AN INTRODUCTION TO GEOMETRIC INVARIANT THEORY

Victoria Hoskins

Plan:

① Moduli problems and group actions

② Mumford's reductive GIT

③ Semistability & Instability

④ Non-reductive GIT (for graded unipotent groups) ← Time Permitting!

Conventions: All schemes are finite type schemes over $k = \bar{k}$

§1 MODULI PROBLEMS AND GROUP ACTIONS

§1.1 Moduli problems

Naively: set $\mathcal{A}$ of objects we want to classify + equivalence rel^n $\sim$ on $\mathcal{A}$

Goal: equip $\mathcal{A}/\sim$ with a geom structure describing how objects vary in families

↳ Need: notion of families & equiv. of families $/S$ + pullback of families, compatible with $\sim$

Formally: Moduli functor $\mathcal{M} : Sch_k^{op} \to Sets$

$$\begin{tikzcd} S \arrow[d, "f"'] & \{\text{families}/S \text{ up to equiv.}\} \\ T & \{\text{families}/T \text{ up to equiv.}\} \arrow[u, "f^*"'] \end{tikzcd}$$

Examples

1) r-dimensional vector subspaces in k^n (Family$/S$: $\mathcal{V} \subset \mathcal{O}_S^{\oplus n}$ rank r subbundle)

2) $n \times n$ matrices up to conjugation

3) smooth projective curves (of fixed genus) up to isomorphism

4) vector bundles on a fixed curve (or sheaves on schemes)

5) projective hypersurfaces up to coordinate changes (or hypersurfaces in a scheme X up to $\mathrm{Aut}(X)$).

Dream: $\exists$ scheme M representing the moduli functor $\mathcal{M}$ $\rightsquigarrow$ "Fine Moduli Space"
i.e. $\mathcal{M} \xrightarrow{\sim} \mathrm{Hom}(-, M)$

Eg. $\mathbb{P}^n$ = fine moduli space for lines in k^{n+1} $\rightsquigarrow$ $\mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$ "universal family"

Otherwise:
$\rightsquigarrow$ Coarse Moduli Space $\simeq$ Best approximation to $\mathcal{M}$ of form $\mathrm{Hom}(-, M)$ s.t $\mathcal{M}(k) = M(k)$
$\rightsquigarrow$ Moduli stack $\rightsquigarrow$ encode automorphisms $\rightsquigarrow$ Good Moduli Space (or adequate) [Alper]

§1.2 Strategy for constructing moduli spaces via group quotients

After fixing any discrete topological invariants

① Describe the moduli problem as a group action:

Find a parameter scheme X with local universal family & gp action $G \times X \to X$ s.t orbits $\longleftrightarrow$ equiv. classes

Typically: overparametrisation ↵ (using some extra data)

↳ ie any other family is locally equivalent to a pullback of this family

② Construct an algebro-geometric quotient (of an open "semistable" subset) → GIT

③ Moduli-theoretically interpret semistability

Ex 1) 1-dim$^{\mathrm{e}}$ subspaces in $V = k^{n+1}$ are parametrised by $X = V \setminus \{0\}$ (choose a basis vector)
modulo scalar multiplication $G = k^\times \curvearrowright X$ (change of basis)
orbits $\longleftrightarrow$ equivalence classes & $\mathbb{P}^n = X/G$ is an orbit space & fine moduli space.

2) For matrices up to conjugation, consider $GL_n \curvearrowright \mathrm{Mat}_{n \times n}$ by conjugation

5) For hypersurfaces in $\mathbb{P}^n$ of degree d up to change of coords,
parameter scheme: $X = \mathbb{P}(k[x_0, \ldots, x_n]_d)$
group action: $G = PGL_{n+1} = \mathrm{Aut}(\mathbb{P}^n) \curvearrowright X$ change of coords

Alternative stacky approach to construct moduli spaces: "Beyond GIT"

① Describe the moduli problem as an algebraic stack

② Apply existence criteria [Alper–Halpern-Leistner–Heinloth]

③ Give moduli-theoretic proof of geometric properties (eg. projectivity)

We implement this for moduli of reps of an acyclic quiver
$\rightsquigarrow$ GIT-free construction & proof of projectivity
with P. Belmans, C. Damiolini, H. Franzen, S. Makarova, T. Tajakka

§2 MUMFORD'S REDUCTIVE GEOMETRIC INVARIANT THEORY

For an algebraic group action $G \times X \to X$

↳ morphism of schemes (usual action axioms encoded by commutative diagrams)

