Stratifications and (in)stability II

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§4 Coherent sheaves

Let $(X, \mathcal{O}_X(1))$ be a projective scheme with an ample line bundle.

Moduli problem: Classify coherent sheaves of $\mathcal{O}_X$-modules on X with fixed Hilbert polynomial P up to isomorphism.

Recall: $P(\mathcal{E}, n) := \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{E}(n))$

Def A sheaf $\mathcal{E}$ is <u>n-regular</u> if $H^i(X, \mathcal{E}(n-i)) = 0 \quad \forall i \geq 1$.

Facts a) Any coh sheaf over X is n-regular for $n \gg 0$ (by Serre's vanishing thm)

b) If $\mathcal{E}$ is n-regular, then

i) $\mathcal{E}$ is m-regular for $m \geq n$

ii) $H^i(\mathcal{E}(n)) = 0 \quad \forall i > 0 \implies \dim H^0(\mathcal{E}(n)) = P(\mathcal{E}, n)$.

iii) eval : $H^0(\mathcal{E}(n)) \otimes \mathcal{O}_X(-n) \to \mathcal{E}$ is surjective,

These facts are proved by first reducing to the case where $X = \mathbb{P}^r$ and then proceeding by induction on r, by restricting to a hyperplane.

Lemma: Every n-regular coherent sheaf with Hilbert poly P is parametrised by a point in an open subscheme

$$Q^{n\text{-reg}} \underset{\text{open}}{\subseteq} \mathrm{Quot}_X(\mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n), P) =: \mathrm{Quot}_n.$$

$$\| \; \{ q : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E} \; : \; \mathcal{E} \text{ is } n\text{-regular and } H^0(q(n)) \text{ is an iso} \}$$

Furthermore, $GL_{P(n)} \curvearrowright \mathrm{Quot}_n$ and we have a 1-1 correspondence:

$$\{ GL_{P(n)}\text{-orbits in } Q^{n\text{-reg}} \} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} n\text{-reg coh sheaves over } X \\ \text{with Hilbert polynomial } P \end{matrix} \right\} \Big/ \underset{\text{isos}}{\cong}.$$

Proof If $\mathcal{E}$ is n-regular, then eval : $H^0(\mathcal{E}(n)) \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ and $\dim H^0(\mathcal{E}(n)) = P(n)$. By choosing an iso $\psi : H^0(\mathcal{E}(n)) \xrightarrow{\simeq} \mathbb{C}^{P(n)}$, we obtain a quotient $q_{\mathcal{E},\psi} : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \cong H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) \overset{ev}{\twoheadrightarrow} \mathcal{E}$ in $Q^{n\text{-reg}}$.

Furthermore, the $GL_{P(n)}$-action accounts for the above choice of iso ψ.

More precisely, if $g \cdot q = q'$, then we have a commutative square

$$\begin{array}{ccccc} \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) & \xrightarrow{g^{-1}} & \mathbb{C}^{P(n)} \otimes \mathcal{O}(-n) & \xrightarrow{q} & \mathcal{E} \\ \| & & & & \wr\| \\ \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) & & \longrightarrow & & \mathcal{E}' \end{array}$$

i.e. $\mathcal{E} \cong \mathcal{E}'$.

Conversely, if $q_{\mathcal{E},\psi}$ and $q_{\mathcal{E}',\psi'} \in Q^{n\text{-reg}}$ and $\mathcal{E} \overset{\varphi}{\cong} \mathcal{E}'$, then $g \cdot q_{\mathcal{E},\psi} = q_{\mathcal{E}',\psi'}$ where $g \in GL_{P(n)}$ is the following iso $g : \mathbb{C}^{P(n)} \underset{\psi^{-1}}{\xrightarrow{\simeq}} H^0(\mathcal{E}(n)) \underset{H^0(\varphi(n))}{\xrightarrow{\simeq}} H^0(\mathcal{E}'(n)) \underset{\psi'}{\xrightarrow{\simeq}} \mathbb{C}^{P(n)}$. ∎

Rmk: To construct a coarse moduli space, we want to take a categorical quotient of this action using GIT.

Semistability for sheaves

There are two notions of semistability:

1) Mumford's slope semistability for torsion free sheaves: $\mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$ slope,

2) Gieseker-Maruyama's reduced Hilbert polynomial semistability for pure sheaves
↳ $p^{red}(\mathcal{E}) := \frac{P(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}}$.

