5.3. **Linearisations.** An abstract projective scheme X does not come with a pre-specified embedding in a projective space. However, an ample line bundle L on X (or more precisely some power of L) determines an embedding of X into a projective space. More precisely, the projective scheme X and ample line bundle L, determine a finitely generated graded k-algebra

$$R(X,L) := \bigoplus_{r\geq 0} H^0(X, L^{\otimes r}).$$

We can choose generators of this k-algebra: $s_i \in H^0(X, L^{\otimes r_i})$ for $i = 0, , ..n$, where $r_i \geq 1$. Then these sections determine a closed immersion

$$X \hookrightarrow \mathbb{P}(r_0, \dots, r_n)$$

into a weighted projective space, by evaluating each point of X at the sections s_i. In fact, if we replace L by $L^{\otimes m}$ for m sufficiently large, then we can assume that the generators s_i of the finitely generated k-algebra

$$R(X, L^{\otimes m}) = \bigoplus_{r\geq 0} H^0(X, L^{\otimes mr})$$

all lie in degree 1. In this case, the sections s_i of the line bundle $L^{\otimes m}$ determine a closed immersion

$$X \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(X, L^{\otimes m})^*)$$

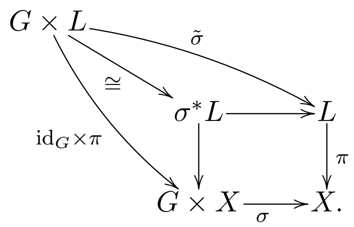
given by evaluation $x \mapsto (s \mapsto s(x))$.

Now suppose we have an action of an affine algebraic group G on X; then we would like to do everything above G-equivariantly, by lifting the G-action to L such that the above embedding is equivariant and the action of G on $\mathbb{P}^n$ is linear. This idea is made precise by the notion of a linearisation.

Definition 5.15. Let X be a scheme and G be an affine algebraic group acting on X via a morphism $\sigma : G \times X \to X$. Then a *linearisation* of the G-action on X is a line bundle $\pi : L \to X$ over X with an isomorphism of line bundles

$$\pi_X^* L = G \times L \cong \sigma^* L,$$

where $\pi_X : G \times X \to X$ is the projection, such that the induced bundle homomorphism $\tilde{\sigma} : G \times L \to L$ defined by



induces an action of G on L; that is, we have a commutative square of bundle homomorphisms

$$\begin{array}{ccc} G \times G \times L & \xrightarrow{\mathrm{id}_G \times \tilde{\sigma}} & G \times L \\ \downarrow{\scriptstyle \mu_G \times \mathrm{id}_L} & & \downarrow{\scriptstyle \tilde{\sigma}} \\ G \times L & \xrightarrow[\tilde{\sigma}]{} & L. \end{array}$$

We say that a linearisation is *(very) ample* if the underlying line bundle is (very) ample.

Let us unravel this definition a little. Since $\tilde{\sigma} : G \times L \to L$ is a homomorphism of vector bundles, we have

i) the projection $\pi : L \to X$ is G-equivariant,
ii) the action of G on the fibres of L is linear: for $g \in G$ and $x \in X$, the map on the fibres $L_x \to L_{g\cdot x}$ is linear.

Remark 5.16.

(1) The notion of a linearisation can also be phrased sheaf theoretically: a linearisation of a G-action on X on an invertible sheaf $\mathcal{L}$ is an isomorphism
$$\Phi : \sigma^*\mathcal{L} \to \pi_X^*\mathcal{L},$$
where $\pi_X : G \times X \to X$ is the projection map, which satisfies the cocycle condition:
$$(\mu \times \mathrm{id}_X)^*\Phi = \pi_{23}^*\Phi \circ (\mathrm{id}_G \times \sigma)^*\Phi$$
where $\pi_{23} : G \times G \times X \to G \times X$ is the projection onto the last two factors. If $\pi : L \to X$ denotes the line bundle associated to the invertible sheaf $\mathcal{L}$, then the isomorphism Φ determines a bundle isomorphism of line bundles over $G \times X$:
$$\Phi : (G \times X) \times_{\pi_X, X, \pi} L \to (G \times X) \times_{\sigma, X, \pi} L$$
and then we obtain $\tilde{\sigma} := \pi_X \circ \Phi$. The cocycle condition ensures that $\tilde{\sigma}$ is an action.

