Motivic mirror symmetry for Higgs bundles

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joint work with
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§1 Moduli spaces of Higgs bundles

C/k smooth projective connected alg. curve

Fix a rank n & degree d s.t $(n,d)=1$

Moduli spaces:

- $N_{n,d}$ moduli space of stable rank n degree d vector bundles on C

smooth projective variety of dim $n^2(g-1)+1$

$T^*_{[E]}N_{n,d} \simeq \mathrm{Ext}^1(E,E)^* \simeq \mathrm{Hom}(E, E\otimes\omega_C)$

RR as $\mathrm{Hom}(E,E)=k$ for E stable

Def. theory — Serre Duality — Higgs fields

Relation to stack of all vector bundles: $\mathrm{Bun}_{n,d} \hookleftarrow \mathrm{Bun}^{ss}_{n,d} \overset{(n,d)=1}{=} \mathrm{Bun}^{s}_{n,d}$

$$\mathrm{Bun}^{s}_{n,d} \xrightarrow{\mathbb{G}_m\text{- gerbe}} N_{n,d}$$

(E is (semi)stable if $\forall\, 0\neq E' \subsetneq E$: $\mu(E') \underset{(=)}{<} \mu(E) := \frac{\deg E}{\mathrm{rk} E}$)

- $M_{n,d} \hookleftarrow T^*N_{n,d}$ moduli space of stable rank n degree d Higgs bundles on C $(E, \Phi: E \to E\otimes\omega_C)$

smooth q-proj variety

$\dim M_{n,d} = 2\dim N_{n,d}$

Geometric features:

* Over $k=\mathbb{C}$, M is a non-compact hyperkähler mfld (In general M is algebraic symplectic)

* Cohomologically M behaves like a compact mfld, due to the existence of a "semi-projective" scaling action: $\mathbb{G}_m = k^\times \curvearrowright M$

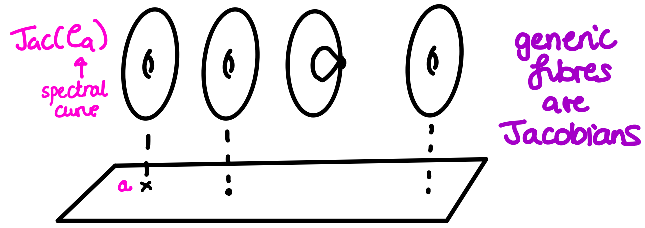
[Hitchin, Simpson, Hausel...]

$t\cdot[E,\Phi] = [E, t\Phi]$

* Hitchin fibration :

Completely integrable Hamiltonian system

$$\begin{array}{ccc} \mathcal{M} & & (E,\Phi) \\ h \downarrow & & \rotatebox[origin=c]{-90}{$\mapsto$} \text{ proper} \\ \text{affine space} \to \mathcal{A} & & \mathrm{Char}(\Phi) \end{array}$$



* Versions for other reductive groups $G \neq GL_n \rightsquigarrow h_G : \mathcal{M}_G \to \mathcal{A}_G$

- SL_n-Higgs bundles : for $L \in \mathrm{Pic}^d(C)$

$$\mathcal{M}_{n,L} = \{ (E,\Phi) \in \mathcal{M}_{n,d} : \det(E) \simeq L, \ \mathrm{Tr}(\Phi) = 0 \} \hookrightarrow \mathcal{M}_{n,d}$$

$h_L \downarrow$ $\circlearrowleft$ $\Gamma := \mathrm{Jac}(C)[n]$ $\qquad (n,d)=1 \Rightarrow \mathcal{M}_{n,L}$ smooth $\qquad \circlearrowright \ \mathrm{Jac}(C)$

$\mathcal{A}_{n,L}$

- PGL_n-Higgs bundles

$$\bar{\mathcal{M}}_{n,d} := [\mathcal{M}_{n,L}/\Gamma] \simeq [\mathcal{M}_{n,d}/T^*\mathrm{Jac}(C)]$$

smooth orbifold (Deligne-Mumford stack)

+ μ_n-gerbe δ_L

whose coarse m.space is singular

Mirror symmetry for SL_n & PGL_n-Higgs moduli spaces $\quad (k = \mathbb{C})$

- Langlands dual groups have isomorphic Hitchin bases :

