Stratifications and (in)stability I

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§1 Summary

Moduli problems

eg ① representations of a quiver,
② coh. sheaves on a proj. variety.

(semi)stability and Harder-Narasimhan filtrations $\to$ HN Stratification (Shatz) $\rightsquigarrow$ cohomology of moduli spaces
$\rightsquigarrow$ wall-crossing formulas

Goal: compare these stratifications

Geometric Invariant Theory (GIT)

$G \curvearrowright Y$ w.r.t. $\mathcal{L}$
reductive group; proj/affine scheme; linearisation (over $\mathbb{C}$)

GIT quotient $Y^{ss}(\mathcal{L}) \to Y/\!/_{\mathcal{L}} G$

(semi)stability
Hilbert-Mumford criterion in terms of 1-param subgps
and adapted 1-param subgps (Kempf) $\to$ Instability Stratifications in GIT (Hesselink) $\rightsquigarrow$ cohomology of GIT quot (Kirwan)
$\rightsquigarrow$ Variation of GIT (Dolgachev-Hu) Thaddeus

Aim: compare the HN and GIT instability stratifications
for 1) quiver reps
2) coh sheaves on a proj. variety.

§2 Geometric Invariant Theory

Motivation: Construct moduli spaces as quotients of group actions in algebraic geometry.

Def Let G be a linear algebraic group acting on a scheme Y. Then a categorical quotient of $G \curvearrowright Y$ is a universal G-invariant morphism $\varphi: X \to Y$ i.e. every other G-inv morphism $Y \to Z$ factors uniquely via φ.

Affine GIT quotient

$G \curvearrowright Y = \operatorname{Spec} A \rightsquigarrow G \curvearrowright A = \mathcal{O}(Y)$ by $g \cdot f(y) := f(g^{-1} \cdot y)$.

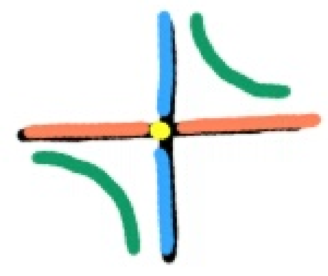
Thm (Hilbert, Nagata)
If G is reductive, then $\mathcal{O}(Y)^G$ is finitely generated.

Def (Over $\mathbb{C}$) G is reductive $\iff$ $G = K_{\mathbb{C}}$ for $K \leq G$ maximal compact.

e.g. $G = GL_n \geq K = U(n)$ is reductive
$G = \mathbb{G}_a = (\mathbb{C}, +) \geq K = \{0\}$ is not reductive.

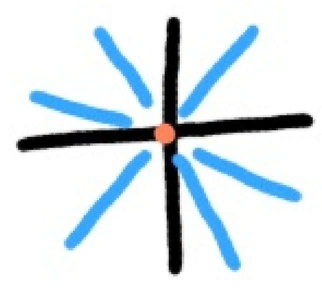
Thm (Hilbert, Mumford)
For G reductive, Y affine, $Y \longrightarrow Y/\!/G := \operatorname{Spec} \mathcal{O}(Y)^G$ is a categorical quotient "affine GIT quotient"

Example Let $G = \mathbb{G}_m \curvearrowright Y = \mathbb{A}^2$ by

1) $t \cdot (x,y) = (tx, t^{-1}y)$  $Y \xrightarrow{\pi} Y/\!/G = \operatorname{Spec} \mathbb{C}[xy] \cong \mathbb{A}^1$

$\pi^{-1}(0) =$ union of 3 orbits

GIT quotient is not necessarily an orbit space

2) $t \cdot (x,y) = (tx, ty)$  $Y \to Y/\!/G = \operatorname{Spec} \mathbb{C} = *$

$\rightsquigarrow$ would like to remove $0 \in \mathbb{A}^2$, to get $\mathbb{P}^1 = (\mathbb{A}^2 - \{0\})/\mathbb{G}_m$.

