Motives of moduli spaces of Higgs bundles

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joint work with
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§1 Moduli spaces of Higgs bundles

C/k smooth projective connected alg. curve with $C(k) \ni x$

Fix a rank n & degree d s.t $(n,d)=1$

Moduli spaces:

- $\mathcal{N}_{n,d}$ moduli space of stable rank n degree d vector bundles on C

smooth proj. variety of dim $n^2(g-1)+1$

$T^*_{[E]}\mathcal{N}_{n,d} \simeq \mathrm{Ext}^1(E,E)^* \simeq \mathrm{Hom}(E, E\otimes\omega_C) \leftarrow$ Higgs fields

RR as $\mathrm{Hom}(E,E)=k$; def. theory ; S.D.

- $\mathcal{M}_{n,d}$ moduli space of stable rank n degree d Higgs bundles on C $\quad (E, \Phi: E \to E\otimes\omega_C)$

$\mathcal{M}_{n,d} \hookleftarrow T^*\mathcal{N}_{n,d}$

smooth q-proj variety $\quad \dim\mathcal{M} = 2\dim\mathcal{N}$

Geometric features:

* Over $k=\mathbb{C}$, $\mathcal{M}$ is a non-compact hyperkähler mfld (via its gauge theor. construction)
In general $\mathcal{M}$ is algebraic symplectic.

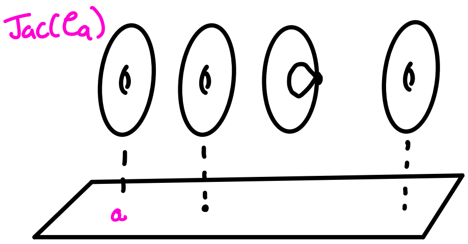
* Although non-compact, it cohomologically behaves like its compact due to the existence of a "semi-projective" scaling action: $\mathbb{G}_m = k^* \curvearrowright \mathcal{M}$
$t\cdot[E,\Phi]=[E,t\Phi]$
inducing a deformation retract to the fixed locus, which is compact

* Hitchin fibration:

$$\begin{array}{ccc} \mathcal{M} & & (E,\Phi) \\ \downarrow & & \downarrow \text{ proper} \\ \mathcal{A} & & \mathrm{Char}(\Phi) \end{array}$$

completely integrable Hamiltonian system

affine space $\to \mathcal{A}$



generic fibres are Jacobians

* Spectral correspondence (Beauville-Narasimhan-Ramanan)
loc. free Higgs sheaves on $C \leftrightsquigarrow$ pure sheaves on T^*C supported on a finite cover of C

* Non-abelian Hodge correspondence ($k = \mathbb{C}$)

For $d = 0$:

$$\underset{\text{flat connections}}{\mathcal{M}^{dR}_{n,d}} \overset{\text{homeos}}{\simeq} \mathcal{M}_{n,0} \simeq \mathcal{M}^{\text{Betti}}_{n,0} = \mathrm{Hom}(\pi_1(C), GL_n) /\!/ GL_n \quad \text{local systems}$$

For d coprime to n, get smooth moduli spaces

$\mathcal{M}^{dR}_{n,d}$ moduli space of semistable logarithmic connections on (C,x) of rk n, deg d w/ residue $-\frac{d}{n}$ at x

$$\mathcal{M}^{dR}_{n,d} \underset{C^\infty}{\simeq} \mathcal{M}_{n,d} := \mathcal{M}^{Dol}_{n,d} \underset{C^\infty}{\simeq} \mathcal{M}^{\text{Betti}}_{n,d}$$

$\mathcal{M}^{\text{Betti}}_{n,d}$ moduli space of twisted semisimple representations of $\pi_1(C,x)$