Both notions coincide if X = curve.

The 2nd notion is used in higher dims to construct moduli spaces. We give a reformulation of 2) due to Rudakov.

Def A coh sheaf $\mathcal{E}$ over X is semistable if $\forall\, 0 \neq \mathcal{E}' \subseteq \mathcal{E}$, we have

$$\left.\frac{P(\mathcal{E}',n)}{P(\mathcal{E}',m)} \leq \frac{P(\mathcal{E},n)}{P(\mathcal{E},m)} \quad \forall\, m >> n >> 0 \right\} \leftarrow \text{Notation: } P(\mathcal{E}') \preccurlyeq P(\mathcal{E}).$$

Rmk
- We can rephrase $\preccurlyeq$ in terms of the coefficients of the Hilbert poly.
- Lower degree polynomials are ranked higher w.r.t. $\preccurlyeq$ $\Rightarrow$ purity of $\mathcal{E}$ is a nec. condition
- For polynomials of the same degree, $\preccurlyeq$ is equivalent to an inequality of reduced Hilbert polynomials.
- The Hilbert polynomial and thus semistability depend on $\mathcal{O}_X(1)$.

Theorem (Le Potier - Simpson) "Boundedness result"

There exists N s.t $\forall\, n \geq N$, every semistable sheaf on X with fixed Hilbert polynomial P is n-regular.

$\Rightarrow$ semistable sheaves are parametrised by $Q_n^{ss} \underset{open}{\subseteq} Q^{n\text{-}reg}$

$Q_n^{ss} = \{q : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E} : \mathcal{E} \text{ is semistable}\}$ in $Q^{n\text{-}reg}$

Let R be the closure of Q_n^{ss} in Quot_n.

Goal: • Construct moduli space for semistable sheaves as quotient of $Gl_{P(n)} \curvearrowright R$

Linearisation of the action: for $m >> n$, use Grothendieck's embedding of Quot_n

$$\mathrm{Quot}_n \hookrightarrow Gr(\mathbb{C}^{P(n)} \otimes H^0(\mathcal{O}(m-n)), P(m)) \xhookrightarrow{\text{Plücker}} \mathbb{P}^r$$

$$[q : \mathbb{C}^{P(n)} \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}] \mapsto [H^0 q(m)) : \mathbb{C}^{P(n)} \otimes H^0(\mathcal{O}(m-n)) \twoheadrightarrow H^0(\mathcal{E}(m))]$$

(pick m so all quotient sheaves $\mathcal{E}$ are m-reg & so are their kernels)

Let $\mathcal{L}_{n,m}$ be the pullback of $\mathcal{O}_{\mathbb{P}^r}(1)$ to Quot_n.

Theorem (Seshadri, Gieseker, Maruyama, Simpson)

For $m >> n >> 0$, $M_P^{ss}(X) = R /\!\!/_{\mathcal{L}_{n,m}} SL_{P(n)}$ is a coarse moduli space for S-equiv. classes of semistable sheaves over X with Hilbert polynomial P.

Exercise: If X is a smooth proj curve and $\mathcal{E}, \mathcal{F}$ are vector bundles on X, show

i) $P(\mathcal{E}) \preccurlyeq P(\mathcal{F}) \iff \mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\mathrm{rk}\,\mathcal{E}} \leq \mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\mathrm{rk}\,\mathcal{F}}$

Hint: $P(\mathcal{E},n) = \mathrm{rk}\,\mathcal{E}.n + \deg \mathcal{E} + \mathrm{rk}\,\mathcal{E}(1-g)$ by Riemann-Roch.

ii) $Z : K_0(\mathrm{Coh}X) \longrightarrow \mathbb{C}$, $[\mathcal{E}] \mapsto -\deg \mathcal{E} + i\,\mathrm{rk}\,\mathcal{E}$ is a Bridgeland stability condition on $\mathrm{Coh}\,X$ s.t Z-semistability $\iff$ μ-semistability

Hint: use the next prop.

Harder-Narasimhan Stratifications

Proposition Every coherent sheaf $\mathcal{E}$ over X has a unique Harder-Narasimhan filtration w.r.t $\mathcal{O}_X(1)$: $0 = \mathcal{E}^{(0)} \subsetneq \mathcal{E}^{(1)} \subsetneq \cdots \subsetneq \mathcal{E}^{(r)} = \mathcal{E}$ s.t. $\mathcal{E}^i = \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ are semistable and $P(\mathcal{E}^1) \succ P(\mathcal{E}^2) \succ \cdots \succ P(\mathcal{E}^r)$.

