Algebra III (Algebraic Geometry II) WS15-16
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Algebraic Geometry II

Exercise Sheet 6

Hand-in date: 10am, Monday 23rd November.

Exercise 1. For the following finite subgroups of $\Gamma \subset \mathrm{SL}_2$, calculate the ring of invariants $\mathcal{O}(\mathbb{A}^2)^\Gamma$ where Γ acts via SL_2 acting by left multiplication on $\mathbb{A}^2$. Assume that $n \in \mathbb{N}$ is fixed and that the characteristic of the field k does not divide $2n$.

$$a) \qquad \Gamma = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} : \eta \in k, \eta^n = 1 \right\}$$

$$b) \qquad \Gamma = \left\{ \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix} : \eta \in k, \eta^{2n} = 1 \right\}$$

In particular, note that the affine GIT quotient of a smooth scheme is not always smooth.

Exercise 2. Let G be an affine algebraic group acting on an irreducible affine variety X over a field of characteristic zero. Suppose that there is a G-invariant morphism $\varphi : X \to Y$ such that Y is a normal irreducible affine variety. Assume that

i) the codimension of $Y - \varphi(X)$ in Y is greater than or equal to 2,

ii) there is a non-empty open subset $U \subset Y$ such that, for all $y \in U$, the fibre $\varphi^{-1}(y)$ contains a unique closed orbit.

Then prove that $\mathcal{O}(Y) = \mathcal{O}(X)^G$ and, in particular, deduce that the ring of G-invariant functions on X is finitely generated.

[Hint: first observe that φ is dominant, and so the induced morphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. Using ii), show that any G-invariant function $f \in \mathcal{O}(X)$ can be lifted to a rational function $\tilde{f}$ on Y such that $\varphi^* \tilde{f} = f$. Finally, use the following result: if $\psi : M \to N$ is a dominant morphism of irreducible affine varieties and h a rational function on N such that $\psi^* h$ is a regular function on M, then h is defined on any normal point of the open set $\psi(M)^0 := \psi(M) - (\overline{\psi(M)} - \psi(M))$ (see Shafarevich *Foundations of Algebraic Geometry*).]

Please turn over for Exercise 3.

Exercise 3. For the following linear representations of the additive group $\mathbb{G}_a$, compute the ring of invariants $\mathcal{O}(V)^{\mathbb{G}_a}$ and note that the ring of invariants is finitely generated.

a) Let $\mathbb{G}_a$ act on $V = \mathbb{A}^3$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}(V)$ given by

$$\rho(c) = \begin{pmatrix} 1 & 2c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Let $\mathbb{G}_a$ act on $V = \mathbb{A}^4$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}(V)$ given by

$$\rho(c) = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

c) (Harder/Optional) In both cases, does the morphism of affine schemes obtained by taking the ring of invariants give a good quotient of the action?