Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 9

Hand-in date: 10am, Monday 14th December.

Exercise 1. Let $G = (\mathbb{G}_m)^2$ act on $\mathbb{P}^3$ linearly by the representation

$$(s,t) \mapsto \mathrm{diag}(st, s^{-1}t, s^{-1}t^{-1}, st^{-1}).$$

For this action, calculate the G-invariant subring in the homogeneous coordinate ring $R(\mathbb{P}^n) = k[x_0, x_1, x_2, x_3]$ and determine the semistable and stable sets for the action. Then explicitly write down the GIT quotient morphism.

Exercise 2. Let G act on a scheme X and consider the forgetful map

$$\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

from the group of isomorphism classes of G-linearisations on X to the group of isomorphism classes of line bundles on X.

a) Show that the group of isomorphism classes of G-linearisations on the trivial line bundle over X is isomorphic to the group

$$Z^1(G, \mathcal{O}(X)^*) := \{\Psi \in \mathcal{O}(G \times X)^* : \Psi(gg', x) = \Psi(g, g' \cdot x)\Psi(g', x)\}.$$

b) Determine when an automorphism of the trivial line bundle on X, which is given by $f \in \mathcal{O}(X)^*$, commutes with the actions defined by Ψ and Ψ'.

c) Let $B^1(G, \mathcal{O}(X)^*)$ be the subgroup of $Z^1(G, \mathcal{O}(X)^*)$ of functions Ψ of the form $\Psi(g, x) := \phi(g \cdot x)/\phi(x)$ for $\phi \in \mathcal{O}(X)^*$. Prove that

$$\mathrm{Ker}(\alpha) \cong Z^1(G, \mathcal{O}(X)^*)/B^1(G, \mathcal{O}(X)^*).$$

Exercise 3. For $n \geq 1$, consider the natural action of PGL_{n+1} on $\mathbb{P}^n$. In this exercise, we will show that $\mathcal{O}_{\mathbb{P}^n}(1)$ does not admit a linearisation of this action. Let $\Delta \subset \mathbb{P}^{n^2+2n} = \mathbb{P}(\mathrm{Mat}_{(n+1)\times(n+1)})$ be the determinant hypersurface, so that $\mathrm{PGL}_{n+1} = \mathbb{P}^{n^2+2n} - \Delta$.

a) Show that there is a rational map $\mathbb{P}^{n^2+2n} \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which restricts to the action morphism $\sigma : \mathrm{PGL}_{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ and that the indeterminacy locus of the rational map is a closed set of codimension greater than or equal to 2 in $\mathbb{P}^{n^2+2n} \times \mathbb{P}^n$.

b) Let π_i be the projection map from $\mathbb{P}^{n^2+2n} \times \mathbb{P}^n$ onto the ith factor. Show that $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is the restriction of $\pi_1^*(\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$ to $\mathrm{PGL}_{n+1} \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{n^2+2n} \times \mathbb{P}^n$.

c) Show that if $\mathcal{O}_{\mathbb{P}^n}(1)$ admits a PGL_{n+1} linearisation, then the restriction of $\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)$ to PGL_{n+1} must be trivial. Finally deduce that this is not possible, by using the fact that, for an irreducible hypersurface $H \subset \mathbb{P}^r$, the restriction map $\mathrm{Pic}(\mathbb{P}^r) \to \mathrm{Pic}(\mathbb{P}^r - H)$ induces an isomorphism $\mathrm{Pic}(\mathbb{P}^r - H) \cong \mathbb{Z}/(\deg H)\mathbb{Z}$.