbb{A}^{n+1}) = k[x_0, \dots, x_n] = \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_r = \bigoplus_{r \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(r))$$

and if $X \subset \mathbb{P}^n$ is the closed subscheme associated to a homogeneous ideal $I(X) \subset k[x_0, \dots, x_n]$, then $\tilde{X} = \operatorname{Spec} R(X)$ where $R(X) = k[x_0, \dots, x_n]/I(X)$.

The k-algebras $\mathcal{O}(\mathbb{A}^{n+1})$ and $R(X)$ are graded by homogeneous degree and, as the G-action on $\mathbb{A}^{n+1}$ is linear it preserves the graded pieces, so that the invariant subalgebra

$$\mathcal{O}(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} k[x_0, \dots, x_n]_r^G$$

is a graded algebra and, similarly, $R(X)^G = \oplus_{r \geq 0} R(X)_r^G$. By Nagata's theorem, $R(X)^G$ is finitely generated, as G is reductive. The inclusion of finitely generated graded k-algebras $R(X)^G \hookrightarrow R(X)$ determines a rational morphism of projective schemes

$$X \dashrightarrow \operatorname{Proj} R(X)^G$$

whose indeterminacy locus is the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G := \oplus_{r > 0} R(X)_r^G$.

Definition 5.2. For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, we define the *nullcone* N to be the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G$ in $R(X)$ (strictly speaking the nullcone is really the affine cone $\tilde{N}$ over N). We define the *semistable set* $X^{ss} = X - N$ to be the open subset of X given by the complement to the nullcone. More precisely, $x \in X$ is *semistable* if there exists a G-invariant homogeneous function

$f \in R(X)^G_r$ for $r > 0$ such that $f(x) \neq 0$. By construction, the semistable set is the open subset which is the domain of definition of the rational map

$$X \dashrightarrow \operatorname{Proj} R(X)^G.$$

We call the morphisms $X^{ss} \to X//G := \operatorname{Proj} R(X)^G$ the *GIT quotient* of this action.

Theorem 5.3. *For a linear action of a reductive group G on a projective scheme $X \subset \mathbb{P}^n$, the GIT quotient $\varphi : X^{ss} \to X//G$ is a good quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X//G$ is a projective scheme.*

Proof. We let $\varphi : X^{ss} \to Y := X//G$ denote the projective GIT quotient. By construction $X//G$ is the projective spectrum of the finitely generated graded k-algebra $R(X)^G$. We claim that $\operatorname{Proj} R(X)^G$ is projective over $\operatorname{Spec} R(X)^G_0 = \operatorname{Spec} k$. If $R(X)^G$ is finitely generated by $R(X)^G_1$ as a k-algebra, this result follows immediately from [14] II Corollary 5.16. If not, then as $R(X)^G$ is a finitely generated k-algebra, we can pick generators $f_1, \dots, f_r$ in degrees $d_1, \dots, d_r$. Let $d := d_1 \cdot \dots \cdot d_r$; then

$$(R(X)^G)^{(d)} = \bigoplus_{l \geq 0} R(X)^G_{dl}$$

is finitely generated by $(R(X)^G)^{(d)}_1$ as k-algebra and so $\operatorname{Proj}\left((R(X)^G)^{(d)}\right)$ is projective over $\operatorname{Spec} k$. Since $X//G := \operatorname{Proj} R(X)^G \cong \operatorname{Proj}\left((R(X)^G)^{(d)}\right)$ (see [14] II Exercise 5.13), we can conclude that $X//G$ is projective.

For $f \in R^G_+$, the open affine subsets $Y_f \subset Y$ form a basis of the open sets on Y. Since $f \in R(X)^G_+ \subset R(X)_+$, we can also consider the open affine subset $X_f \subset X$ and, by construction of φ, we have that $\varphi^{-1}(Y_f) = X_f$. Let $\tilde{X}_f$ (respectively $\tilde{Y}_f$) denote the affine cone over X_f (respectively Y_f). Then

$$\mathcal{O}(Y_f) \cong \mathcal{O}(\tilde{Y}_f)_0 \cong ((R(X)^G)_f)_0 \cong ((R(X)_f)_0)^G \cong (\mathcal{O}(\tilde{X}_f)_0)^G \cong \mathcal{O}(X_f)^G$$

and so the corresponding morphism of affine schemes $\varphi_f : X_f \to Y_f \cong \operatorname{Spec} \mathcal{O}(X_f)^G$ is an affine GIT quotient, and so also a good quotient by Theorem 4.30. The morphism $\varphi : X^{ss} \to Y$ is obtained by gluing the good quotients $\varphi_f : X_f \to Y_f$. Since Y_f cover Y (and X_f cover X^{ss}) and being a good quotient is local on the target Remark 3.34, we can conclude that φ is also a good quotient. $\square$