SL-Higgs $\qquad$ PGL-Higgs

$$\begin{tikzcd} \mathcal{M}_{n,L} \arrow[dr, "h_L"'] & & \bar{\mathcal{M}}_{n,d} \arrow[dl, "\bar{h}"] \\ & \mathcal{A}_{n,L} & \end{tikzcd}$$

[Hausel-Thaddeus]: generic fibres of h_L & $\bar{h}$ are dual abelian varieties

$$h_L^{-1}(a) \circlearrowleft \mathrm{Prym}(C_a/C) \xrightarrow{\sim} \mathrm{Prym}(C_a/C)/\Gamma \circlearrowright \bar{h}^{-1}(a)$$

$$\mathrm{Prym}(C_a/C)/\Gamma \cong \mathrm{Prym}(C_a/C)^\vee$$

- Expected derived equivalence

$$D^b_{coh}(\mathcal{M}_{n,L}) \simeq D^b_{coh}(\bar{\mathcal{M}}_{n,d}, \delta_L) \quad \text{relative to } \mathcal{A}_{n,L}$$

[Donagi-Pantev]: For G & ${}^LG$-Higgs bundles, there is such a derived equivalence over an open subset of the Hitchin base

• Topological mirror symmetry (Conjectured by Hausel & Thaddeus)

Proved by Groechenig-Wyss-Ziegler (for Hodge numbers) and Maulik-Shen (for pure Hodge structures)

$$(*) \qquad \boxed{H^*(\mathcal{M}_{n,L};\mathbb{C}) \simeq H^*_{orb}(\bar{\mathcal{M}}_{n,d}, \delta_L;\mathbb{C})}$$

RHS = Orbifold cohomology twisted by the gerbe δ_L:

$$H^*_{orb}(\bar{\mathcal{M}}_{n,d},\delta_L) := \bigoplus_{\substack{\gamma\in\Gamma \\ \| \\ \mathrm{Jac}(C)[n]}} H^{*-2d_\gamma}(\mathcal{M}_\gamma)_{\kappa_\gamma}(-d_\gamma)$$

κ_γ-isotypical piece (for Γ-action)

where: i) Under the Weil pairing: $\Gamma \simeq \hat{\Gamma}$ $\quad \gamma \leftrightarrow \kappa_\gamma$

ii) $\mathcal{M}_\gamma := (\mathcal{M}_{n,L})^\gamma \curvearrowleft \Gamma$ $\quad$ γ-fixed locus with induced Γ-action

iii)
$$\begin{array}{ccc} \mathcal{M}_\gamma & \longrightarrow & \mathcal{M}_{n,L} \\ {\scriptstyle h_\gamma}\downarrow & & \downarrow{\scriptstyle h_L} \\ h_L(\mathcal{M}_\gamma) =: \mathcal{A}_\gamma & \xrightarrow[\text{codim } d_\gamma]{i_\gamma} & \mathcal{A}_{n,L} \end{array}$$

Since $H^*(\mathcal{M}_{n,L}) = \bigoplus_{\kappa\in\hat{\Gamma}} H^*(\mathcal{M}_{n,L})_\kappa$, $(*)$ follows by summing isos
(isotypical decomposition)

$$(*)_\gamma \qquad \boxed{H^*(\mathcal{M}_{n,L})_{\kappa_\gamma} \simeq H^{*-2d_\gamma}(\mathcal{M}_\gamma)_{\kappa_\gamma}(-d_\gamma)} \qquad \forall \gamma\in\Gamma$$

Maulik & Shen:

• Construct $(*)_\gamma$ from an isomorphism relative to the Hitchin base

• Use perverse sheaves, the decomposition theorem & analysis of supports

• Work with D-twisted Higgs bundles to simplify the topology of the Hitchin map
↳ For a divisor D of degree $> 2g-2$

& use vanishing cycles to pass from D-Higgs bundles to K_C-Higgs bundles

• For D-twisted Higgs bundles, $(*)_\gamma$ is constructed from an endoscopic correspondence following Ngô & Yun

Goal: prove a motivic refinement of topological mirror symmetry

motive: in the sense of Grothendieck

(realised by Voevodsky's triangulated category $DM(k,\mathbb{Q})$)