G affine algebraic group

eg. $\mathbb{G}_m = \operatorname{Spec} k[t,t^{-1}]$ multiplicative gp

$\mathbb{G}_a = \operatorname{Spec} k[t]$ additive group

$GL_n = \operatorname{Spec} k[x_{ij}, 1/\det(x_{ij})]$ general linear gp

Problem: Orbit space X/G does not admit a scheme structure

↳ algebraic groups are often non-compact ⇝ non-closed orbits

Def: A categorical quotient for $G \times X \to X$ is a universal G-invariant morphism

↳ constant on orbits (and their closures)

ie

$$\begin{tikzcd} \varphi : X \arrow[r, "G\text{-inv}"] \arrow[dr, "\forall\, G\text{-inv}"'] & Y \arrow[d, dashed, "\exists !"] \\ & Z \end{tikzcd}$$

Ex: $\mathbb{G}_m \curvearrowright X = \mathbb{A}^n$ scalar multiplication.

Orbits ↗ Punctured lines through the origin
↘ The origin

→ all contain 0 in their closure

⇒ Categorical quotient is $\varphi : X \to \operatorname{Spec} k$.

§2.1 Hilbert's 14th Problem

For an action $\sigma : G \times X \to X$ of an affine algebraic group on an affine scheme, we get an induced coaction homomorphism

$$\sigma^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(G \times X) \simeq \mathcal{O}(G) \otimes_k \mathcal{O}(X)$$

⇝ linear representation $G \longrightarrow GL(\mathcal{O}(X))$

Concretely $(g \cdot f)(x) = f(\sigma(g^{-1}, x))$ for $f \in \mathcal{O}(X)$, $g \in G$ and $x \in X$.

Any G-invariant morphism $\phi : X \to Z$ induces $\phi^* : \mathcal{O}(Z) \longrightarrow \mathcal{O}(X)$ with

$\operatorname{Im}(\phi^*) \subset \mathcal{O}(X)^G = \{ f \in \mathcal{O}(X) : g \cdot f = f \ \forall g \in G \}$

Question: Does the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ correspond to a morphism of schemes? (of finite type/k)

Question: (Hilbert's 14^{th} Problem) Is $\mathcal{O}(X)^G$ finitely generated?

Answer: (Hilbert) Yes for GL_n over $k = \mathbb{C}$

(Nagata) No in general, but yes for "reductive" groups G!

§2.2 Reductive groups

Def: A affine alg. group G over k is

i) unipotent if its isomorphic to a subgroup of $\mathbb{U}_n = \left\{\begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & & \vdots \\ 0 & & & 1 \end{pmatrix}\right\} < GL_n$

ii) reductive if its smooth & every smooth connected unipotent normal subgroup is trivial

iii) geometrically reductive if $\forall$ finite dim^e linear representation $G \to GL(V)$ and $v^{\neq 0} \in V^G$, $\exists$ non-constant homogeneous polynomial $f \in \mathcal{O}(V)^G$ s.t $f(v) \neq 0$.

iv) linearly reductive if —״—, $\exists$ linear $f \in \mathcal{O}(V)^G$ s.t $f(v) \neq 0$

Rmk: 1) (Weyl, Nagata, Mumford, Haboush) For smooth affine alg groups over k:

$$\text{linearly reductive} \underset{\Longleftarrow}{\overset{\text{def}^n}{\Longrightarrow}} \text{geometrically reductive} \Longleftrightarrow \text{reductive}$$

In char. 0 [Weyl]

Reduces to $k=\mathbb{C}$ & uses the rep. theory of compact Lie groups ↵

↳ Conj. [Mumford]

⇒ [Nagata]

⇐ [Haboush]

2) G is unipotent $\Longleftrightarrow$ $\forall$ non-trivial linear representation $G \to GL(V)$ $\exists$ $v \neq 0 \in V^G$

3) G is linearly reductive $\Leftrightarrow$ every finite dim^e linear representation is completely irred

$\Leftrightarrow$ taking G-invariants on finite dim^e linear G-reps is exact.

Ex 1) $\mathbb{G}_a \cong \mathbb{U}_2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} < GL_2$ is unipotent

2) Any torus $T = \mathbb{G}_m^n$ is linearly reductive

(Every finite dim^e linear representation $T \to GL(V)$ is completely reducible, as $\exists$ weight space decomposition $V = \bigoplus_{\chi \in X^*(T)} V_\chi$ for $V_\chi := \{ v \in V : t \cdot v = \chi(t) v \}$)

3) GL_n, SL_n and PGL_n are reductive but not linearly reductive for $n > 1$ in positive characteristic

§2.3 Finitely generated rings of invariants

Thm (Nagata) For a reductive group G acting on an affine scheme X, the ring of invariants $\mathcal{O}(X)^G$ is finitely generated.