We recall that as $\varphi : X^{ss} \to X//G$ is a good quotient, for two semistable points x_1, x_2 in X^{ss}, we have

$$\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset \iff \varphi(x_1) = \varphi(x_2).$$

Furthermore, the preimage of each point in $X//G$ contains a unique closed orbit. The presence of non-closed orbits in the semistable locus will prevent the good quotient $\varphi : X^{ss} \to X//G$ from being a geometric quotient.

We can now ask if there is an open subset X^s of X^{ss} on which this quotient becomes a geometric quotient. For this we want the action to be closed on X^s. This motivates the definition of stability (see also Definition 4.35).

Definition 5.4. Consider a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$. Then a point $x \in X$ is

(1) *stable* if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)^G_+$ such that $x \in X_f$ and the action of G on X_f is closed.
(2) *unstable* if it is not semistable.

We denote the set of stable points by X^s and the set of unstable points by $X^{us} := X - X^{ss} = N$.

We emphasise that, somewhat confusingly, unstable does not mean not stable, but this terminology has long been accepted by the mathematical community.

Lemma 5.5. *The stable and semistable sets X^s and X^{ss} are open in X.*

Proof. By construction, the semistable set is open in X as it is the complement to the nullcone N, which is closed. To prove that the stable set is open, we consider the subset $X_c := \cup X_f$ where the union is taken over $f \in R(X)^G_+$ such that the action of G on X_f is closed; then X_c is open in X and it remains to show X^s is open in X_c. By Proposition 3.21, the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X and so the set of points with zero dimensional stabiliser is open. Hence, we have open inclusions $X^s \subset X_c \subset X$. □

Theorem 5.6. *For a linear action of a reductive group G on a closed subscheme $X \subset \mathbb{P}^n$, let $\varphi : X^{ss} \to Y := X//G$ denote the GIT quotient. Then there is an open subscheme $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and that the GIT quotient restricts to a geometric quotient $\varphi : X^s \to Y^s$.*

Proof. Let Y_c be the union of Y_f for $f \in R(X)^G_+$ such that the G-action on X_f is closed and let X_c be the union of X_f over the same index set so that $X_c = \varphi^{-1}(Y_c)$. Then $\varphi : X_c \to Y_c$ is constructed by gluing $\varphi_f : X_f \to Y_f$ for $f \in R(X)^G_+$ such that the G-action on X_f is closed. Each φ_f is a good quotient and as the action on X_f is closed, φ_f is also a geometric quotient by Corollary 3.32. Hence $\varphi : X_c \to Y_c$ is a geometric quotient by Remark 3.34.

By definition, X^s is the open subset of X_c consisting of points with zero dimensional stabilisers and we let $Y^s := \varphi(X^s) \subset Y_c$. It remains to prove that Y^s is open. As $\varphi : X_c \to Y_c$ is a geometric quotient and X^s is a G-invariant subset of X, $\varphi^{-1}(Y^s) = X^s$ and also $Y_c - Y^s = \varphi(X_c - X^s)$. As $X_c - X^s$ is closed in X_c, property iv) of good quotient gives that $\varphi(X_c - X^s) = Y_c - Y^s$ is closed in Y_c and so Y^s is open in Y_c. Since Y_c is open in Y, we can conclude that $Y^s \subset Y$ is open. Finally, the geometric quotient $\varphi : X_c \to Y_c$ restricts to a geometric quotient $\varphi : X^s \to Y^s$ by Corollary 3.33. □

Remark 5.7. We see from the proof of this theorem that to get a geometric quotient we do not have to impose the condition $\dim G_x = 0$ and in fact in Mumford's original definition of stability this condition was omitted. However, the modern definition of stability, which asks for zero dimensional stabilisers, is now widely accepted. One advantage of the modern definition is that if the semistable set is nonempty, then the dimension of the geometric quotient equals its expected dimension. A second advantage of the modern definition of stability is that it is better suited to moduli problems.