Mumford's linearised GIT

Suppose $\underset{\text{red}}{G} \curvearrowright Y$ w.r.t. a linearisation $\begin{matrix} \mathcal{L} \curvearrowleft G \\ \downarrow \\ Y \end{matrix}$

and either

① $\underset{\text{affine}}{Y \subseteq \mathbb{A}^n}$, G acts linearly via $G \to GL_n$, $\mathcal{L} = \mathcal{O}_Y^{\chi}$ where $\chi: G \to \mathbb{G}_m$ character and $G \curvearrowright \mathcal{L} \cong Y \times \mathbb{A}^1$ by $g \cdot (y,z) = (g \cdot y, \chi(g) z)$

② Y is projective & $\mathcal{L}$ is v. ample ($\Leftrightarrow$ $Y \overset{|\mathcal{L}|}{\hookrightarrow} \mathbb{P}^n$ and G acts via $G \to GL_{n+1}$)

These assumptions are needed in order to have a Hilbert-Mumford criterion for semistability. In fact, it suffices to assume Y is proj. over affine & $\mathcal{L}$ is rel. ample.

$$G \curvearrowright \begin{matrix} \mathcal{L} \\ \downarrow \\ Y \end{matrix} \rightsquigarrow G \curvearrowright \underbrace{R(Y,\mathcal{L}) = \bigoplus_{r \geq 0} H^0(Y, \mathcal{L}^{\otimes r})}_{\text{graded ring}} \supseteq \underbrace{R(Y,\mathcal{L})^G}_{\text{invariant ring}}$$

Def $Y \dashrightarrow Y/\!/_{\mathcal{L}} G = \operatorname{Proj} R(Y,\mathcal{L})^G \rightarrow$ proj over $Y/\!/G$ in case ①; projective $/\mathbb{C}$ in case ②

open $\cup$ $Y^{ss}(\mathcal{L})$ $\nearrow$ GIT quotient w.r.t. $\mathcal{L}$ (which is a categorical quotient of $G \curvearrowright Y^{ss}(\mathcal{L})$)

The semistable set w.r.t $\mathcal{L}$ is the domain of definition of this rational map.

The Hilbert-Mumford Criterion for (semi)stability

$\underset{\text{closed point}}{y \in Y}$ is semistable $\Leftrightarrow$ $\mu^{\mathcal{L}}(y, \lambda) \geq 0$ $\forall$ 1-parameter subgps $\lambda: \mathbb{G}_m \to G$ s.t. $\lim_{t \to 0} \lambda(t) \cdot y$ exists

"Hilbert-Mumford weight"

where $\mu^{\mathcal{L}}(y,\lambda) =$ weight of $\lambda(\mathbb{G}_m) \curvearrowright \mathcal{L}_{y_0}$ where $y_0 = \lim_{t \to 0} \lambda(t) \cdot y$ is fixed by λ

$= (\chi, \lambda)$ in case ①, where $(\ ,\ ) =$ natural pairing between characters & cocharacters

Rmk: $\mu^{\mathcal{L}}(y, \lambda^n) = n \mu^{\mathcal{L}}(y, \lambda)$ for $n > 0$. Hence, it suffices to check for primitive 1-parameter subgps.

Instability stratifications

Idea: Stratify $Y^{us}(\mathcal{L}) = Y - Y^{ss}(\mathcal{L})$ using a normalised Hilbert-Mumford weight

$\rightsquigarrow$ associate to each unstable pt, a 1-PS* that is "most responsible" for its instability.

(* more precisely, a conj class of a rat$^{\text{l}}$ 1-PS.)

Def A norm on the set of conjugacy classes of 1-PSs of G is a norm $\|-\|$ on the space of 1-PSs $X_*(T)_{\mathbb{R}}$ of a max^e torus $T \leq G$ which is invariant for the Weyl group of T.
If $\lambda : \mathbb{G}_m \to G$ is a 1-PS, then $\exists\, g \in G$ s.t $g\lambda g^{-1}$ is a 1-PS of T and we let $\|\lambda\| := \|g\lambda g^{-1}\|$.

