Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Eva Martínez

Algebraic Geometry II

Exercise Sheet 14

Hand-in date: 10am, Monday 1st February.

Throughout this sheet, we let X denote a smooth projective curve.

Exercise 1. Let L be a line bundle and E a vector bundle over X; then

a) L is stable.
b) If E is stable (resp. semistable), then $E \otimes L$ is stable (resp. semistable).
c) If E is semistable and the rank and degree of E are coprime, then E is stable.

Exercise 2. Let $f : E \to F$ be a non-zero homomorphism of vector bundles over X; then

a) If E and F are semistable, $\mu(E) \leq \mu(F)$.
b) If E and F are stable of the same slope, then f is an isomorphism.
c) Every stable vector bundle E is simple i.e. $\operatorname{End} E = k$.

Exercise 3. In this exercise, we will classify all (semi)stable vector bundles over $\mathbb{P}^1$ by proving Grothendieck's Theorem on vector bundles over $\mathbb{P}^1$.

a) Prove that there is an isomorphism $\mathbb{Z} \cong \operatorname{Pic}(\mathbb{P}^1)$ given by $d \mapsto \mathcal{O}_{\mathbb{P}^1}(d)$.
b) Prove by induction on the rank n of a locally free sheaf $\mathcal{E}$ over $\mathbb{P}^1$ that there exists a sequence of integers $d_1 \geq d_2 \geq \cdots \geq d_n$ such that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$.
c) Deduce that the only stable vector bundles on $\mathbb{P}^1$ are line bundles and classify all semistable vector bundles of a given rank n and degree d.