Eg For an elliptic curve E: $\mathcal{N}_{1,0} = \mathrm{Jac}(E)$

$$\mathcal{M}_{1,0} \cong T^* \mathcal{N}_{1,0} \cong E \times \mathbb{C} \underset{\text{homeo}}{\simeq} (S^1)^2 \times \mathbb{R}^2 \simeq (S^1 \times \mathbb{R}^2)^2$$

$$\mathcal{M}^{\text{Betti}}_{1,0} = \mathrm{Hom}(\pi_1(E), \mathbb{C}^*) /\!/ \mathbb{C}^* = (\mathbb{C}^*)^2 \underset{\text{homeo}}{\simeq} (S^1 \times \mathbb{R})^2$$

* (Over $k = \mathbb{C}$) Schiffmann computed the Betti no's of $\mathcal{M}_{n,d}$ by counting absolutely indecomposable vector bundles on curves over finite fields

* Versions for other reductive groups $G \neq GL_n$
+ mirror symmetry predictions for Higgs bundles under Langland's dual groups

Goal: Describe the motives of these moduli spaces

<u>motive</u>: in the sense of Grothendieck
(realised by Voevodsky's triangulated category $DM(k, \mathbb{Q})$)

* encodes cohomology groups:
  - ($k = \mathbb{C}$) singular cohomology + mixed Hodge structure
  - ℓ-adic cohomology + Galois representation
* and algebraic cycles (Chow groups)

Results:

Thm 1 [H-Pepin Lehalleur]

The motive $M(\mathcal{M}^{Dol}_{n,d}) \in DM(k,\mathbb{Q})$ lies in the thick tensor subcategory $\mathcal{C}$ generated by the motive $M(C)$ of C.

Following geometric techniques of Hitchin, García-Prada-Heinloth-Schmitt

Corollary

i) $M(\mathcal{M}^{Dol}_{n,d})$ is a pure abelian motive. (so we can view this as a Chow motive)

ii) $M(\mathcal{M}^{Dol}_{n,d})$ is a direct factor of $M(C^m)$ for $m \gg 0$.

iii) $k=\mathbb{F}_q$: The e-values of $Fr \curvearrowright H^*(\mathcal{M}^{Dol}_{n,d}, \mathbb{Q}_\ell)$ are monomials in the Weil numbers of C.

Rmk: In [Fu-H.-Pepin Lehalleur], we show

1) For a general curve $C/\mathbb{C}$, the motive of the (twisted) SL_n-Higgs m. space $\mathcal{M}_{n,L}$ (fixed determinant L, trace-free Higgs field) is not generated by C: $M(\mathcal{M}_{n,L}) \notin \mathcal{C}$.

2) We compute formulae for $\mathcal{N}$ & $\mathcal{M}$ in low rank $n=2$ & 3

$$M(\mathcal{N}_{2,d}) \simeq M(Jac(C)) \otimes \left(\bigoplus_{i=0}^{g-2} M(Sym^i(C)) \otimes (\mathbb{Q}\{i\} \oplus \mathbb{Q}\{3g-3-2i\}) \oplus M(Sym^{g-1}(C))\{g-1\} \right)$$

Tate twists

$$M(\mathcal{N}_{3,d}) \simeq M(Jac(C)) \otimes \left(\bigoplus_{\substack{i,j \geq 0 \\ i+j \leq 2g-2}} M(Sym^i(C)) \otimes M(Sym^j(C)) \otimes T_{i,j} \right)$$

explicit sum of Tate twists

$$M(\mathcal{M}_{2,d}) \simeq M(\mathcal{N}_{2,d}) \oplus \bigoplus_{j=1}^{g-1} M(Jac(C)) \otimes M(Sym^{2j-1}(C))\{3g-2j-2\}$$

$$M(\mathcal{M}_{3,d}) \simeq M(\mathcal{N}_{3,d}) \oplus \bigoplus_{i \in I} M(J_C) \otimes M(C^{(i)}) \otimes T_i \oplus \bigoplus_{(i,j) \in J} M(J_C) \otimes M(C^{(i)} \times C^{(j)}) \otimes T_{i,j}$$

$Sym^i(C)$

$Jac(C)$

explicit sums of Tate twists

Generalises cohomological results of Hitchin ($n=2$) and Gothen ($n=3$).

