Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 12

Hand-in date: 10am, Monday 18th January.

Exercise 1. Show that $Y_{d,n} := \mathbb{P}(k[x_0, \dots, x_n]_d)$ parametrises a family of degree d hypersurfaces in $\mathbb{P}^n$ which has the local universal property. Is this a universal family? Deduce that any moduli space of such hypersurfaces is a categorical quotient of SL_{n+1} acting on $Y_{d,n}$.

Exercise 2. Fix a non-zero homogeneous polynomial

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j$$

of degree 3 and let C be the corresponding plane cubic curve defined by $F = 0$. For $p = [1 : 0 : 0] \in \mathbb{P}^2$, show the following statements hold.

i) $p \in C$ if and only if $a_{00} = 0$.

ii) p is a singular point of F if and only if $a_{00} = a_{10} = a_{01} = 0$.

iii) p is a triple point of F if and only if $a_{00} = a_{10} = a_{01} = a_{11} = a_{20} = a_{02} = 0$.

iv) If $p = [1 : 0 : 0]$ is a double point of F, then its tangent lines are defined by

$$a_{20}y^2 + a_{11}yz + a_{02}z^2 = 0.$$

Exercise 3. (Optional/Harder) Prove that if we define families of hypersurfaces of degree d in $\mathbb{P}^n$ using only coefficients in the trivial line bundle, then the moduli functor is a separated presheaf but not a sheaf (and so in particular cannot be represented by a scheme). Show that the sheafification of this functor is the moduli functor defined in the course, where families are allowed to have coefficients in any line bundle.

[For the definition of sheaf and separated presheaf, see the notes from Lecture 2].