Rmk For linearly reductive groups, proof goes back to 19th invariant thy (Hilbert) & involves constructing a "Reynolds operator" $\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(X)^G$

Thm (Popov) For any non-reductive group G, $\exists$ affine G-scheme X s.t $\mathcal{O}(X)^G$ is non-finitely generated

Thm (Weitzenbock) (Char $k=0$) Any linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ extends to SL_2 via $\left(\begin{array}{c}\mathbb{G}_a \hookrightarrow SL_2 \\ u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\end{array}\right)$ and $\mathcal{O}(\mathbb{A}^n)^{\mathbb{G}_a}$ is finitely generated

Proof: The action $\sigma : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n$ & coaction $\sigma^* : \mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(\mathbb{A}^n) \otimes k[t]$ induces a locally nilpotent derivation $D : \mathcal{O}(\mathbb{A}^n) \to \mathcal{O}(\mathbb{A}^n)$ ⚠ need Char 0

$$f \longmapsto \frac{\partial}{\partial t}(\sigma^*(f))\Big|_{t=0}$$

Since $\mathbb{G}_a$ acts linearly, we can extend to SL_2 by putting D in JNF.
(length r Jordan block $\rightsquigarrow$ SL_2-rep $\mathrm{Sym}^{r-1}(k^2)$)

Borel transfer principle: $\mathcal{O}(\mathbb{A}^n)^{\mathbb{G}_a} \cong \mathcal{O}(\mathbb{A}^n \times^{\mathbb{G}_a} SL_2)^{SL_2} \underset{\substack{\uparrow \\ \mathbb{G}_a \curvearrowright \mathbb{A}^n \\ \text{extends to } SL_2}}{\cong} \mathcal{O}(\underbrace{\mathbb{A}^n \times SL_2/\mathbb{G}_a}_{\text{only quasi-affine}})^{SL_2}$

$$\begin{tikzcd}
\text{Claim: } SL_2/\mathbb{G}_a \cong \mathbb{A}^2\setminus\{0\} \arrow[r] & \rotatebox{90}{$\cong$} \\
 & \mathcal{O}(\mathbb{A}^n \times (\mathbb{A}^2\setminus\{0\}))^{SL_2} \arrow[d, phantom, "\rotatebox{90}{$\cong$}", "\text{codim } 2 \text{ complement}"', "\text{Hartogs' Lemma}"] \\
 & \mathcal{O}(\mathbb{A}^n \times \mathbb{A}^2)^{SL_2}
\end{tikzcd}$$

f. generated by Nagata's Thm ∎

Pf: $SL_2 \to \mathbb{A}^2\setminus\{0\}$ $\mathbb{G}_a$-inv.
bottom row

For $(1,0) \in \mathbb{A}^2$ see orbit is $\mathbb{A}^2\setminus\{0\}$
stabiliser is $\mathbb{G}_a$.

Rmk: $\mathcal{O}(SL_2)^{\mathbb{G}_a} = k[x_{21}, x_{22}]$, but

$$\begin{tikzcd}
SL_2 \arrow[r] \arrow[dr] & \mathbb{A}^2 := \operatorname{Spec} \mathcal{O}(SL_2)^{\mathbb{G}_a} \\
 & \mathbb{A}^2\setminus\{0\} \arrow[u, phantom, "\rotatebox{90}{$\subsetneq$}"]
\end{tikzcd}$$

is not surjective

ie. Spec of invariants does not give categorical quotient

⚠ image could even be constructible!

§2.4 Affine Geometric Invariant Theory (for reductive groups)

Assume $\underset{\text{reductive}}{G} \curvearrowright \underset{\text{affine scheme}}{X}$ ⇝ $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$

take Spec

Thm (Mumford) The affine GIT quotient $\varphi : X \to X/\!/G := \operatorname{Spec} \mathcal{O}(X)^G$ is a categorical quotient

↳ Notation emphasises: not an orbit space in general

Rmk: 1) In fact, Mumford shows φ is a good quotient, which is a stronger notion:

i) φ is G-invariant, surjective & affine

ii) $\forall\ \underset{\text{open}}{U \subset} X/\!/G$, we have $\mathcal{O}_{X/\!/G}(U) \xrightarrow{\sim} \mathcal{O}_X(\varphi^{-1}(U))^G$

iii) $\forall$ disjoint closed G-inv. subsets $W_1, W_2 \subset X$, the images $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint & closed

The proof uses Nagata's Thm: $\mathcal{O}(X)^G$ is f.gen & that there are enough reductive invariants to separate orbits.