The idea of the proof is the same as for quiver representations.

Notation: The HN type of $\mathcal{E}$ is $\tau(\mathcal{E}) = (P(\mathcal{E}^1), \dots, P(\mathcal{E}^r))$.

Rmk: In general, there are infinitely many HN types for sheaves with a fixed Hilbert polynomial.

eg $X = \mathbb{P}^1$ and $P(t) = 2(t+1) \leftarrow$ H. poly of rk 2 deg 0 sheaf on X.

$\mathcal{E}_n = \mathcal{O}(n) \oplus \mathcal{O}(-n)$ has HN filtr $\mathcal{O}(n) \subseteq \mathcal{E}_n$ and type $\tau_n = (t+n+1, t-n+1)$.

Theorem (Shatz, Nitsure)

Let $\mathcal{F}$ be a family over a (finite type) scheme S of coh sheaves on X. Then the HN type is upper semi-continuous
$$\tau : S \longrightarrow HNT.$$
$$s \longmapsto \tau(\mathcal{F}_s)$$

Furthermore, there is a finite stratification
$$S = \coprod_\tau S_\tau \quad \text{into locally closed subschemes } S_\tau$$
such that $s \in S_\tau \iff \mathcal{F}_s$ has HN type τ.

Eg. Let $\mathcal{U}_n$ be the universal quotient sheaf on the Quot scheme
$$\begin{array}{c}\mathcal{U}_n \\ \downarrow \\ Q^{n\text{-reg}} \times X\end{array}$$
(recall: Quot is a fine moduli space)

Then we have a stratification by HN types: $Q^{n\text{-reg}} = \coprod_\tau Q^{n\text{-reg}}_\tau$

§5 Comparison Results

We want to compare the HN stratification $Q^{n\text{-reg}} = \coprod_\tau Q^{n\text{-reg}}_\tau$ with the GIT instability stratification for $SL_{P(n)} \curvearrowright \mathrm{Quot}_n$ wrt $\mathcal{L}_{n,m}$ and the Euclidean norm on 1-PS of $SL_{P(n)}$, restricted to $Q^{n\text{-reg}} \underset{\substack{\text{open} \\ SL_{P(n)}\text{-inv}}}{\subset} \mathrm{Quot}_n$:
$$Q^{n\text{-reg}} = \coprod_{\beta \in \mathcal{B}_{n,m}} S_\beta^{n,m}$$

Recall: GIT unstable indices $\beta = (\underset{\substack{\text{conj class of} \\ \text{1-PS of } SL_{P(n)}}}{[\lambda]}, \underset{\text{neg number}}{d}) \iff$ conj class $[\lambda_\beta]$ of a rat^e 1-PS of $SL_{P(n)}$
$$(\lambda, d) \longleftarrow\!\!\shortmid \lambda_\beta$$
where $d = -\|\lambda_\beta\|$ & $\lambda = n\lambda_\beta$ for $n \in \mathbb{N}_+$ s.t λ is primitive

May expect stratifications to agree following quiver result + Atiyah-Bott's results on the Yang-Mills stratification

Def Let $\nu = (P_1, \dots, P_s)$ be a tuple of Hilbert Polynomials s.t $\sum_{i=1}^{s} P_i = P$.
For $m, n \in \mathbb{N}$, define $\beta_{n,m}(\nu)$ to be the conjugacy class of the rational 1-PS of $SL_{P(n)}$

$$t \longmapsto \begin{pmatrix} t^{r_1} I_{P_1(n)} & & \\ & \ddots & \\ & & t^{r_s} I_{P_s(n)} \end{pmatrix} \quad \text{where } r_i = \frac{P(m)}{P(n)} - \frac{P_i(m)}{P_i(n)}.$$

Rmk: If ν is a HN type, then $P_1 \succ \cdots \succ P_s$
ie for $m >> n >> 0$, $r_1 > \cdots > r_s$.

Thm (H.-Kirwan)
Let τ be a HN type for a sheaf on X with Hilbert poly P.
Then for $m >> n >> 0$ (depending on τ),

$$\underset{\text{HN Stratum}}{Q^{n\text{-reg}}_{\tau}} \underset{\substack{\text{closed}\\ \text{subscheme}}}{\subseteq} \underset{\text{GIT Stratum}}{S^{n,m}_{\beta_{n,m}(\tau)}}$$

Idea of proof: give an inclusion of the limit sets.

