Algebra III (Algebraic Geometry II) WS15-16
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Algebraic Geometry II

Exercise Sheet 2

Hand-in date: 10am, Monday 26th October.

Exercise 1. Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Prove that (M, η) is a fine moduli space if and only if

1. there exists a family $\mathcal{U}$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$,
2. for families $\mathcal{F}$ and $\mathcal{G}$ over a scheme S, we have $\mathcal{F} \sim_S \mathcal{G} \iff \eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Exercise 2. Let $\mathcal{M}_{0,n}$ denote the moduli functor classifying n ordered distinct points on $\mathbb{P}^1$ up to the automorphisms of $\mathbb{P}^1$.

i) Show that $M_{0,4} := \mathbb{P}^1 - \{0, 1, \infty\}$ is a fine moduli space for the moduli problem $\mathcal{M}_{0,4}$ using the following fact.
Fact. If $\pi : \mathcal{X} \to S$ is a proper flat morphism with fibres isomorphic to $\mathbb{P}^1$ and $\sigma_1, \sigma_2, \sigma_3$ are distinct sections of π, then there is an isomorphism $\mathcal{X} \cong S \times \mathbb{P}^1$ over S which sends $(\sigma_1, \sigma_2, \sigma_3)$ to the constant sections $(0, 1, \infty)$.

ii) Let $M_{0,5} := M_{0,4} \times M_{0,4} - \Delta(M_{0,4})$, where $\Delta : M_{0,4} \to M_{0,4} \times M_{0,4}$ denotes the diagonal. Prove that $M_{0,5}$ is a fine moduli space for $\mathcal{M}_{0,5}$ and determine the universal family $\mathcal{U}_{0,5}$ over $M_{0,5}$.

iii) For $n > 5$, determine the fine moduli space $M_{0,n}$ for $\mathcal{M}_{0,n}$.
[Hint: $M_{0,n}$ can be realised as a subvariety of $n - 3$ copies of $M_{0,4}$.]

iv) (Harder - optional) Prove the above fact stated in i).
[Hint: Use σ_1 to show that $\mathcal{X} \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E} := \pi_*(\mathcal{O}(\sigma_1(S)))$ is a rank 2 vector bundle on S. Then use the additional distinct sections σ_2, σ_3 to show $\mathcal{E} \cong \mathcal{O}_S^{\oplus 2}$.]

Please turn over for Exercise 3.

Exercise 3. Let $\mathrm{Gr}_{d,n}(k)$ be the Grassmannian variety of d-dimensional subspaces of k^n, as constructed in Sheet 1. Consider the map

$$\varphi : \mathrm{Gr}_{d,n}(k) \to \mathbb{P}(\wedge^d(k^n)) \quad \text{given by} \quad W \mapsto [w_1 \wedge \cdots \wedge w_d]$$

where $w_1, \ldots, w_d$ is a basis of W.

i) Check that φ is well-defined, injective and a morphism of varieties.

ii) We say that $w \in \wedge^d(k^n)$ is totally decomposable if $w = w_1 \wedge \cdots \wedge w_d$ where $w_1, \ldots, w_d$ are linearly independent. Show that $w \in \wedge^r(k^n)$ is totally decomposable if and only if the map $\phi_w : k^n \to \wedge^{d+1}(k^n)$, given by $v \mapsto w \wedge v$, has rank $n - d$.

iii) Show that $[w] \in \mathbb{P}(\wedge^d(k^n))$ is in the image of φ if and only if all $(n-d+1) \times (n-d+1)$ minors of the map ϕ_w vanish. Conclude that $\mathrm{Gr}_{d,n}(k)$ is a projective variety.

iv) Explicitly determine the single relation defining $\varphi : \mathrm{Gr}_{2,4} \hookrightarrow \mathbb{P}^5$.

The morphism $\varphi : \mathrm{Gr}_{d,n} \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$ is called the *Plücker embedding* and the relations given by the vanishing of these minors are called the *Plücker relations.*