* encodes cohomology groups:
    - ($k=\mathbb{C}$) singular cohomology + mixed Hodge structure
    - ℓ-adic cohomology + Galois representation
* and algebraic cycles (Chow groups)

Main Theorem [H - Pepin Lehalleur] "Motivic Mirror Symmetry"

Let $k=\bar{k}$ be a field with $\mathrm{char}(k)=0$ & let $\Lambda := \mathbb{Q}(\zeta_n)$ primitive n^{th} root of unity

In Voevodsky's category $DM(k,\Lambda)$ of motives over k with Λ-coeffs there is an isomorphism of (twisted orbifold) motives:

$$M(\mathcal{M}_{n,L}) \simeq M_{orb}(\overline{\mathcal{M}}_{n,d}, \delta_L)$$

Corollary: Mirror symmetry for (twisted orbifold) Chow groups with Λ-coeffs

§2 Motives

Grothendieck: envisaged motives as a universal coh theory for k-varieties.

Voevodsky: There is a category $DM(k,\mathbb{Q})$ of mixed motives $/k$ with $\mathbb{Q}$-coefficients together with a functor

$$M: \mathrm{Var}_k \longrightarrow DM(k,\mathbb{Q})$$
$$X \longmapsto M(X)$$

realising part of Grothendieck's vision.

Properties

* Künneth isomorphism $\quad M(X \times Y) \simeq M(X) \otimes M(Y)$

  unit $\mathbb{1} = M(\operatorname{Spec} k)$

* $\mathbb{A}^1$-homotopy invariance: $\underset{\text{vector bundle}}{E \to X} \rightsquigarrow M(E) \simeq M(X)$

* Projective bundle formula: $M(\mathbb{P}(E)) \simeq M(X) \otimes M(\mathbb{P}^{n-1})$

$$M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Q}\{i\}$$ ← Tate twists $\qquad n = \operatorname{rk}(E)$

* Gysin triangles: for $Z \underset{\text{codim } c}{\hookrightarrow} X$ both smooth k-varieties

$$M(X \setminus Z) \to M(X) \to M(Z)\{c\} \xrightarrow{+1}$$

* Chow gps: X smooth k-variety

$$CH^i(X)_{\mathbb{Q}} \simeq \operatorname{Hom}_{DM}(M(X), \mathbb{Q}\{i\})$$

* Realisation functors:

  Betti / de Rham / ℓ-adic cohomology factor via $M: \mathrm{Var}_k \to DM$

  $+$ MHS $\qquad$ $+$ Galois rep.

  eg. For $k \hookrightarrow \mathbb{C}$, $\exists$ Betti realisation $R_B : DM(k, \mathbb{Q}) \to D(\mathbb{Q}\text{-vect})$

* Relative motives & six operations formalism: for varieties over S, there are categories of relative motives $DM(S, \mathbb{Q})$

$f: T \to S$ separated finite type $\rightsquigarrow$ $f^*: DM(S, \mathbb{Q}) \rightleftarrows DM(T, \mathbb{Q}) : f_*$

as well as $f_!, f^!, \otimes, \underline{\operatorname{Hom}}$

§3 Outline of the proof

The motivic mirror symmetry isomorphism is obtained by summing isos

$$\alpha_\gamma : M(\mathcal{M}_\gamma)_{\kappa_\gamma}\{d_\gamma\} \xrightarrow{\sim} M(\mathcal{M}_{n,L})_{\kappa_\gamma}$$

over all $\gamma \in \Gamma = \mathrm{Jac}(C)[n]$.

Recall: $\Gamma \simeq \hat{\Gamma}$, $\gamma \leftrightarrow \kappa_\gamma$ (Weil pairing)

and

$$\begin{tikzcd} \Gamma \circlearrowright \mathcal{M}_\gamma := (\mathcal{M}_{n,L})^\gamma \arrow[r, hook] \arrow[d, "h_\gamma"] & \mathcal{M}_{n,L} \circlearrowleft \Gamma \arrow[d, "h_L"] \\ \mathcal{A}_\gamma := h_L(\mathcal{M}_\gamma) \arrow[r, hook, "\text{codim } d_\gamma"'] & \mathcal{A}_L \end{tikzcd}$$

Part A: Lift the construction of Maulik-Shen to DM

- Construct a relative morphism in $DM(\mathcal{A}_L, \Lambda)$:

$$\beta_\gamma : (h_{L*}\mathbb{1})_{\kappa_\gamma} \longrightarrow i_{\gamma *}(h_{\gamma *}\mathbb{1})_{\kappa_\gamma}\{-d_\gamma\}$$

whose Betti realisation $R_B(\beta_\gamma)$ is the (relative) iso of Maulik-Shen.

