Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Eva Martínez

Algebraic Geometry II

Exercise Sheet 13

Hand-in date: 10am, Monday 25th January.

Throughout this sheet, we let X denote a smooth projective curve.

Exercise 1. Prove the following statements.

a) A subsheaf of a locally free sheaf over X is locally free.

b) A non-zero homomorphism $f : \mathcal{L} \to \mathcal{E}$ of locally free sheaves over X with $\operatorname{rk} \mathcal{L} = 1$ is injective.

Exercise 2. Let $\mathcal{E}$ be a locally free sheaf of rank r over X. In this exercise, we will prove that there exists a short exact sequence of locally free sheaves over X

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

such that $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ has rank $r-1$.

a) Show that for any effective divisor D with $r \deg D > h^1(X, \mathcal{E})$, the vector bundle $\mathcal{E}(D)$ admits a section by considering the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to k_D \to 0$$

tensored by $\mathcal{E}$ (here k_D denotes the skyscraper sheaf with support D). Deduce that $\mathcal{E}$ has an invertible subsheaf.

b) For an invertible sheaf $\mathcal{L}$ with $\deg \mathcal{L} > 2g-2$, prove that $h^1(X, \mathcal{L}) = 0$ using Serre duality.

c) Show that the degree of an invertible subsheaf $\mathcal{L}$ of $\mathcal{E}$ is bounded above, by using the Riemann–Roch formula for invertible sheaves and part b).

d) Let $\mathcal{L}$ to be an invertible subsheaf of $\mathcal{E}$ of maximal degree; then verify that the quotient $\mathcal{F}$ of $\mathcal{L} \subset \mathcal{E}$ is locally free.

Exercise 3. In this exercise, we will prove for locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over X that

$$\deg(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rk}\mathcal{E} \deg \mathcal{F} + \operatorname{rk}\mathcal{F} \deg \mathcal{E}$$

by induction on the rank of $\mathcal{E}$.

a) Prove the base case where $\mathcal{E} = \mathcal{O}_X(D)$ by splitting into two cases. If D is effective, use the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D \to 0$$

and the Riemann–Roch Theorem to prove the result. If D is not effective, write D as $D_1 - D_2$ for effective divisors D_i and modify $\mathcal{F}$ by twisting by a line bundle.

b) For the inductive step, use Exercise 2.