2) φ restricts to an orbit space on the stable locus:

$$X^s = \{x \in X : G\cdot x \underset{\text{closed}}{\subset} X \text{ and } \dim G_x = 0\}$$

Ex1 $\mathbb{G}_m \curvearrowright X = \mathbb{A}^2$
$t\cdot(x,y) = (tx, t^{-1}y)$

Orbits:
- ↗ conics $xy = \alpha$ for $\alpha \neq 0$
- → punctured axes ← both contain 0 in their closure
- ↘ origin

$\mathcal{O}(X)^{\mathbb{G}_m} = k[x,y]^{\mathbb{G}_m} = k[xy]$ ⇝ GIT quotient $X \to \mathbb{A}^1$, $(x,y) \mapsto xy$

Ex2 $GL_2 \curvearrowright \mathrm{Mat}_{2\times 2}$ by conjugation. Orbits ⟷ JNFs: $\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$ or $\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$

Note $k[\mathrm{tr}, \det] \subset \mathcal{O}(\mathrm{Mat}_{2\times 2})^{GL_2}$. We claim this is an equality:

$$\mathcal{O}(\mathrm{Mat}_{2\times2})^{GL_2} \underset{\uparrow}{\subset} k[x_{11}, x_{22}]^{S_2} = k[\overset{\text{trace}}{x_{11} + x_{22}}, \overset{\text{det}}{x_{11}x_{22}}]$$

any orbit closure contains a diagonal matrix by considering JNFs:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}t^{-1} & 0\\ 0 & t\end{pmatrix} = \begin{pmatrix}\alpha & t^2\\ 0 & \alpha\end{pmatrix} \xrightarrow[\lim_{t\to 0}]{} \begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$$

The GIT quotient $\varphi: X \to \mathbb{A}^2 = \mathrm{Mat}_{2\times2}/\!/GL_2$ is given by $M \mapsto (\mathrm{tr} M, \det M)$

and identifies non-equivalent JNFs $\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$ & $\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$

§2.5 Projective Geometric Invariant Theory

Let $X \subset \mathbb{P}^n$ be a projective scheme and assume G acts via $G \to GL_{n+1}$
reductive

$$R(X,\mathcal{O}(1)) = \bigoplus_{r\geq 0} H^0(X,\mathcal{O}(r)) = k[x_0,\dots,x_n]/I_X$$

finitely generated graded k-algebra

↑ homogeneous coordinate ring

Def: The inclusion $R(X,\mathcal{O}(1))^G \hookrightarrow R(X,\mathcal{O}(1))$ induces a rational map

$$X \dashrightarrow X/\!/G := \operatorname{Proj} R(X,\mathcal{O}(1))^G$$

whose domain of definition is the GIT semistable locus

$$X^{ss} = \{x \in X : \exists\ f \in R(X,\mathcal{O}(r))^G \text{ for } r>0 \text{ s.t } f(x) \neq 0\}.$$

The restriction $\varphi : X^{ss} \longrightarrow X/\!/G$ is the projective GIT quotient

Thm (Mumford) The projective GIT quotient is a good quotient.
& restricts to an orbit space on $X^s = \{x \in X^{ss} : G\cdot x \underset{\text{closed}}{\subset} X^{ss} \text{ and } \dim G_x = 0\}$

"Proof" The projective GIT quotient is constructed by gluing affine GIT quotients $X_f \to X_f/\!/G$ for $f \in R(X,\mathcal{O}(1))^G_+$

Rmk: Since the homogeneous coordinate ring depends on the embedding $X \hookrightarrow \mathbb{P}^n$, the GIT quotient and semistable locus depend on this too.

More generally: GIT quotient depends on G-linearisation $\mathcal{L}$ (G-equiv. line bundle on X)

↳ v.ample $\mathcal{L}$ induces G-equiv. embedding $X \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(X,\mathcal{L})^*)$

$$\rightsquigarrow G \curvearrowright R(X,\mathcal{L}) := \bigoplus_{r\geq 0} H^0(X,\mathcal{L}^{\otimes r})$$

$$\begin{tikzcd} X \arrow[r, dashed] & X/\!/_{\mathcal{L}} G := \operatorname{Proj}\left(R(X,\mathcal{L})^G\right) \\ X^{ss}(\mathcal{L}) \arrow[u, phantom, sloped, "\subset"] \arrow[ur] & \end{tikzcd}$$

$$X^{ss}(\mathcal{L}) = \{x \in X : \exists s \in H^0(X,\mathcal{L}^{\otimes r})^G \text{ for } r>0 \text{ s.t } s(x) \neq 0\}$$

VGIT: The effect of changing $\mathcal{L}$ determines a birational transformation of GIT quotients described by variation of GIT flips (Dolgachev-Hu Thaddeus)

More generally
- can construct GIT quotients for q-proj. schemes with ample linearisation
- can construct relative GIT quotients over general base (Seshadri)

§3 SEMISTABILITY AND INSTABILITY

§3.1 Semistability and the Hilbert-Mumford criterion

The Hilbert-Mumford criterion reduces G-semistability to $\mathbb{G}_m$-actions

For $\mathbb{G}_m < G$, we have x is G-(semi)stable $\Rightarrow$ x is $\mathbb{G}_m$-(semi)stable

$\rightarrow$ The Hilbert-Mumford criterion gives a converse!