- Pushforward β_γ along the structure map $\mathcal{A}_L \to \mathrm{Spec}\, k$ & dualise to get an absolute morphism in $DM(k, \Lambda)$:

$$\alpha_\gamma : M(\mathcal{M}_\gamma)_{\kappa_\gamma}\{d_\gamma\} \longrightarrow M(\mathcal{M}_{n,L})_{\kappa_\gamma}$$

which we want to show is an isomorphism.

Here β_γ is relative to the Hitchin base $\mathcal{A}$ and is first constructed for D-twisted Higgs bundles via a motivic endoscopic correspondence.

Next we go from D-twisted Higgs bundles to classical K_C-Higgs bundles using motivic vanishing cycles for certain stacks (pass from D to $D-p$)

This boils down to computing the motivic vanishing cycles functor for the Killing form

$$\left[\mathfrak{sl}_n / SL_n\right] \longrightarrow \mathbb{A}^1$$

stack of SL_n-Higgs bundles over a point $p \in C$

$$g \longmapsto \mathrm{Tr}(g^2)$$

Part B: Prove that α_γ is an isomorphism via a conservativity argument

- Prove that both sides are abelian motives (Following geom. ideas of García-Prada, Heinloth & Schmitt)

$$DM_c^{ab}(k,\mathbb{Q}) := \langle M(A) : A \text{ abelian variety} \rangle$$
$$= \langle M(C) : C \text{ smooth proj. curve} \rangle$$

- Apply Wildeshaus' conservativity thm for Betti realisation on abelian motives:

Thm [Wildeshaus] For $k \hookrightarrow \mathbb{C}$, the Betti realisation
$$R_B : DM_c^{ab}(k,\mathbb{Q}) \longrightarrow D(\mathbb{Q}\text{-vect}) \text{ is conservative.}$$

- By construction $R_B(\alpha_\gamma)$ is the isomorphism $(*)_\gamma$ of Maulik-Shen

§4 Part B: The motives involved are abelian

Thm 1 [H - Pepin Lehalleur '19]

The motive of the GL-Higgs moduli space is generated by $M(C)$:
For $(n,d)=1$, we have $M(\mathcal{M}_{n,d}) \in \langle M(C) \rangle \subset DM^{ab}(k,\mathbb{Q})$.

Our proof uses geom. ideas of Hitchin & García-Prada, Heinloth & Schmitt

Thm 2 [H - Pepin Lehalleur '22]

Assume $C(k) \neq \phi$ & degree $D \geqslant 2g-2$. For $(n,d)=1$, we have

i) $M(\mathcal{M}^D_{n,d}) \in \langle M(C) \rangle \subset DM^{ab}(k,\mathbb{Q})$

ii) $M(\mathcal{M}^D_{n,L}) \in \langle M(\tilde{C}) : \tilde{C} \to C \text{ certain finite étale covers} \rangle \subset DM^{ab}(k,\mathbb{Q})$

On cohomology, Hitchin observed this for rank $n=2$ $(D=K_C)$

⚠ $M(\mathcal{M}_{n,L}) \notin \langle M(C) \rangle$ in general [Lie Fu - H. - Pepin Lehalleur]

Whereas for vector bundle moduli: $M(\mathcal{N}_{n,d})$ and $M(\mathcal{N}_{n,L}) \in \langle M(C) \rangle$.

In fact $M(\mathcal{N}_{n,d}) \simeq M(\mathcal{N}_{n,L}) \otimes M(\mathrm{Jac}\, C)$ [Fu - H - Pepin Lehalleur]

motivic version of cohomological thm of Harder & Narasimhan.