(2) The above notion of a linearisation of a G-action on X can be easily modified to larger rank vector bundles (or locally free sheaves) over X. However, we will only work with linearisations for line bundles (or equivalently invertible sheaves).

Exercise 5.17. For an action of an affine algebraic group G on a scheme X, the tensor product of two linearised line bundles has a natural linearisation and the dual of a linearised line bundle also has a natural linearisation. By an isomorphism of linearisations, we mean an isomorphism of the underlying line bundles that is G-equivariant; that is, commutes with the actions of G on these line bundles. We let $\mathrm{Pic}^G(X)$ denote the group of isomorphism classes of linearisations of a G-action on X. There is a natural forgetful map $\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$.

Example 5.18. (1) Let us consider $X = \mathrm{Spec}\, k$ with necessarily the trivial G-action. Then there is only one line bundle $\pi : \mathbb{A}^1 \to \mathrm{Spec}\, k$ over $\mathrm{Spec}\, k$, but there are many linearisations. In fact, the group of linearisations of X is the character group of G. If $\chi : G \to \mathbb{G}_m$ is a character of G, then we define an action of G on $\mathbb{A}^1$ by acting by $G \times \mathbb{A}^1 \to \mathbb{A}^1$. Conversely, any linearisation is given by a linear action of G on $\mathbb{A}^1$; that is, by a group homomorphism $\chi : G \to \mathrm{GL}_1 = \mathbb{G}_m$.

(2) For any scheme X with an action of an affine algebraic group G and any character $\chi : G \to \mathbb{G}_m$, we can construct a linearisation on the trivial line bundle $X \times \mathbb{A}^1 \to X$ by
$$g \cdot (x, z) = (g \cdot x, \chi(g)z).$$
More generally, for any linearisation $\tilde{\sigma}$ on $L \to X$, we can twist the linearisation by a character $\chi : G \to \mathbb{G}_m$ to obtain a linearisation $\tilde{\sigma}^\chi$.

(3) Not every linearisation on a trivial line bundle comes from a character. For example, consider $G = \mu_2 = \{\pm 1\}$ acting on $X = \mathbb{A}^1 - \{0\}$ by $(-1) \cdot x = x^{-1}$. Then the linearisation on $X \times \mathbb{A}^1 \to X$ given by $(-1) \cdot (x, z) = (x^{-1}, xz)$ is not isomorphic to a linearisation given by a character, as over the fixed points $+1$ and -1 in X, the action of $-1 \in \mu_2$ on the fibres is given by $z \mapsto z$ and $z \mapsto -z$ respectively.

(4) The natural actions of GL_{n+1} and SL_{n+1} on $\mathbb{P}^n$ inherited from the action of GL_{n+1} on $\mathbb{A}^{n+1}$ by matrix multiplication can be naturally linearised on the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. To see why, we note that the trivial rank $n+1$-vector bundle on $\mathbb{P}^n$ has a natural linearisation of GL_{n+1} (and also SL_{n+1}). The tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times \mathbb{A}^{n+1}$ is preserved by this action and so we obtain natural linearisations of these actions on $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$. However, the action of PGL_{n+1} on $\mathbb{P}^n$ does not admit a linearisation on $\mathcal{O}_{\mathbb{P}^n}(1)$ (see Exercise Sheet 9), but we can always linearise any G-action on $\mathbb{P}^n$ to $\mathcal{O}_{\mathbb{P}^n}(n+1)$ as this is isomorphic to the nth exterior power of the cotangent bundle, and we can lift any action on $\mathbb{P}^n$ to its cotangent bundle.