Recall: $\underset{\text{reductive}}{G} \curvearrowright \mathbb{P}(V)$ via a linear representation $G \rightarrow GL(V)$

Prop (Topological Hilbert-Mumford criterion)

$[v] \in \mathbb{P}(V)$ is G-semistable $\iff$ $0 \notin \overline{G \cdot v} \subset V$

$[v] \in \mathbb{P}(V)$ is G-stable $\iff$ $G \cdot v \subset V$ closed and $\dim G_v = 0$.

Pf: $[v]$ is G-semistable $\underset{\text{def.}}{\iff}$ $\exists$ non-constant homogeneous polynomial $f \in k[V]^G$ s.t $f(v) \neq 0$

clear $\Downarrow$ $\Updownarrow$ $\Uparrow$ as G is geom. reductive

The closed G-inv sets 0 & $\overline{G \cdot v}$ are disjoint $\boxtimes$

Def: $T = \mathbb{G}_m^n \longrightarrow GL(V)$ f.dim^e torus rep $\rightsquigarrow$ $V = \bigoplus_{\chi \in X^*(T)} V_\chi$ weight decomposition

If $v = \sum_{\chi \in X^*(T)} v_\chi$, $T\text{-wt}(v) := \{\chi : v_\chi \neq 0\}$ $\quad V_\chi = \{v \in V : t \cdot v = \chi(t) v\}$

$\uparrow$ T-weights of v

Prop: For a finite-dim^e linear rep $\mathbb{G}_m \rightarrow GL(V)$, we have that $[v] \in \mathbb{P}(V)$ is GIT semistable for the $\mathbb{G}_m$-action $\iff$ $0 \in \operatorname{Conv}(\mathbb{G}_m\text{-wt}(v))$

resp. stable —״— $\iff$ $0 \in \operatorname{Int}\operatorname{Conv}(\mathbb{G}_m\text{-wt}(v))$

Proof: By the topological Hilbert-Mumford criterion, we have

$[v]$ is $\mathbb{G}_m$-semistable $\iff 0 \notin \overline{\mathbb{G}_m \cdot v}$ $\leftarrow$ This contains $\lim_{t \to 0} t \cdot v$ and $\lim_{t \to \infty} t \cdot v$ (if they exist)

Not^n: If $\underset{t \,\mapsto\, t \cdot v}{\mathbb{G}_m \xrightarrow{f} V}$ extends to $\mathbb{A}^1 \xrightarrow{\tilde{f}} V$ write:

$0 \longmapsto \tilde{f}(0) \doteqdot \lim_{t \to 0} t \cdot v$

$\lim_{t\to 0} t\cdot v = 0 \Leftrightarrow \mathbb{G}_m\text{-wt}(v) \subset \mathbb{Z}_{>0}$

$\lim_{t\to \infty} t\cdot v = 0 \Leftrightarrow \mathbb{G}_m\text{-wt}(v) \subset \mathbb{Z}_{<0}$

$\Leftrightarrow \quad \lim_{t\to 0} t\cdot v \neq 0$ and $\lim_{t\to\infty} t\cdot v \neq 0$

$\Leftrightarrow \quad \exists$ weights r_0 & $r_\infty \in \mathbb{G}_m\text{-wt}(v)$ s.t $r_0 \leq 0$ and $r_\infty \geq 0$

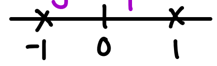
$\Leftrightarrow \quad 0 \in \mathrm{Conv}(\mathbb{G}_m\text{-wt}(v))$ $\boxed{}$

Ex: For $\mathbb{G}_m \curvearrowright \mathbb{P}^n$ via $t \mapsto \mathrm{diag}(t^{-1}, t, \ldots, t)$

$x \in \mathbb{P}^n$ is $\mathbb{G}_m$-(semi)stable $\Leftrightarrow x_0 \neq 0$ and $\exists\, i > 0$ s.t $x_i \neq 0$

$(\mathbb{P}^n)^{\mathbb{G}_m\text{-ss}} \cong \mathbb{A}^n \setminus \{0\} \longrightarrow \mathbb{P}^n /\!/ \mathbb{G}_m = \mathbb{P}^{n-1}$

T-weight picture:



Thm: (Hilbert-Mumford criterion) For $G \curvearrowright X \subset \mathbb{P}^n$ linearly