Thm 2 [H-Pepin Lehalleur] (Motivic non-abelian Hodge correspondence)

In Char $k=0$, we have $M(\mathcal{M}^{Dol}_{n,d}) \simeq M(\mathcal{M}^{dR}_{n,d})$ in $DM(k, \underset{\text{integral coeff.}}{\mathbb{Z}})$

Consequently these have isomorphic Chow rings.

Rmk: $M(\underset{\text{pure}}{\mathcal{M}^{Dol}_{n,d}}) \neq M(\mathcal{M}^{Betti}_{n,d})$ ← not a pure motive (non-trivial weight filtration)

⇝ P = W conjecture

Thm 3 [H-Pepin Lehalleur]

The motive of the SL_n-Higgs moduli space $\mathcal{M}_{n,L}$ for $(n, \deg L) = 1$ is pure and abelian.

The proofs of these theorems start with Hitchin's scaling action.

Thm 4 [H-Pepin Lehalleur] (Motivic mirror symmetry for SL & PGL-Higgs bdles)

$M(\mathcal{M}_{SL_n}) \simeq M_{orb}(\mathcal{M}_{PGL_n})$ ↖ orbifold motive wrt a gerbe

in $DM(k, \mathbb{Q}(\zeta_n))$

$\mathcal{M}_{SL_n} := \mathcal{M}_{n,L}$ as above ($\deg L$ & n coprime)

$\mathcal{M}_{PGL_n} := [\mathcal{M}_{n,L} / Jac(C)[n]]$

The proof uses a cohomological version of Maulik-Shen, involving correspondences & vanishing cycles (+perverse sheaves & decomp. thm), with conservativity of the Betti realisation for abelian motives.

§2 Motives

Grothendieck: envisaged motives as a universal coh theory for k-varieties not necessarily smooth or projective.

Voevodsky: There is a category $DM(k,\mathbb{Q})$ of mixed motives $/k$ with $\mathbb{Q}$-coefficients together with a functor

$$M: Var_k \longrightarrow DM(k,\mathbb{Q})$$
$$X \longmapsto M(X)$$

realising part of Grothendieck's vision.

[$DM(k,\mathbb{Q})$ is a tensor triangulated category constructed from $D^b(Sh(SmVar_k, \mathbb{Q}))$ by imposing certain axioms.]

Expectation: $DM(k,\mathbb{Q})$ = derived category of an abelian category of mixed motives

Properties

* Künneth isomorphism $\quad M(X\times Y)\simeq M(X)\otimes M(Y)$

* $\mathbb{A}^1$-homotopy invariance: $\underset{\text{vector bundle}}{E\to X} \rightsquigarrow M(E)\simeq M(X)$

* Projective bundle formula: $M(\mathbb{P}(E))\simeq M(X)\otimes M(\mathbb{P}^{n-1})$ $\quad n=\mathrm{rk}(E)$

$$M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Q}\{i\} \leftarrow \text{Tate twists}$$

* Gysin triangles: for $\underset{\text{codim } c}{Z\hookrightarrow X}$ both smooth k-varieties

$$M(X\setminus Z)\longrightarrow M(X)\longrightarrow M(Z)\{c\}\xrightarrow{+1}$$

* Chow gps: X smooth k-variety

$$CH^i(X)_{\mathbb{Q}} \simeq \mathrm{Hom}_{DM}(M(X),\mathbb{Q}\{i\})$$

* Realisation functors:

Betti / de Rham / ℓ-adic cohomology factor via $M: \mathrm{Var}_k\to DM$
$+$ MHS $\quad\quad$ $+$ Galois rep.

§3 Hitchin's scaling action

Idea: Use the $\mathbb{G}_m$-action on $\mathcal{M}^{Dol}_{n,d}$ $\quad$ scaling the Higgs field

$$t\cdot[E,\Phi] := [E,t\Phi]$$

to stratify $\mathcal{M}^{Dol}_{n,d}$ into simpler pieces

Key properties of this action [Hitchin, Simpson]

* fixed locus is projective
* flow as $t\to 0$ exists for all points

} "semi-projective $\mathbb{G}_m$-action" [Hausel]

Fixed points: $\Phi=0$ or $\exists\ \mathbb{G}_m\subset \mathrm{Aut}(E) \rightsquigarrow E=\bigoplus_i E_i$ $\quad$ weight decomposition

$\downarrow$ $\mathcal{N}_{n,d}$ $\quad$ $\downarrow$ "chain" $\quad E_0\to E_1\otimes\omega_C\to E_2\otimes\omega_C^{\otimes 2}\to\cdots$

Moduli of chains [Alvarez-Consul, García-Prada, Schmitt...]

