Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Eva Martínez

Algebraic Geometry II

Exercise Sheet 4

Hand-in date: 10am, Monday 9th November.

Exercise 1. Let $\varphi\colon X \to Y$ be a categorical quotient of the G-action on X. Show that if X is connected (resp. irreducible, resp. reduced) then Y is also connected (resp. irreducible, resp. reduced).

Exercise 2. Let G be an affine group variety acting on a scheme X over k and let $H \subset G$ be a subgroup. Prove the following statements.

i) If Y and Z are subschemes of X such that Z is closed, then $\{g \in G : gY \subset Z\}$ is a closed subset of G.

ii) If X is a variety, the fixed point locus $X^H = \{x \in X : H \cdot x = x\}$ is closed in X.

[Hint: express these subsets as intersections of preimages of closed subschemes under morphisms associated to the action.]

Exercise 3. Consider the action of GL_n on the set $M_n := \mathrm{Mat}_{\mathrm{n\times n}}$ of $n \times n$ matrices given by conjugation.

1. Describe the orbits of the action, which orbits are closed and when two orbits closures meet.
   [Hint: consider the Jordan–Normal form of the matrices.]
2. Show that there is a GL_n-invariant morphism
   $$p : M_n \to \mathbb{A}^n$$
   given by taking the coefficients $c_i(M)$ of the characteristic polynomial of M.
3. Prove that the ring of GL_n-invariant functions $\mathcal{O}(M_n)^{\mathrm{GL}_n}$ is isomorphic to the polynomial ring in the functions c_i given by the coefficients of the characteristic polynomial.
   [Hint: From the description of the orbits of this action, we can conclude that a GL_n-invariant function on M_n is determined by its values on diagonal matrices.]