Version 1: $x \in X$ is G-(semi)stable $\Leftrightarrow$ x is $\mathbb{G}_m$-(semi)stable $\forall\, \mathbb{G}_m < G$

Version 2: Fix a maximal torus T

$x \in X$ is G-(semi)stable $\Leftrightarrow$ $g\cdot x$ is T-(semi)stable $\forall\, g \in G$

i.e $x \in \bigcap_{g\in G} g X^{T\text{-(s)s}}$

Moreover $x \in X^{T\text{-ss}} \Leftrightarrow 0 \in \mathrm{Conv}(T\text{-wt}(x))$

Version 3: x is G-semistable $\Leftrightarrow$ $\mu(x,\lambda) \geq 0$ $\forall$ 1-param subgp $\lambda: \mathbb{G}_m \to G$ (1-PS)

Hilbert-Mumford weight

Def (Hilbert-Mumford weight)

For $G \curvearrowright \mathbb{P}(V)$ linearly, the H-M weight of $x \in \mathbb{P}(V)$ w.r.t $\lambda: \mathbb{G}_m \to G$ is

$$\mu(x,\lambda) = -\min\{\lambda(\mathbb{G}_m)\text{-wt}(x)\} = -(\lambda(\mathbb{G}_m)\text{-wt}(x_0)),\ x_0 = \lim_{t\to 0} \lambda(t)\cdot x$$

Rmk: For G acting on a projective scheme X wrt. linearisation $\mathcal{L}$, we have $\mu^{\mathcal{L}}(x,\lambda) := -(\lambda(\mathbb{G}_m)\text{-wt on } \mathcal{L}_{x_0})$ for $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$

for action $\mathcal{L}_{t\cdot x_0} \xrightarrow{\cdot t^{-1}} \mathcal{L}_{x_0}$

The Hilbert-Mumford criterion boils down to the following result:

Thm (Fundamental Thm of GIT)

Let G be a reductive group acting on $\mathbb{A}^n$. If $0 \in \overline{G\cdot x}$, then $\exists$ 1-PS $\lambda: \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x = 0$

§3.2 Instability: What about the unstable points?

Set-up $G \curvearrowright X = \mathbb{P}(V)$ via a linear representation (as above)
reductive

Hilbert-Mumford: $x \in X^{ss} \iff \mu(x,\lambda) \geq 0 \quad \forall$ 1-PS $\lambda : \mathbb{G}_m \to G$

Idea: Stratify $X^{us} = X \setminus X^{ss}$ using normalised H-M weight

(Kempf, Hesselink, Kirwan, Ness) $\rightsquigarrow$ associate to each unstable orbit a conjugacy class of "adapted" 1-PS which are "most responsible" for its instability

↳ Determined using a norm $\|-\|$ on 1-PS of G

Thm [Kempf, Hesselink, Kirwan, Ness]

For $G \curvearrowright X = \mathbb{P}(V)$ linearly (& norm $\|-\|$), there is a finite stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into G-invariant locally closed subvarieties with a partial order on the index set $\mathcal{B}$

s.t.

i) The lowest stratum is $S_0 = X^{ss}$

ii) $\overline{S_\beta} \subset \bigsqcup_{\beta' \geq \beta} S_{\beta'}$

iii) $\mathcal{B}$ is determined combinatorially by the weights of a max^e torus

iv) The strata S_β are determined by simpler limit sets (= GIT ss loci for smaller red gps)

→ combinatorial flavour

Structure of unstable stratum S_β:

$\beta \leftrightarrow [\lambda_\beta]$ conj class of (rational) 1-PS. Choose a representative λ_β

$$\begin{tikzcd}
\text{"Blades"} & Y_\beta^{ss} = \{x \in S_\beta : \lambda_\beta \text{ adapted to } x\} \arrow[r, phantom, "\subset"] \arrow[d, "p_\beta = \lim_{t\to 0} \lambda_\beta(t)\cdot(-)"] & Y_\beta \arrow[d, "p_\beta", "\text{Zar. loc trivial } \mathbb{A}^N\text{-fibration}"'] & P_\beta = \{g \in G : \lim_{t\to 0} \lambda_\beta(t) g \lambda_\beta(t)^{-1} \text{exists}\} \arrow[l, bend right, "\curvearrowleft"'] \arrow[d, "q_\beta"] & \text{parabolic subgp (depend on } \lambda_\beta) \\
\text{(depend on } \lambda_\beta) & & & & \\
\text{"limit sets"} & Z_\beta^{ss} = Y_\beta^{ss} \cap X^{\lambda_\beta} \arrow[r, hook, "\circ" marking] & Z_\beta & L_\beta \arrow[l, bend right, "\curvearrowleft"'] & \text{Levi subgp}
\end{tikzcd}$$

↑ λ_β-fixed components s.t $\mu(-,\lambda_\beta) = -\|\beta\|^2$

Z_β^{ss} = GIT semistable set for $L_\beta \curvearrowright Z_\beta$ wrt twisted linearisation $\mathcal{L}_\beta$ (twist by character $\chi_{-\beta} : L_\beta \to \mathbb{G}_m$ corr. to λ_β)

$Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$

$$\boxed{S_\beta = G Y_\beta^{ss} \simeq G \times^{P_\beta} Y_\beta^{ss}}$$

Question: Can we construct quotients of the unstable strata?