A chain $F_\bullet = (F_0 \to F_1 \to \cdots \to F_r)$ is <u>semistable</u> wrt $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{R}^{r+1}$ if $\mu_\alpha(F_\bullet') \leq \mu_\alpha(F_\bullet) := \dfrac{\sum_i \deg F_i + \alpha_i \operatorname{rk} F_i}{\sum_i \operatorname{rk} F_i}$ for all $F_\bullet' \subset F_\bullet$

i) Every chain has a ! α-Harder-Narasimhan filtration

$\rightsquigarrow$ α-HN Stratification on the stack of all chains: with ranks $\underline{n}$ & degrees $\underline{d}$

$$\mathcal{C}h_{\underline{n},\underline{d}} = \coprod_\tau \mathcal{C}h_{\underline{n},\underline{d}}^{\alpha,\tau}$$

ii) $\exists$ projective moduli spaces $Ch_{\underline{n},\underline{d}}^{\alpha\text{-ss}}$ of α-ss chains

smooth if α-ss $= \alpha$-s and $\alpha \in \Delta_r \subset \mathbb{R}^{r+1}$
$\Delta_r = \{\alpha : \alpha_i - \alpha_{i+1} \geqslant 2g-2\}$

iii) $\exists$ Higgs stability parameter

$$\alpha_H = (r(2g-2), \ldots, (2g-2), 0) \in \partial\Delta_r$$

s.t

$$\left(\mathcal{M}_{n,d}^{Dol}\right)^{\mathbb{G}_m} = \mathcal{N}_{n,d} \sqcup \coprod_{(\underline{m},\underline{e})} Ch_{\underline{m},\underline{e}}^{\alpha_H\text{-ss}}$$

Back to our $\mathbb{G}_m$-action:

[Bialynicki-Birula] Any semi-projective $\mathbb{G}_m$-action on a smooth quasi-proj. variety X induces a BB-decomposition: $X = \coprod_{i=1}^m X_i^+$ where

$$X^{\mathbb{G}_m} = \coprod_{i=1}^m X_i \quad \text{and} \quad X_i^+ = \{x \in X : \lim_{t\to 0} t\cdot x \in X_i\} \xrightarrow[\mathbb{A}^{r_i}\text{-fibration}]{} X_i$$

Moreover: the Gysin l.e.s on Betti cohomology associated to this decomposition split $\rightsquigarrow$ Betti no.s of X can be computed from those of X_i

Furthermore: get a motivic Bialynicki-Birula decomposition!

§4 Sketch of Theorem 1

<u>Thm 1</u>: $M(\mathcal{M}_{n,d}) \in \mathcal{C} := \langle M(C) \rangle^{\otimes} \subset DM(k, \mathbb{Q})$

Geometric strategy: Hitchin, García-Prada-Heinloth-Schmitt

Step 1: Hitchin's scaling action $\mathbb{G}_m \curvearrowright \mathcal{M}_{n,d}$ gives a motivic BB decomposition [Brosnan, Karpenko]

$$M(\mathcal{M}_{n,d}) \simeq \bigoplus_{(\underline{m},\underline{e})\in I} M(Ch^{\alpha_H\text{-}ss}_{\underline{m},\underline{e}})\{codim_{\underline{m},\underline{e}}\}$$

$\rightsquigarrow$ suffices to show $M(Ch^{\alpha_H\text{-}ss}_{\underline{m},\underline{e}}) \in \mathcal{C}$