Eg a) $G = GL_n \geq T =$ diagonal matrices, $W = S_n$
The Euclidean norm $\|-\|_E$ on $\mathbb{R}^n \cong X_*(T)_{\mathbb{R}}$ is W-invariant.

b) $G = GL_{n_1} \times \cdots \times GL_{n_r}$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_{>0}$

$$\|\underset{(\lambda_1,\ldots,\lambda_r)}{\underset{\parallel}{\lambda}}\|^2_\alpha := \sum_{i=1}^r \alpha_i \|\lambda_i\|^2_E \quad \text{"}\alpha\text{-scaled Euclidean norm."}$$

Theorem (Hesselink, Kempf, Kirwan, Ness)
Let $G \curvearrowright Y$ w.r.t. $\mathcal{L}$ as in case ① or ② above and fix a norm $\|-\|$ on the conj classes of 1-PSs of G. Then there is a finite stratification

$Y = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$ into G-invariant locally closed subschemes $Y_\beta \subseteq Y$

such that
- $\overline{S_\beta} \subseteq \bigsqcup_{\beta' \geq \beta} S_{\beta'}$ and the lowest stratum is $S_0 = Y^{ss}(\mathcal{L})$.
- $\mathcal{B}$ can be computed from the weights of the action of a maximal torus $T \leq G$.
- The strata S_β can be constructed from simpler limit sets, which are GIT semistable sets for smaller reductive group actions.

Construction of unstable strata:
Let $M^{\mathcal{L}}(y) := \inf\left\{ \dfrac{\mu^{\mathcal{L}}(y,\lambda)}{\|\lambda\|} : \lambda \text{ 1-PS of } G \text{ and } \lim_{t\to 0} \lambda(t)\cdot y \text{ exists} \right\}$

HM criterion: y is semistable $\iff M^{\mathcal{L}}(y) \geq 0$.

Def For $y \in Y^{us}(\mathcal{L})$, a primitive 1-PS achieving this inf is said to be adapted to y.

For $\beta = (\underset{\substack{\uparrow \\ \text{conj class of} \\ \text{1-PS of } G}}{[\lambda]}, \underset{\substack{\uparrow \\ \text{negative} \\ \text{number}}}{d})$, let $S_\beta = \left\{ y \in Y : M^{\mathcal{L}}(y) = d \;\&\; \exists\, \lambda \in [\lambda] \text{ adapted to } y \right\}$

(can also give strata a natural scheme structure)

Rmk • Strata are G-inv: if λ is adapted to y, $g\lambda g^{-1}$ is adapted to $g\cdot y$.
• In case ①, $d = \dfrac{\langle \chi, \lambda \rangle}{\|\lambda\|}$ is redundant.

Applications

- If Y is smooth, then strata are also smooth: the inclusion of a stratum in its closure gives a Gysin long exact seq in G-equiv. rat^e cohomology, which split into s.e.s. If $Y/\!/_{\mathcal{L}} G$ is an orbit space, $Y/\!/_{\mathcal{L}} G = Y^{ss}(\mathcal{L})/G$ and $H^*_G(Y^{ss}(\mathcal{L})) \cong H^*(Y/\!/_{\mathcal{L}} G)$, and so the above gives a recursive formula for the Betti numbers of $Y/\!/_{\mathcal{L}} G$ (Kirwan).

- Variation of GIT, which studies the birational transformations between GIT quotients as one varies the linearisation on Y, can be described using the above GIT instability stratifications: when crossing a wall, certain unstable strata become stable (Dolgachev-Hu, Thaddeus).

§3 Quiver representations

Let $Q = (V, A, h, t)$ be a quiver with vertex set V, arrow set A and head and tail maps $h, t : A \longrightarrow V$.

Def: A rep of Q is $W = (W_v, v \in V;\ \phi_a : W_{t(a)} \to W_{h(a)}, a \in A)$ where W_v are vector spaces and ϕ_a are linear maps.

Moduli problem: Classify representations of Q of dimension vector $d \in \mathbb{N}^V$ up to isomorphism.

Lemma: On $\mathrm{Rep}_d Q := \bigoplus_{a \in A} \mathrm{Hom}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}})$, there is an action of

$G_d(Q) = \prod_{v \in V} GL_{d_v}$ given by $(g_v)_{v \in V} \cdot (\phi_a)_{a \in A} = (g_{h(a)} \circ \phi_a \circ g_{t(a)}^{-1})_{a \in A}$

such that the orbits are in 1-1 correspondence with the isomorphism classes of reps of Q of dim vector d.