Sketch proof of Thm 1:

① Hitchin's scaling action $\mathbb{G}_m \curvearrowright \mathcal{M}_{n,d}$ $\quad t \cdot [E, \Phi] = [E, t\Phi]$

$\rightsquigarrow$ deformation retract to fixed points (Białynicki-Birula)

$\updownarrow$

chains of vector bundles

$F_0 \to F_1 \to \cdots \to F_r$

$\mathbb{G}_m \curvearrowright E \rightsquigarrow E = \bigoplus E_i$

$F_i = E_i \otimes \omega_C^{\otimes i}$

$\mathcal{M}_{n,d}$

$\downarrow$ flow $t \to 0$

Higgs stability param. for chains

$$(\mathcal{M}_{n,d})^{\mathbb{G}_m} = \coprod_{(\underline{m},\underline{e})} Ch_{\underline{m},\underline{e}}^{\alpha_H\text{-ss}}$$

$\leftarrow$ Chain moduli spaces (smooth projective varieties)

finitely many tuples of ranks & degrees

$\rightsquigarrow$ motivic BB decomposition $\Rightarrow$ suffices to describe motives of chain moduli spaces

② Use wall-crossing on space of chain stability parameters to reduce to injective chains of constant rank $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_r$

Based on [García-Prada, Heinloth, Schmitt] : for virtual motive in $\hat{K}_0(\mathrm{Var})$

* Deformation theory for α-ss chains best understood for $\alpha \in \overset{\circ}{\Delta}_r \subset \mathbb{R}^{r+1}$ cone
[Álvarez-Cónsul, García-Prada, Schmitt]

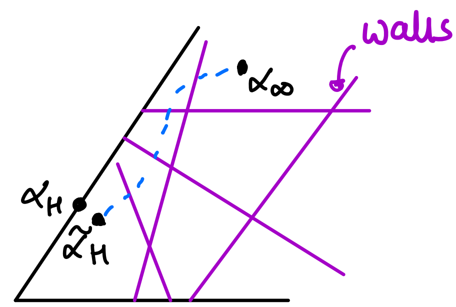
* Thm [García-Prada, Heinloth, Schmitt]

$\exists$ path $\{\alpha_t\}_{t \geq 0}$ in $\overset{\circ}{\Delta}_r$ from $\tilde{\alpha}_H$ to $\alpha_\infty = \alpha_t$ for $t \gg 0$ s.t

i) If $\underline{m}$ is non-constant, $Ch_{\underline{m},\underline{e}}^{\alpha_\infty\text{-ss}} = \phi$

ii) If $\underline{m}$ is constant, $Ch_{\underline{m},\underline{e}}^{\alpha_\infty\text{-ss}} \subset Ch_{\underline{m},\underline{e}}^{inj}$ $\leftarrow$ stack of injective chains

$\nwarrow$ union of α_∞-HN strata



* Gysin triangles for Harder-Narasimhan strata at each wall-crossing
+ HN-recursion.

$\rightsquigarrow$ suffices to show $M(Ch_{\underline{m},\underline{e}}^{inj}) \in \langle\langle M(C) \rangle\rangle$

③ Explicit formula for the motive of the stack of injective chains

Inspired by work of Heinloth & Laumon (on cohomology) using stacks of Hecke modifications & motivic descriptions of small maps (de Cataldo - Migliorini)

Thm [H.-Pepin Lehalleur]

For $\underline{m} = (m_0, \dots, m_r)$ constant rank vector

$$M(\mathrm{Ch}^{inj}_{\underline{m},\underline{e}}) \simeq \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{l_i}(C \times \mathbb{P}^{n-1})) \otimes M(\mathrm{Bun}_{m_r, e_r}) \text{ in } DM(k, \mathbb{Q})$$