Example 5.8. Consider the linear action of $G = \mathbb{G}_m$ on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n].$$

In this case $R(X) = k[x_0, \dots, x_n]$ which is graded into homogeneous pieces by degree. It is easy to see that the functions $x_0x_1, \dots, x_0x_n$ are all G-invariant. In fact, we claim that these functions generate the ring of invariants. To prove the claim, suppose we have $f \in R(X)$; then

$$f = \sum_{\underline{m}=(m_0,\dots,m_n)} a(\underline{m}) x_0^{m_0} x_1^{m_1} \dots x_n^{m_n}$$

and, for $t \in \mathbb{G}_m$,

$$t \cdot f = \sum_{\underline{m}=(m_0,\dots,m_n)} a(\underline{m}) t^{m_0 - m_1 - \cdots - m_n} x_0^{m_0} x_1^{m_1} \dots x_n^{m_n}.$$

Then f is G-invariant if and only if $a(\underline{m}) = 0$ for all $\underline{m} = (m_0, \dots, m_n)$ such that $m_0 \neq \sum_{i=1}^n m_i$. If m satisfies $m_0 = \sum_{i=1}^n m_i$, then

$$x_0^{m_0} x_1^{m_1} \dots x_n^{m_n} = (x_0x_1)^{m_1} \dots (x_0x_n)^{m_n};$$

that is, if f is G-invariant, then $f \in k[x_0x_1, \dots x_0x_n]$. Hence

$$R(X)^G = k[x_0x_1, \dots, x_0x_n] \cong k[y_0, \dots, y_{n-1}]$$

and after taking the projective spectrum we obtain the projective variety $X//G = \mathbb{P}^{n-1}$. The explicit choice of generators for $R(X)^G$ allows us to write down the rational morphism

$$\varphi : X = \mathbb{P}^n \dashrightarrow X//G = \mathbb{P}^{n-1}$$
$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0x_1 : \cdots : x_0x_n]$$

and its clear from this description that the nullcone

$$N = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 = 0 \text{ or } (x_1, \cdots, x_n) = 0\}$$

is the projective variety defined by the homogeneous ideal $I = (x_0x_1, \cdots, x_0x_n)$. In particular,

$$X^{ss} = \bigcup_{i=1}^{n} X_{x_0x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \dots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

where we are identifying the affine chart on which $x_0 \neq 0$ in $\mathbb{P}^n$ with $\mathbb{A}^n$. Therefore

$$\varphi : X^{ss} = \mathbb{A}^n - \{0\} \dashrightarrow X/\!/G = \mathbb{P}^{n-1}$$

is a good quotient of the action on X^{ss}. As the preimage of each point in $X/\!/G$ is a single orbit, this is also a geometric quotient. Moreover, every semistable point is stable as all orbits are closed in $\mathbb{A}^n - \{0\}$ and have zero dimensional stabilisers.

In general it can be difficult to determine which points are semistable and stable as it is necessary to have a description of the graded k-algebra of invariant functions. The ideal situation is as above where we have an explicit set of generators for the invariant algebra which allows us to write down the quotient map. However, finding generators and relations for the invariant algebra in general can be hard. We will soon see that there are other criteria that we can use to determine (semi)stability of points.

Lemma 5.9. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is stable if and only if x is semistable and its orbit $G \cdot x$ is closed in X^{ss} and its stabiliser G_x is zero dimensional.*

Proof. Suppose x is stable and $x' \in \overline{G \cdot x} \cap X^{ss}$; then $\varphi(x') = \varphi(x)$ and so $x' \in \varphi^{-1}(\varphi(x)) \subset \varphi^{-1}(Y^s) = X^s$. As G acts on X^s with zero-dimensional stabiliser, this action must be closed as the boundary of an orbit is a union of orbits of strictly lower dimension. Therefore, $x' \in G \cdot x$ and so the orbit $G \cdot x$ is closed in X^{ss}.