Rmk: Categorical quotient for $S_\beta \simeq G \times^{P_\beta} Y_\beta^{ss}$ $\Longleftrightarrow$ cat. quotient for $P_\beta \curvearrowright Y_\beta^{ss}$

Prop: $Y_\beta^{ss} \xrightarrow{p_\beta} Z_\beta^{ss} \xrightarrow{\pi} Z_\beta /\!/_{\mathcal{L}_\beta} L_\beta$ is a categorical P_β-quotient.

Pf: Since p_β is q_β-equivariant & π is L_β-inv, $\pi \circ p_\beta$ is P_β-invariant.
Suppose $f : Y_\beta^{ss} \longrightarrow T$ is P_β-invariant, then

$$\begin{tikzcd}
Y_\beta^{ss} \arrow[r, "p_\beta"] \arrow[dr, "\circledast"'] & Z_\beta^{ss} \arrow[r, "\pi"] \arrow[d, "f|_{Z_\beta^{ss}}"] & Z_\beta /\!/_{\mathcal{L}_\beta} L_\beta \arrow[dl, "\exists !"] \\
& T &
\end{tikzcd}$$

$\circledast$ commutes as f is constant on orbit closures & $p_\lambda(y) \in \overline{P_\lambda \cdot y}$

by the universal property of π
(as $f|_{Z_\beta^{ss}}$ is L_β-invariant) ☒

Rmk: Under this quotient every $y \in Y_\beta^{ss}$ is identified with its image
$p_\beta(y) = \lim_{t \to 0} \lambda(t) \cdot y \in Z_\beta^{ss}$ $\rightsquigarrow$ far from being a geometric quotient.

To obtain a reasonabe quotient of Y_β^{ss}, we'd like Z_β^{ss} to be unstable

Key observation: It's better to work with $P_\beta \curvearrowright Y_\beta^{ss}$ than $G \curvearrowright S_\beta$

↳ P_β has more characters we can use to twist the linearisation

Goal (Joint project with J. Jackson)
Construct better quotients of unstable strata using non-reductive GIT

Potential applications moduli of unstable objects
eg. vector bundles / Higgs bundles / sheaves / quiver representations of fixed Harder-Narasimhan type.
↳ E. Hamilton

§4 NON-REDUCTIVE GIT IN A NUTSHELL

Assume

Char $k = 0$

(After F. Kirwan, B. Doran, G. Bérczi, T. Hawes . . . .)

Recall: Non-reductive group actions may have non-finitely generated invariant rings

However, even if invariant rings are finitely generated, there are other issues

- Taking Spec of invariants may give a non-surjective morphism with only constructible image!
- There are not enough invariants to separate closed orbits
- Invariant may not extend to ambient affine

Miraculously Many good features of GIT can be extended to groups whose unipotent radical $\mathcal{U}$ is graded by a multiplicative group $\mathbb{G}_m$

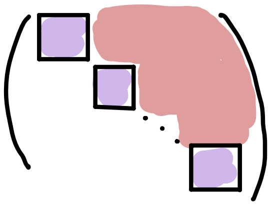
↳ ie. $\mathbb{G}_m \underset{\text{conj.}}{\curvearrowright} \text{Lie}\,\mathcal{U}$ with weights > 0

Many applications involve such graded group actions:

① Unstable strata $S_\beta \cong G \times^{P_\beta} Y_\beta^{ss}$

If $G = GL_n$ & $\lambda_\beta(t) = \begin{pmatrix} t^{r_1} I_{n_1} & & \\ & \ddots & \\ & & t^{r_\ell} I_{n_\ell} \end{pmatrix}$, $P_\beta =$

$r_1 > r_2 > \cdot \cdot > r_\ell$



$= U_\beta \rtimes L_\beta$

↳ graded by $\lambda_\beta : \mathbb{G}_m \to P_\beta$

② Moduli of hypersurfaces in $\underbrace{\text{weighted projective spaces}}$

[Bunnett & Jackson]

non-reductive automorphism gps but unipotent radical is graded.

