Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 10

Hand-in date: 10am, Monday 11th January.

Note: There will be no exercise class on the 5th January.

Exercise 1. Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. For a 1-PS λ of G and $x \in X(k)$ and a non-zero lift $\tilde{x} \in \tilde{X}$, show that the Hilbert–Mumford weight has the following properties.

a) $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t\to 0} t^\mu \lambda(t) \cdot \tilde{x}$ exists and is non-zero.

b) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive integers n.

c) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G(k)$.

d) $\mu(x, \lambda) = \mu(y, \lambda)$ where $y = \lim_{t\to 0} \lambda(t) \cdot x$.

Exercise 2. Let SL_2 act on $V = k^2$ by left multiplication and consider the induced linear action of SL_2 on $\mathrm{Sym}^d(V)$. To simplify notation we identify $V \cong k[x, y]_1$ and $\mathrm{Sym}^d(V) \cong k[x, y]_d$. The goal of this exercise is to determine the (semi)stable k-points for the linear SL_2-action on $\mathbb{P}^{d+1} \cong \mathbb{P}(\mathrm{Sym}^d(V))$.

a) Show that any primitive 1-PS (that is, a 1-PS which is not a positive multiple of any other 1-PS) of SL_2 is conjugate to the 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_2$

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

b) Prove using Exercise 1, that a k-point $p \in \mathbb{P}^{d+1}$ is SL_2-semistable (resp. stable) if and only if $\mu(g \cdot p, \lambda) \geq 0$ (resp. > 0) for the above 1-PS λ and for all $g \in \mathrm{SL}_2(k)$.

c) Calculate the Hilbert–Mumford weight $\mu(p, \lambda)$ for any $p \in \mathbb{P}^{d+1}$ (for this, let $\tilde{p} = \sum_{i=0}^d a_i x^i y^{d-i}$ be a non-zero lift of p and note that the $\lambda(\mathbb{G}_m)$-action with respect to the basis $x^i y^{d-i}$ is diagonal).

d) Deduce that $\mu(p, \lambda) \geq 0$ if and only if $(1, 0)$ occurs as a zero of $\sum_{i=0}^d a_i x^i y^{d-i}$ with multiplicity at most $d/2$. Determine an analogous statement for stability.

e) Determine the (semi)stability of p in terms of the multiplicities of the zeros of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ in $\mathbb{P}^1$.

f) For $d = 3, 4$, determine the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ and the geometric quotient of the stable locus (recall the closed points of the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ are in bijection with the polystable orbits).

g) (Optional/Harder.) Determine a moduli problem for which the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ is a coarse moduli space.