"Proof": Any rep of Q of dim d is isomorphic to a representation parametrised by a point in $\mathrm{Rep}_d Q$, by choosing isomorphisms at each vertex to the standard vector space $\mathbb{C}^{d_v}$. Furthermore, the $G_d(Q)$-action accounts for these choices of isomorphism. ☒

Rmk:
- Any coarse moduli space is a categorical quotient of this action (in fact, a cat quotient is a coarse m. space if it's an orbit space).
- The affine GIT quotient $\mathrm{Rep}_d Q /\!/ G_d(Q)$ does not always give a satisfactory quotient: $\mathcal{O}(\mathrm{Rep}_d Q)^{G_d(Q)}$ is generated by traces of oriented cycles in Q (Le Bruyn - Procesi).

King's construction of moduli via linearised GIT

characters $\chi : G_d(Q) \to \mathbb{G}_m \quad \longleftrightarrow \quad \theta \in \mathbb{Z}^V$

$$\chi_\theta(g_v) = \prod_{v \in V} \det(g_v)^{\theta_v} \quad \longleftarrow\!\shortmid \quad \theta = (\theta_v)_{v \in V}$$

Assume $\sum_{v \in V} \theta_v d_v = 0$ (as $\exists\ \mathbb{G}_m \hookrightarrow G_d(Q)$ which acts trivially, this ensures χ descends to the quotient group.)

Def A rep W of Q of dim d is $\underline{\theta\text{-semistable}}$ if $\forall\, W' \subseteq W$, $\theta(W') = \sum_{v \in V} \theta_v \dim W'_v \geq 0$

Ex $Q = \bullet \to \bullet$, $d = (n, m)$; $\theta = (\theta m, -\theta n)$, $n, m > 0$.

interesting semistability — 0 — everything is unstable $\to \theta$

everything is semistable.

Thm (King) $M_d^{\theta\text{-ss}}(Q) = \mathrm{Rep}_d Q /\!\!/_{\chi_\theta} G_d(Q)$ is a coarse m. space for S-equiv. classes of θ-semistable reps of Q of dim d.

(GIT semistability wrt $\chi_\theta \iff \theta$-semistability)

Harder-Narasimhan Stratifications

Fix $\alpha \in \mathbb{Z}^V_{>0}$ additional stability parameter, used to give slope type stability condition for all reps of Q

Let $\mu_{\theta,\alpha}(W) := \dfrac{\theta(W)}{\alpha(W)} = \dfrac{\sum_{v \in V} \theta_v \dim W_v}{\sum_{v \in V} \alpha_v \dim W_v}$ $(= 0$ if $\dim W = d)$. extends θ-ss for d-dim^e reps.

W: non-zero rep of Q

Def A rep $W \neq 0$ of Q is $\underline{(\theta,\alpha)\text{-semistable}}$ if $\forall\, 0 \neq W' \subseteq W$, $\mu_{\theta,\alpha}(W') \geq \mu_{\theta,\alpha}(W)$.

Prop Every rep. $W \neq 0$ of Q has a ! $\underline{\text{Harder-Narasimhan}}$ filtration w.r.t. (θ, α): $\quad 0 = W^{(0)} \subsetneq W^{(1)} \subsetneq \cdots \subsetneq W^{(r)} = W$ s.t. $W^i = W^{(i)}/W^{(i-1)}$ are (θ,α)-semistable and $\mu_{\theta,\alpha}(W^1) < \cdots < \mu_{\theta,\alpha}(W^r)$.

Proof: Let $\mu_{min}(W) = \inf_{0 \neq W' \subseteq W} \mu_{\theta,\alpha}(W')$ and take $W^{(1)} \subseteq W$ of max^e dimension amongst the subreps of W with slope $\mu_{min}(W)$.

Claim $W^{(1)}$ is unique (called the max^e destab. subrep of W).

Pf: If $W' \subseteq W$ and $\mu_{\theta,\alpha}(W') = \mu_{min}(W)$, we'll show $W' \subseteq W^{(1)}$.