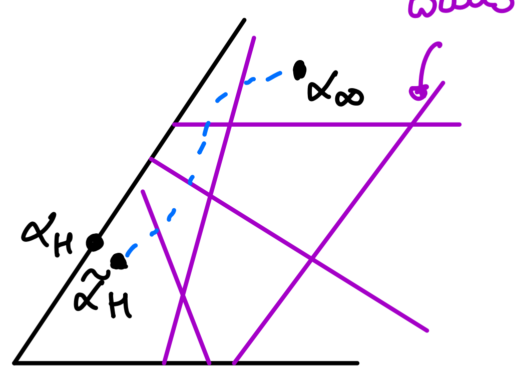
* Motivic BB decompositions are also used to prove Thm 4 (motivic NAHT)

Step 2: $\mathcal{C}h^{\alpha_H\text{-}ss}_{\underline{m},\underline{e}} = \mathcal{C}h^{\alpha_H\text{-}s}_{\underline{m},\underline{e}} \xrightarrow[\text{trivial } \mathbb{G}_m\text{-gerbe}]{} Ch^{\alpha_H\text{-}ss}_{\underline{m},\underline{e}}$ $\left(M(B\mathbb{G}_m) = \bigoplus_{j\geq 0} \mathbb{Q}\{j\}\right)$

stack

$\rightsquigarrow$ suffices to show $M(\mathcal{C}h^{\alpha_H\text{-}ss}_{\underline{m},\underline{e}}) \in \mathcal{C}$ (or technically its ind-completion)

Step 3: Wall-crossing & Harder-Narasimhan recursion



Thm [García-Prada – Heinloth – Schmitt]

$\exists$ path $\{\alpha_t\}_{t\geq 0}$ in Δ°_r from $\tilde{\alpha}_H$ to $\alpha_\infty = \alpha_t$ for $t \gg 0$ s.t

i) If $\underline{m}$ is non-constant, $\mathcal{C}h^{\alpha_\infty\text{-}ss}_{\underline{m},\underline{e}} = \phi$

ii) If $\underline{m}$ is constant, $\mathcal{C}h^{\alpha_\infty\text{-}ss}_{\underline{m},\underline{e}} \subset \mathcal{C}h^{inj}_{\underline{m},\underline{e}}$ $\leftarrow$ injective chain homomorphisms

$\nwarrow$ union of α_∞-HN strata

At each wall-crossing, the semistable loci are related by finitely many Gysin triangles involving higher HN strata.

By a HN recursion (for chains)

$\rightsquigarrow$ suffices to show $M(\mathcal{C}h^{inj}_{\underline{m},\underline{e}}) \in \mathcal{C}$

$\underline{m}$ constant

Step 4: Explicit formula for $M(\mathcal{C}h^{inj}_{\underline{m},\underline{e}})$ $\quad$ $\underline{m} = (m, \dots, m)$ constant with $\mathbb{Q}$-coeffs

using stacks of Hecke modifications and motivic descriptions of certain small maps

$\hookrightarrow$ Following ideas of Laumon & Heinloth in cohomology

Thm 6: For $\underline{m}=(m_0,\dots,m_r)$ constant, the forgetful maps

$$\mathcal{C}h^{inj}_{\underline{m}\geq 0,\underline{e}\geq 0} \longrightarrow \mathcal{C}h^{inj}_{\underline{m}\geq 1,\underline{e}\geq 1} \longrightarrow \cdots\cdots \longrightarrow \mathcal{C}h^{inj}_{\underline{m}\geq r,\underline{e}\geq r} = \mathrm{Bun}_{m_r,e_r}$$

$$(E_0\to E_1\to\cdots\to E_r) \mapsto (E_1\to\cdots\to E_r) \mapsto \cdots\cdots \mapsto E_r$$

are Hecke modification stacks. Moreover,

$$M(\mathcal{C}h^{inj}_{\underline{m},\underline{e}}) \simeq \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{e_i-e_{i-1}}(C\times\mathbb{P}^{n-1}))\otimes M(\mathrm{Bun}_{m_r,e_r})$$

