Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Eva Martínez

Algebraic Geometry II

Exercise Sheet 1

Hand-in date: 10am, Monday 19th October.

Exercise 1. Let $\mathcal{C}$ be any category and denote by $\mathrm{Psh}(\mathcal{C})$ the category of presheaves on $\mathcal{C}$.

i) Prove that the Yoneda embedding $h\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is fully faithful.

ii) Show that if there is a natural isomorphism $h_C \to h'_C$ then there is a canonical isomorphism $C \to C'$.

Exercise 2. Let R be a commutative ring. Let A, B, and C be R-Modules. Prove that fixing any two of A, B, C, each of the following

$$\tau_{A,B,C}\colon \mathrm{Hom_R(A \otimes_R B, C) \to Hom_R(A, Hom(B, C))}$$

is a natural isomorphism.
(Hint: For $f \in \mathrm{Hom_R(A \otimes_R B, C)}$, $a \in A$ and $b \in B$ consider $\tau_{A,B,C}(f)(a)\colon b \mapsto f(a \otimes b)$).

Exercise 3. Let $\mathrm{Gr}_{d,n}(k)$ be the set of d-dimensional subspaces of k^n, for an algebraically closed field k. We will show this is an abstract variety called the Grassmannian variety.

i) Let $\mathrm{GL}_d(k)$ act on the set $W := M^d_{d\times n}(k)$ of $d \times n$ matrices of rank d by left multiplication. Show that there is a bijection between the orbits of this action and $\mathrm{Gr}_{d,n}(k)$.

ii) For any $J = \{1 \leq j_1 < j_2 < \cdots < j_d \leq n\}$, let $U_J \subset \mathrm{Gr}_{d,n}(k)$ be the set of d-dimensional subspaces $W \subset k^n$ which are complementary to the $n-d$-dimensional space spanned by $\{e_j\}_{j \notin J}$ and let $W_J \subset W$ be the closed set of matrices whose Jth submatrix is I_r. Show that we can identify $U_J \cong W_J \cong \mathbb{A}_k^{d(n-d)}$.
(Hint: for $M \in M^d_{d\times n}(k)$, the $\mathrm{GL}_d(k)$-orbit of M meets W_J if and only if the Jth minor of M is non-zero and then, by using the $\mathrm{GL}_d(k)$-action, we can assume the Jth submatrix is I_d.)

iii) For $J \neq K \subset \{1, \dots, n\}$ of cardinality d, let $W_{J,K} \subset W_J$ be the open set of matrices whose Kth minor is non-zero. Show that there are isomorphisms $\phi_{J,K} : W_{J,K} \to W_{K,J}$.

iv) Conclude that we can construct $\mathrm{Gr}_{d,n}(k)$ as an abstract variety by gluing the affine spaces W_J along the maps $\phi_{J,K}$.