If $W' \not\subseteq W^{(1)}$, then $W^{(1)} \subsetneq W' + W^{(1)} \subseteq W$. We have a s.e.s.

$$0 \to W' \cap W^{(1)} \longrightarrow W' \oplus W^{(1)} \longrightarrow W' + W^{(1)} \longrightarrow 0$$

$$\mu_{\theta,\alpha}(W' \cap W^{(1)}) \geq \mu(W' \oplus W^{(1)}) = \mu_{min}(W) \leq \mu_{\theta,\alpha}(W' + W^{(1)}) \implies \text{must all be equalities}$$

$\Rightarrow$ the existence of $W' + W^{(1)}$ with slope $\mu_{min}(W)$, contradicts the maximality of the choice of $W^{(1)}$.

Moreover $W^{(1)}$ is (θ,α)-semistable. We then construct the HN filtration by iterating: let $W^{(2)}$ be the preimage under $W \twoheadrightarrow W/W^{(1)}$ of the maximal destabilising subrepresentation of $W/W^{(1)}$ and so on. $\boxed{3}$

<u>Notation</u> $\gamma(W) = (\dim W^1, \dots, \dim W^r)$ is the <u>HN type</u> of W wrt (θ,α)

<u>Thm</u> (Reineke, Shatz)

There is a stratification of $\mathrm{Rep}_d Q$ by HN types w.r.t (θ,α)

$$\mathrm{Rep}_d Q = \coprod_\gamma HN_\gamma \quad \text{s.t. } HN_\gamma \subseteq \mathrm{Rep}_d Q \text{ are locally closed.}$$

<u>Exercise</u> For Q acyclic, $K_0(\mathrm{Rep}\,Q) \cong \mathbb{Z}^V$ and $Z_{\theta,\alpha} : K_0(\mathrm{Rep}\,Q) \longrightarrow \mathbb{C}$

(Grothendieck gp of the category $\mathrm{Rep}\,Q$)

$$[W] \mapsto \sum_{v\in V} (\theta_v + i\alpha_v)\dim W_v$$

is a Bridgeland stability condition with heart $\mathcal{A} = \mathrm{Rep}\,Q$, such that $Z_{\theta,\alpha}$-semistability $\iff$ (θ,α)-semistability.

<u>Hint</u>: $\phi' = \phi_{Z_{\theta,\alpha}}(W') \leq \phi = \phi_{Z_{\theta,\alpha}}(W) \iff \tan(\pi\phi' + \frac{\pi}{2}) \leq \tan(\pi\phi + \frac{\pi}{2}) \iff \mu_{\theta,\alpha}(W') \geq \mu_{\theta,\alpha}(W)$

+ use prop above.

tan is increasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$

$\tan(x+\frac{\pi}{2}) = -\cot(x)$

<u>Theorem</u> (H.)

Fix θ and α as above. Then the HN stratification on $\mathrm{Rep}_d Q$ wrt (θ,α) coincides with the GIT instability stratification for $G_d(Q) \curvearrowright \mathrm{Rep}_d Q$ w.r.t the linearisation given by χ_θ and α-scaled Euclidean norm $\|-\|_\alpha$.

<u>Sketch of proof</u>:

- Bijection between index sets of stratifications:

$$\text{HN type } \gamma = (d^1, \dots, d^r) \longleftrightarrow \text{GIT index } \beta_\gamma = ([\lambda_\gamma], d_\gamma)$$

redundant in case ①

$$\lambda_{\gamma,v}(t) = \begin{pmatrix} t^{a_1} I_{d_v^1} & & \\ & \ddots & \\ & & t^{a_r} I_{d_v^r} \end{pmatrix} \quad \text{where } a_i = -\mu_{\theta,\alpha}(d^i)$$

(techically $a_i \in \mathbb{Q}$, so this only gives a rat'l 1-PS $\rightsquigarrow$ multiply by positive integer to get a primitive (integral) 1-PS of $G_d(Q)$)

- King proves the lowest strata coincide.
- Show higher strata coincide, by proving their limit sets agree (inductively use King's result).