Question: Why don't these stratifications coincide?

1) For fixed n and m, the assignment HN types $\longrightarrow$ Conj classes of rat^{e} 1-PSs of $SL_{P(n)}$, $\tau \longmapsto \beta_{n,m}(\tau)$, is not injective (unless $\dim X = 1$).
However, if $\tau \neq \tau'$, then $\beta_{n,m}(\tau) \neq \beta_{n,m}(\tau')$ $\forall\, m >> n >> 0$.

2) For each τ, the choice of $m >> n >> 0$ in this theorem depends on τ, but there are infinitely many HN types
$\longrightarrow$ can't pick finite m and n so that we have an inclusion for all HN types.

Moral reason: HN stratification should be infinite, whereas GIT stratification is finite.

$\rightsquigarrow$ the Quot scheme does not parametrise all coh. sheaves with Hilbert poly P; it is a truncated parameter space.
Want to compare Quot_n for all n.

Solutions 1) Can refine GIT strata by gathering together some of the connected components of $S^{n,m}_{\beta}$. $\rightsquigarrow$ $S^{n,m}_{\nu} \subseteq S^{n,m}_{\beta}$ refined by $\nu = (P_1, \dots P_s)$ s.t $\beta_{n,m}(\nu) = \beta$

2) Construct an asymptotic GIT stratification using the stratifications on $Q^{n\text{-reg}} \subseteq \mathrm{Quot}_n$ for all n.

Recall: every n-regular sheaf is n'-regular for all $n' > n$ $\rightsquigarrow$ $Q^{n\text{-reg}} \to Q^{n'\text{-reg}}$
$\rightsquigarrow$ open immersion of quotient stacks $\mathcal{C}oh^{n\text{-reg}}_{X,P} = \left[Q^{n\text{-reg}} / GL_{P(n)} \right] \hookrightarrow \mathcal{C}oh^{n'\text{-reg}}_{X,P} = \left[Q^{n'\text{-reg}} / GL_{P(n')} \right]$
The limit of this diagram is the stack $\mathcal{C}oh_{X,P}$ of coherent sheaves.

Since the HN stratifications and GIT stratifications on each $Q^{n\text{-reg}}$ are $GL_{P(n)}$-invariant, they descend to the stack quotients:

$$\mathcal{C}oh^{n\text{-reg}}_{X,P} = \coprod_{\tau} \mathcal{C}oh^{n\text{-reg}}_{X,P,\tau} \qquad\qquad \mathcal{C}oh^{n\text{-reg}}_{X,P} = \coprod_{\nu} \mathcal{S}^n_\nu$$

HN stratification (refined) GIT stratification (can suppress m from notation)

Def For ν a tuple of Hilbert polynomials which sum to P, we let $\mathcal{S}^{n,n'}_\nu$ for $n' > n$ be the following fibre product

$$\begin{tikzcd}
\mathcal{S}^{n,n'}_\nu \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}^{n'}_\nu \arrow[d, hook] \\
\mathcal{S}^n_\nu \arrow[r, hook] \arrow[dr, hook] & \mathcal{C}oh^{n'\text{-reg}} \\
& \mathcal{C}oh^{n\text{-reg}} \arrow[u, hook]
\end{tikzcd}$$

Theorem (H.)

1) For each tuple ν and $n' \gg n \gg 0$, the stacks $\mathcal{S}^{n,n'}_\nu$ stabilise to an asymptotic GIT stratum $\mathcal{S}_\nu$
s.t. $\mathcal{F} \in \mathcal{S}_\nu \iff \mathcal{F} \in \mathcal{S}^n_\nu \ \forall\, n \gg 0$.
2) $\mathcal{S}_\nu \neq \phi \iff \nu$ is a HN type.
3) For a HN type τ, we have that the asymptotic GIT stratum S_τ equals the HN stratum $\mathcal{C}oh_{X,P,\tau} = \bigcup_n \mathcal{C}oh^{n\text{-reg}}_{X,P,\tau}$.

Hence, the asymptotic GIT stratification on $\mathcal{C}oh_{X,P}$ coincides with the HN stratification on $\mathcal{C}oh_{X,P}$.

In particular, $\mathcal{C}oh_{X,P}$ is the correct place to compare these stratifications, as it parametrises <u>all</u> coherent sheaves on X with Hilbert polynomial P.