eg $\text{Aut}(\mathbb{P}(1,1,2)) \cong ((GL_2 \times \mathbb{G}_m) \ltimes \mathbb{G}_a^3)/\mathbb{G}_m$

↳ $z \mapsto z + ax^2 + bxy + cy^2$ $\quad a, b, c \in \mathbb{G}_a$

③ Moduli of jets of map germs ⇝ applications to hyperbolicity

[Bérczi-Kirwan]

X sm proj variety/$\mathbb{C}$ ⇝ $J_k X$ bundle of k-jet germs $(\mathbb{C},0) \to (X,p)$

$$\begin{tikzcd} J_k X \arrow[d] \\ X \end{tikzcd}$$

reparametrisation

$\curvearrowleft \text{Diff}_k = \{(\mathbb{C},0) \to (\mathbb{C},0) \text{ gular k-jets}\}$

$\parallel$

$\mathcal{U} \rtimes \mathbb{C}^*$ ‿ grading $\mathcal{U}$

§4.1 $\mathbb{G}_a$-actions & $\mathbb{G}_m$-gradings

$$\{\mathbb{G}_a\text{-actions on affine variety } X\} \xleftrightarrow{\text{char } 0} \{\text{loc. nilpotent derivations on } \mathcal{O}(X)\}$$

$$\sigma : \mathbb{G}_a \times X \to X \quad \longmapsto \quad D(f) := \frac{\partial}{\partial t}\sigma^*(f)\Big|_{t=0}$$

(equiv. $\sigma^* : \mathcal{O}(X) \to \mathcal{O}(X) \otimes k[t]$)

$$\sigma^* = \exp(tD) \quad \longleftarrow\!\shortmid \quad D$$

Furthermore : $\boxed{\mathcal{O}(X)^{\mathbb{G}_a} = \ker D}$

Prop: If D has a slice (ie. $\exists\, s \in \mathcal{O}(X)$ s.t $D(s) = 1$), then the $\mathbb{G}_a$-action has a geometric slice $S = \{s = 0\} \subset X$ ie

$$\begin{tikzcd} \mathbb{G}_a \times S \arrow[r, phantom, "\simeq"] \arrow[dr] & X \arrow[d] \arrow[d, phantom, shift left=6, "\text{trivial } \mathbb{G}_a\text{-quotient}"] \\ & S \end{tikzcd}$$

In particular $S = \operatorname{Spec} \underset{\sim\sim}{\mathcal{O}(X)^{\mathbb{G}_a}}$ ← finitely generated!

Proof: Show $\mathcal{O}(X)^{\mathbb{G}_a} = \ker D = \operatorname{Im}(\Phi : \mathcal{O}(X) \to \mathcal{O}(X))$
$$f \mapsto \exp(tD)|_{t=-s}$$

Thus $\mathcal{O}(X) = \mathcal{O}(X)^{\mathbb{G}_a}[s]$ & $\mathcal{O}(X)^{\mathbb{G}_a} = \mathcal{O}(X)/(s)$ ∎

$$f = \sum_{i \geq 0} \Phi(D^i(f)) \frac{s^i}{i!}$$

Observe: Slice of action gives $\mathbb{Z}_{\leq 0}$-grading of $\mathcal{O}(X)$ ie $\mathbb{G}_m$-action on X grading $\mathbb{G}_a$-action
↳ by − degree of s

The starting point of non-reductive GIT is a converse statement:

Key Prop [Bérczi-Doran-Hawes-Kirwan] For a $\mathbb{G}_a$-action on an affine scheme X, assume:

i) The action extends to $\mathbb{G}_a \rtimes \mathbb{G}_m$ s.t $\mathbb{G}_m \curvearrowright \operatorname{Lie} \mathbb{G}_a$ wt $n > 0$

ii) $\lim_{t \to 0} t \cdot x$ exists $\forall\, x \in X$ ie $\mathbb{G}_m$-action $\rightsquigarrow$ $\mathbb{Z}_{\leq 0}$-grading on $\mathcal{O}(X)$

iii) $\operatorname{Stab}_{\mathbb{G}_a}(x) = \{e\}$ $\forall\, x \in X^{\mathbb{G}_m}$.

Then there exists a $\mathbb{G}_a$-slice: $\mathcal{O}(X)^{\mathbb{G}_a}$ f.gen & $X \to \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$ trivial $\mathbb{G}_a$-quotient

Rmk: These 3 assumptions govern the 3 key assumptions in nrGIT:

i) There is a $\mathbb{G}_m$-action grading the unipotent radical

ii) The linearisation is chosen so the $\mathbb{G}_m$-weights satisfy certain inequalities
↳ "adapted" linearisation   fixes VGIT chamber for $\mathbb{G}_m$-action ↵

iii) Certain unipotent stabilisers must be trivial.