Conversely, we suppose x is semistable with closed orbit in X^{ss} and zero dimensional stabiliser. As x is semistable, there is a homogeneous $f \in R(X)^G_+$ such that $x \in X_f$. As $G \cdot x$ is closed in X^{ss}, it is also closed in the open affine set $X_f \subset X^{ss}$. By Proposition 3.21, the G-invariant set

$$Z := \{z \in X_f : \dim G_z > 0\}$$

is closed in X_f. Since Z is disjoint from $G \cdot x$ and both sets are closed in the affine scheme X_f, by Lemma 4.29, there exists $h \in \mathcal{O}(X_f)^G$ such that

$$h(Z) = 0 \quad \text{and} \quad h(G \cdot x) = 1.$$

We claim that from the function h, we can produce a G-invariant homogeneous polynomial $h' \in R(X)^G_+$ such that $x \in X_{fh'}$ and $X_{fh'}$ is disjoint from Z, as then all orbits in $X_{fh'}$ have zero dimensional stabilisers and so must be closed in $X_{fh'}$ (as the closure of an orbit is a union of lower dimensional orbits), in which case we can conclude that x is stable. The proof of the above claim follows from Lemma 5.10 below and uses the fact that G is geometrically reductive. More precisely, we have that $\mathcal{O}(X_f) = \mathcal{O}(\tilde{X}_f)_0$ is a quotient of $A := (k[x_0, \dots, x_n]_f)_0$ and we take I to be the kernel. Then $h^r = h'/f^s \in A^G/(I \cap A^G)$ for some homogeneous polynomial h' and positive integers r and s. □

Lemma 5.10. *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. For a G-invariant ideal I of A and $a \in (A/I)^G$, there is a positive integer r such that $a^r \in A^G/(I \cap A^G)$.*

Proof. Let $b \in A$ be an element whose image in A/I is a and we can assume $a \neq 0$. As the action is rational, b is contained in a finite dimensional G-invariant linear subspace $V \subset A$ spanned by the translates $g \cdot b$. Then $b \notin V \cap I$ as $a \neq 0$; however, $g \cdot b - b \in V \cap I$ for all $g \in G$ as a is G-invariant. Therefore $\dim V = \dim(V \cap I) + 1$ and every element in V can be uniquely written as $\lambda b + b'$ for $\lambda \in k$ and $b' \in V \cap I$. Consider the linear projection $l : V \to k$ onto the line spanned by b, which is G-equivariant. In terms of the dual representation $V^\vee$, the projection l

corresponds to a non-zero fixed point l^* and so, as G is geometrically reductive, there exists a G-invariant homogeneous function $F \in \mathcal{O}(V^\vee)$ of positive degree r which is not vanishing at l^*. We can take a basis of V (and dual basis of $V^\vee$) where the first basis vector corresponds to b. Then the coefficient λ of x_1^r in F is non-zero. Consider the algebra homomorphism

$$\mathcal{O}(V^\vee) = \mathrm{Sym}^* V \to A$$

and let $b_0 \in A^G$ be the image of $F \in \mathcal{O}(V^\vee)^G$. Then $b_0 - \lambda b^r \in I$, as this belongs to the ideal generated by a choice of basis vectors for $V \cap I$. Hence $a^r \in A^G/(I \cap A^G)$ as required. □

Remark 5.11. If G is linearly reductive, then taking G-invariants is exact, and so we can take $r = 1$ in the above lemma.

5.2. **A description of the k-points of the GIT quotient.**

Definition 5.12. Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. A k-point $x \in X(k)$ is said to be *polystable* if it is semistable and its orbit is closed in X^{ss}. We say two semistable k-points are *S-equivalent* if their orbit closures meet in X^{ss}. We write this equivalence relation on $X^{ss}(k)$ as $\sim_{\text{S-equiv.}}$ and let $X^{ss}(k)/\sim_{\text{S-equiv.}}$ denote the S-equivalence classes of semistable k-points.

By Lemma 5.9 above, every stable k-point is polystable.

Lemma 5.13. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$ and let $x \in X(k)$ be a semistable k-point; then its orbit closure $\overline{G \cdot x}$ contains a unique polystable orbit. Moreover, if x is semistable but not stable, then this unique polystable orbit is also not stable.*

Proof. The first statement follows from Corollary 3.32: φ is constant on orbit closures and the preimage of a k-point under φ contains a orbit which is closed in X^{ss}; this is the polystable orbit. For the second statement we note that if a semistable orbit $G \cdot x$ is not closed, then the unique closed orbit in $\overline{G \cdot x}$ has dimension strictly less than $G \cdot x$ by Proposition 3.15 and so cannot be stable. □

Corollary 5.14. *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For two semistable points $x, x' \in X^{ss}$, we have $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, there is a bijection of sets*

$$X/\!/G(k) \cong X^{ps}(k)/G(k) \cong X^{ss}(k)/\sim_{\textit{S-equiv.}}$$

where $X^{ps}(k)$ is the set of polystable k-points.