Lemma 5.19. *Let G be an affine algebraic group acting on a scheme X via $\sigma : G \times X \to X$ and let $\tilde{\sigma} : G \times L \to L$ be a linearisation of the action on a line bundle L over X. Then there is a natural linear representation $G \to \mathrm{GL}(H^0(X, L))$.*

Proof. We construct the co-module $H^0(X,L) \to \mathcal{O}(G) \otimes_k H^0(X,L)$ defining this representation by the composition

$$H^0(X,L) \xrightarrow{\sigma^*} H^0(G\times X, \sigma^* L) \cong H^0(G \times X, G \times L) \cong H^0(G, \mathcal{O}_G) \otimes H^0(X,L)$$

where the final isomorphism follows from the Künneth formula and the middle isomorphism is defined using the isomorphism $G \times L \cong \sigma^* L$. □

Remark 5.20. Suppose that X is a projective scheme and L is a very ample linearisation. Then the natural evaluation map

$$H^0(X,L) \otimes_k \mathcal{O}_X \to L$$

is G-equivariant. Moreover, this evaluation map induces a G-equivariant closed embedding

$$X \hookrightarrow \mathbb{P}(H^0(X,L)^*)$$

such that L is isomorphic to the pullback of the Serre twisting sheaf $\mathcal{O}(1)$ on this projective space. In this case, we see that we have an embedding of X as a closed subscheme of a projective space $\mathbb{P}(H^0(X,L)^*)$ such that the action of G on X comes from a linear action of G on $H^0(X,L)^*$. In particular, we see that a linearisation naturally generalises the setting of G acting linearly on $X \subset \mathbb{P}^n$.

5.4. **Projective GIT with respect to an ample linearisation.** Let G be a reductive group acting on a projective scheme X and let L be an ample linearisation of the G-action on X. Then consider the graded finitely generated k-algebra

$$R := R(X,L) := \bigoplus_{r\geq 0} H^0(X, L^{\otimes^r})$$

of sections of powers of L. Since each line bundle $L^{\otimes r}$ has an induced linearisation, there is an induced action of G on the space of sections $H^0(X, L^{\otimes^r})$ by Lemma 5.19. We consider the graded algebra of G-invariant sections

$$R^G = \bigoplus_{r\geq 0} H^0(X, L^{\otimes^r})^G.$$

The subalgebra of invariant sections R^G is a finitely generated k-algebra and $\operatorname{Proj} R^G$ is projective over $R_0^G = k^G = k$ following a similar argument to above.

Definition 5.21. For a reductive group G acting on a projective scheme X with respect to an ample line bundle, we make the following definitions.

1) A point $x \in X$ is *semistable* with respect to L if there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$.
2) A point $x \in X$ is *stable* with respect to L if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes^r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and the action of G on $X_\sigma := \{x \in X : \sigma(x) \neq 0\}$ is closed.

We let $X^{ss}(L)$ and $X^s(L)$ denote the open subset of semistable and stable points in X respectively. Then we define the *projective GIT quotient with respect to* L to be the morphism

$$X^{ss} \to X/\!/_L G := \operatorname{Proj} R(X,L)^G$$

associated to the inclusion $R(X,L)^G \hookrightarrow R(X,L)$.

Exercise 5.22. We have already defined notions of semistability and stability when we have a linear action of G on $X \subset \mathbb{P}^n$. In this case, the action can naturally be linearised using the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Show that the two notions of semistability agree; that is,

$$X^{(s)s} = X^{(s)s}(\mathcal{O}_{\mathbb{P}^n}(1)|_X).$$

Theorem 5.23. *Let G be a reductive group acting on a projective scheme X and L be an ample linearisation of this action. Then the GIT quotient*

$$\varphi : X^{ss}(L) \to X//_L G = \mathrm{Proj} \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^G$$

is a good quotient and $X//_L G$ is a projective scheme with a natural ample line bundle L' such that $\varphi^* L' = L^{\otimes n}$ for some $n > 0$. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.

Proof. As L is ample, for each $\sigma \in R(X, L)_+^G$, the open set X_σ is affine and the above GIT quotient is obtained by gluing affine GIT quotients (we omit the proof as it is very similar to that of Theorem 5.3 and Theorem 5.6). □

Remark 5.24. In fact, the graded homogeneous ring $R(X, L)^G$ also determines an ample line bundle L' on its projectivisation $X//_L G$ such that $R(X//_L G, L') \cong R(X, L)^G$. Furthermore, $\phi^*(L') = L^{\otimes r}$ for some $r > 0$ (for example, see [4] Theorem 8.1 for a proof of this statement).