stack of vector bundles

where $l_i = e_i - e_{i-1}$

Proof: Based on ideas of Heinloth, Laumon and de Cataldo & Migliorini

$$\begin{tikzcd}
\mathrm{Ch}^{inj}_{\underline{m},\underline{e}} \arrow[r, "\mathrm{gr}\ (\text{smooth})"] & \prod_{i=1}^{r} \mathrm{Coh}_{0,l_i} \times \mathrm{Bun}_{m_r,e_r} \arrow[r, "\mathrm{supp}"] & \prod_{i=1}^{r} C^{(l_i)} \times \mathrm{Bun}_{m_r,e_r} \\
\widetilde{\mathrm{Ch}}^{inj}_{\underline{m},\underline{e}} \arrow[u, "\substack{\text{small} \\ [\text{Heinloth}]}"] \arrow[r, "\widetilde{\mathrm{gr}}"] \arrow[ru, phantom, "\Leftarrow"] \arrow[rr, bend right, "(\Delta)"'] & \prod_{i=1}^{r} \widetilde{\mathrm{Coh}}_{0,l_i} \times \mathrm{Bun}_{m_r,e_r} \arrow[u, "\substack{\text{small map} \\ [\text{Laumon}]}"] \arrow[r, "\widetilde{\mathrm{supp}}"] & \prod_{i=1}^{r} C^{l_i} \times \mathrm{Bun}_{m_r,e_r} \arrow[u]
\end{tikzcd}$$

$(F_0 \hookrightarrow \dots \hookrightarrow F_r) \longmapsto (F_1/F_0, \dots, F_r/F_{r-1}, F_r)$

$\widetilde{\mathrm{Ch}}^{inj}_{\underline{m},\underline{e}} = \{F_0 \subset F_0^1 \subset \dots \subset F_0^{l_1} = F_1 \subset \dots F_{r-1}^{l_r} = F_r\}$

elementary Hecke modifications

$\underbrace{\widetilde{\mathrm{Coh}}_{0,l_i}}_{} = \{0 \subset T_1 \subset \dots \subset T_{l_i} : T_j \in \mathrm{Coh}_{0,j}\}$

$\sum_{i=1}^{r} l_i$-iterated $\mathbb{P}^{m-1}$-bdle

(A Hecke modification of a rk m bdle F at $p \in C(k)$ $\Leftrightarrow$ $F_p \twoheadrightarrow k$)

$$(\Delta) \Rightarrow M(\widetilde{Fh}^{inj}_{\underline{m},\underline{e}}) \simeq M(\mathrm{Bun}_{m_r,e_r}) \times \bigotimes_{i=1}^{r} M(C \times \mathbb{P}^{m-1})^{\otimes \ell_i}$$

Laumon & Heinloth: These small maps are torsors under $\prod_{i=1}^{r} S_{\ell_i}$ on a dense open

where $\forall i$ $\mathrm{supp}(F_i/F_{i-1})$ consists of ℓ_i distinct pts

Following ideas of de Cataldo & Migliorini:

$$M(Fh^{inj}_{\underline{m},\underline{e}}) = M(\widetilde{Fh}^{inj}_{\underline{m},\underline{e}})^{\prod_{i=1}^{r} S_{\ell_i}} \underset{(\Delta)}{\simeq} M(\mathrm{Bun}_{m_r,e_r}) \otimes \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{\ell_i}(C \times \mathbb{P}^{m-1}))$$

Here we need to work with $\mathbb{Q}$-coefficients (to take invariant piece)

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④ Explicit formula for the stack Bun of vector bundles:

<u>Thm</u> [H-Pepin Lehalleur '18] Assume $C(k) \neq \emptyset$. For any m, e:

$$M(\mathrm{Bun}_{m,e}) \simeq M(\mathrm{Jac}\, C) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{m-1} \underbrace{Z(C, \mathbb{Q}\{i\})}_{\shortparallel \; \bigoplus_{j \geq 0} M(\mathrm{Sym}^j(C))\{ij\}}$$

In particular, $M(\mathrm{Bun}_{m,e}) \in \langle\langle M(C) \rangle\rangle$.

<u>Idea of proof</u>: Rigidify using ind-schemes built from flag-Quot schemes

fix $x \in C(k) \neq \emptyset$

$$\mathrm{Bun}_{m,e} \longleftarrow \mathrm{Div}_{m,e} = \lim_{\ell} \mathrm{Div}_{m,e}(\ell x) \longleftarrow \mathrm{FDiv}_{m,e} = \lim_{\ell} \mathrm{FDiv}_{m,e}(\ell x)$$

almost a vector bundle

Quot scheme of torsion quotients

small map and torsor under a symmetric group for each ℓ $\rightsquigarrow$ similar to step ③.

Flag-Quot scheme

$$E \longleftarrow\!\shortmid \; (E \hookrightarrow \mathcal{O}(\ell x)^{\oplus m} \twoheadrightarrow T)$$