Proof: Based on ideas of Heinloth, Laumon and de Cataldo & Migliorini

$l_i = e_i - e_{i-1}$

$$\begin{tikzcd}
\mathcal{C}h^{inj}_{\underline{m},\underline{e}} \arrow[r, "gr\ (\text{smooth})"] & \prod_{i=1}^{r} \mathcal{C}oh_{0,l_i}\times \mathrm{Bun}_{m_r,e_r} \arrow[r, "\mathrm{supp}"] & \prod_{i=1}^{r} C^{(l_i)}\times \mathrm{Bun}_{m_r,e_r} \\
\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e}} \arrow[u, "\text{small [Heinloth]}"'] \arrow[r, "\widetilde{gr}"] & \prod_{i=1}^{r} \widetilde{\mathcal{C}oh}_{0,l_i}\times \mathrm{Bun}_{m_r,e_r} \arrow[u, "\text{small map [Laumon]}"'] \arrow[r, "\widetilde{\mathrm{supp}}"] & \prod_{i=1}^{r} C^{l_i}\times \mathrm{Bun}_{m_r,e_r} \arrow[u]
\end{tikzcd}$$

$\ni (F_0\hookrightarrow\cdots\hookrightarrow F_r) \longmapsto (F_1/F_0,\dots,F_r/F_{r-1},F_r) \in$

small

$\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e}} = \{F_0\subset F_0^1\subset\cdots\subset F_0^{l_1}=F_1\subset\cdots F_{r-1}^{l_r}=F_r\}$

$\sum_{i=1}^{r} l_i$ elementary Hecke modifications

$\widetilde{\mathcal{C}oh}_{0,l_i} = \{0\subset T_1\subset\cdots\subset T_{l_i} : T_j\in \mathcal{C}oh_{0,j}\}$

$\sum_{i=1}^{r} l_i$-iterated $\mathbb{P}^{m-1}$-bdle (A Hecke modification of a rk m bdle F at $p\in C(k)$ $\Longleftrightarrow$ $F_p\twoheadrightarrow k$)

$$\Rightarrow M(\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e}}) \simeq M(\mathrm{Bun}_{m_r,e_r}) \times \bigotimes_{i=1}^{r} M(C\times\mathbb{P}^{m-1})^{\otimes l_i}$$

Moreover, these small maps are torsors under $\prod_{i=1}^{r} S_{l_i}$ on a dense open where $\forall i$ $\mathrm{supp}(F_i/F_{i-1})$ consists of l_i distinct pts

Following ideas of de Cataldo & Migliorini:

$$M(\mathcal{C}h^{inj}_{\underline{m},\underline{e}}) = M(\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e}})^{\prod_{i=1}^{r} S_{l_i}} \simeq M(\mathrm{Bun}_{m_r,e_r})\otimes\bigotimes_{i=1}^{r} M(\mathrm{Sym}^{l_i}(C\times\mathbb{P}^{m-1})) \qquad \blacksquare$$

$\rightsquigarrow$ suffices to show $M(\mathrm{Bun}_{m,e})\in\mathcal{C}$

Step 5: Explicit formula for $M(\mathrm{Bun}_{m,e})$

<u>Thm</u> [H-Pepin Lehalleur] Assume $C(k) \neq \phi$. For any m, e:

$$M(\mathrm{Bun}_{m,e}) \simeq M(\mathrm{Jac}\, C) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{m-1} \underbrace{Z(C, \mathbb{Q}\{i\})}_{= \bigoplus_{j \geq 0} M(\mathrm{Sym}^j(C))\{ij\}}$$

In particular, $M(\mathrm{Bun}_{m,e}) \in \mathcal{C} = \langle M(C) \rangle^{\otimes}$.

The proof involves rigidifications by ind-(flag)-Quot schemes of matrix divisors (after Bifet-Ghione-Letizia)

$$\begin{tikzcd} \mathrm{Bun}_{m,e} & \mathrm{Div}_{m,e} \arrow[l] & \mathrm{FDiv}_{m,e} \arrow[l] \end{tikzcd}$$

stack of vector bundles

"almost" vector bundle

ind-Quot scheme of matrix divisors

small map & torsor for symmetric group on large open

ind-flag-Quot scheme (Hecke modifications)

We compute $M(\mathrm{Bun}_{m,e}) \simeq M(\mathrm{Div}_{m,e})$ from $M(\mathrm{FDiv}_{m,e})$ in a similar way to Step 4.