Remark 5.25 (Variation of geometric invariant theory quotient)**.** We note that the GIT quotient depends on a choice of linearisation of the action. One can study how the semistable locus $X^{ss}(L)$ and the GIT quotient $X//_L G$ vary with the linearisation L; this area is known as variation of GIT. A key result in this area is that there are only finitely many distinct GIT quotients produced by varying the ample linearisation of a fixed G-action on a projective normal variety X (for example, see [5] and [41]).

Remark 5.26. For an ample linearisation L, we know that some positive power of L is very ample. By definition $X^{ss}(L) = X^{ss}(L^{\otimes n})$ and $X^s(L) = X^s(L^{\otimes n})$ and $X//_L G \cong X//_{L^{\otimes n}} G$ (as abstract projective schemes), we can assume without loss of generality that L is very ample and so $X \subset \mathbb{P}^n$ and G acts linearly. However, we note that the induced ample line bundles on $X//_L G$ and $X//_{L^{\otimes n}} G$ are different, and so these GIT quotients come with different embeddings into (weighted) projective spaces.

Definition 5.27. We say two semistable k-points x and x' in X are S-equivalent if the orbit closures of x and x' meet in the semistable subset $X^{ss}(L)$. We say a semistable k-point is polystable if its orbit is closed in the semistable locus $X^{ss}(L)$.

Corollary 5.28. *Let x and x' be k-points in $X^{ss}(L)$; then $\varphi(x) = \varphi(x')$ if and only if x and x' are S-equivalent. Moreover, we have a bijection of sets*

$$(X//_L G)(k) \cong X^{ps}(L)(k)/G(k) \cong X^{ss}(L)(k)/ \sim_{\text{S-equiv.}}$$

where $X^{ps}(L)(k)$ is the set of polystable k-points.

5.5. **GIT for general varieties with linearisations.** In this section, we give a more general theorem of Mumford for constructing GIT quotients of reductive group actions on quasi-projective schemes with respect to (not necessarily ample) linearisations.

Definition 5.29. Let X be a quasi-projective scheme with an action by a reductive group G and L be a linearisation of this action.

1) A point $x \in X$ is *semistable* with respect to L if there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and $X_\sigma = \{x \in X : \sigma(x) \neq 0\}$ is affine.
2) A point $x \in X$ is *stable* with respect to L if $\dim G \cdot x = \dim G$ and there is an invariant section $\sigma \in H^0(X, L^{\otimes r})^G$ for some $r > 0$ such that $\sigma(x) \neq 0$ and X_σ is affine and the action of G on X_σ is closed.

The open subsets of stable and semistable points with respect to L are denoted $X^s(L)$ and $X^{ss}(L)$ respectively.

Remark 5.30. If X is projective and L is ample, then this agrees with Definition 5.21 as X_σ is affine for any non-constant section σ (see [14] III Theorem 5.1 and II Proposition 2.5).

In this setting, the GIT quotient $X//_L G$ is defined by taking the projective spectrum of the ring $R(X,L)^G$ of G-invariant sections of powers of L. One proves that $\varphi : X^{ss}(L) \to Y := X//_L G$ is a good quotient by locally showing that this morphism is obtained by gluing affine GIT quotients $\varphi_\sigma : X_\sigma \to Y_\sigma$ in exactly the same way as Theorem 5.3. Then similarly to Theorem 5.6, one proves that this restricts to a geometric quotient on the stable locus. In particular, we have the following result.

Theorem 5.31. *(Mumford) Let G be a reductive group acting on a quasi-projective scheme X and L be a linearisation of this action. Then there is a quasi-projective scheme $X//_L G$ and a good quotient $\varphi : X^{ss}(L) \to X//_L G$ of the G-action on $X^{ss}(L)$. Furthermore, there is an open subset $Y^s \subset X//_L G$ such that $\varphi^{-1}(Y^s) = X^s(L)$ and $\varphi : X^s(L) \to Y^s$ is a geometric quotient for the G-action on $X^s(L)$.*

The only part of this theorem which remains to be proved is the statement that the GIT quotient $X//_L G$ is quasi-projective. To prove this, one notes that the GIT quotient comes with an ample line bundle L' which can be used to give an embedding of X into a projective space.