∎

§ Proof of Theorem 2

inspired by work of Nakajima & Hausel

Thm 2 is a corollary of:

Thm 5 [H.-Pepin Lehalleur]

For a semi-proj $\mathbb{G}_m$-action on a smooth quasi-projective variety X with a smooth $\mathbb{G}_m$-equivariant map $f : X \longrightarrow \mathbb{A}^1 \circlearrowleft \mathbb{G}_m$ with positive wt we have $M(X_\lambda) \simeq M(X)$ in $DM(k,\mathbb{Z})$ $\forall \lambda \in k$

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{A}^1 \\ X_\lambda \arrow[u, hook] \arrow[r] & \lambda \arrow[u, phantom, "\in"] \end{tikzcd}$$

Application: Deligne-Simpson Hodge moduli space of ss log λ-connections

$$\begin{tikzcd} \mathbb{G}_m \circlearrowright \mathcal{M}^{Hodge}_{n,d} \arrow[d, "\text{smooth } \mathbb{G}_m\text{-equiv}"'] & \mathcal{M}^{Dol}_{n,d} \arrow[d] & \mathcal{M}^{dR}_{n,d} \arrow[d] & \simeq \mathcal{M}^{\lambda\text{-}dR}_{n,d} \arrow[d] \\ \mathbb{A}^1 & 0 & 1 & \lambda \neq 0 \end{tikzcd}$$

$\Rightarrow$ Thm 2.

Proof of Thm 5:

- $\lambda = 0$ $\quad X_0 \hookrightarrow X$ $\mathbb{G}_m$-equivariant & BB "transverse"

  & $X_0^{\mathbb{G}_m} = X^{\mathbb{G}_m} = \coprod_{i=1}^{m} X_i$ $\qquad X_{0,i}^{+} = X_0 \cap X_i^{+}$

  Comparing the BB decompositions $\quad c_i = \mathrm{codim}(X_{0,i}^{+}, X_0) = \mathrm{codim}(X_i^{+}, X)$

$$M(X_0) \simeq \bigoplus M(X_i)\{c_i\} \simeq M(X)$$

- $\lambda \neq 0$: suffices to take $\lambda = 1$

$$\begin{tikzcd} X_1 \times \mathbb{G}_m \arrow[d, hook, "i_1 \times id"'] \arrow[r, phantom, "\simeq"] & X \setminus X_0 \arrow[d, "(id,f)|_{X\setminus X_0} = F"] \\ X \times \mathbb{G}_m \arrow[r, "\sim", "\mathbb{G}_m\text{-action}"'] & X \times \mathbb{G}_m \\ (x,t) \arrow[r, mapsto] & (t\cdot x, t) \end{tikzcd}$$

$\Rightarrow X_1 \cong X_\lambda \quad \forall \lambda \neq 0$

To show $M(i_1)$ is an iso, it suffices to show $M(F)$ is an iso. For this,

$$\begin{tikzcd} X_0 \arrow[r, hook, "i_0"] \arrow[d, hook, "i_0"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "(id,f)"] & X \setminus X_0 \arrow[l, hook'] \arrow[d, "F"] \\ X \arrow[r, hook] & X \times \mathbb{A}^1 & X \times \mathbb{G}_m \arrow[l, hook'] \end{tikzcd}$$

induces a morphism of Gysin Δs

$$\begin{tikzcd} M(X\setminus X_0) \arrow[r] \arrow[d, "\wr", "M(F)"'] & M(X) \arrow[r] \arrow[d, "\wr", "\mathbb{A}^1\text{-homo inv.}"'] & M(X_0)\{1\} \arrow[r, "+1"] \arrow[d, "\wr", "\lambda=0 \text{ case}"'] & {} \\ M(X \times \mathbb{G}_m) \arrow[r] & M(X \times \mathbb{A}^1) \arrow[r] & M(X)\{1\} \arrow[r, "+1"] & {} \end{tikzcd}$$

$\Leftarrow$

∎

§ Explicit formulas

: determine chain invariants $\underline{m}, \underline{e}$ in $(\mathcal{M}_{n,d})^{\mathbb{G}_m}$

Rank 2: • $\underline{m} = (2)$ & $\underline{e} = (d)$ $\rightsquigarrow$ $\mathcal{N}_{2,d}$

(Hitchin) • $\underline{m} = (1,1)$ & $\underline{e} = (e_1, e_2)$ $e_2 = d - e_1 + 2g - 2$

stability & $\phi \neq 0$ $\Rightarrow$ finitely many possible values of e_1

$L_0 \xrightarrow[\phi]{} L_1 \otimes \omega_C$ $\rightsquigarrow$ param by $\mathrm{Pic}^{e_1}(C) \times C^{(e_2 - e_1)}$

Tate twist = codim of BB stratum (calculate using fact that downward flow is Lagrangian)

Rmk: BB decomposition for $\mathcal{M}_{2,L} = \mathcal{N}_{2,L} \amalg \coprod_{j=1}^{g-1} \widetilde{C}^{(2j-1)}$ [Hitchin]

where

$$\begin{tikzcd} \widetilde{C}^{(j)} \arrow[r, "\deg\ 2^{2g}"', "\text{étale cover}" below] \arrow[d] & C^{(j)} \arrow[d] \\ \mathrm{Jac}(C) \arrow[r, "\cdot 2"] & \mathrm{Jac}(C) \end{tikzcd}$$

$\rightsquigarrow$ $h(\mathcal{M}_L(2,d)) \in \langle h(\widetilde{C}) \rangle^{\otimes}$

For a general complex curve C: $\langle h(C) \rangle^{\otimes} \subsetneq \langle h(\widetilde{C}) \rangle^{\otimes}$

Rank 3: • $\underline{m} = (3)$ & $\underline{e} = (d)$ $\rightsquigarrow$ $\mathcal{N}_{3,d}$

(Gothen)

finitely many possibilities in each case

• $\underline{m} = (1,1,1)$ & $\underline{e} = (e_1, e_2, e_3)$ $e_1 + e_2 + e_3 = d + 6(g-1)$

$L_0 \to L_1 \otimes \omega_C \longrightarrow L_2 \otimes \omega_C^{\otimes 2}$

$\rightsquigarrow$ param. by $\mathrm{Pic}^{e_0}(C) \times C^{(e_2 - e_1)} \times C^{(e_3 - e_2)}$

• $\underline{m} = (1,2)$ & $\underline{e} = (e_1, e_2)$

$L \to E \otimes \omega_C$ $\rightsquigarrow$ rk 2 pair $(F = E \otimes \omega_C \otimes L^{-1}, \phi)$
  (E: rk 2; ϕ: section)

[Bradlow, Thaddeus]

$\rightsquigarrow$ param. by $\mathrm{Pic}^{e_1}(C) \times \underbrace{\mathcal{P}^{\sigma\text{-ss}}_{2,f}}$ ⟵ moduli space of semistable pairs of rk 2 degree f

Motives computed by pairs wall-crossings (explicit flips)
(varying σ)

(Easy in $K_0(\mathrm{Var})$ but harder in DM, where we can't "subtract")

• $m = (2,1)$ & $\underline{e} = (e_1, e_2)$ (dual to $\underline{m} = (1,2)$)

$F \to L \otimes \omega_C$ $\rightsquigarrow$ param by $\mathrm{Jac}(C) \times \mathcal{P}^{\sigma\text{-ss}}_{2,f}$

§ Outline of Theorem 3

Thm 2 $M(\mathcal{M}_{n,L})$ is pure and abelian, but not generated by $M(C)$.
$\uparrow$ SL_n-Higgs moduli space $\qquad (n, \deg L) = 1$

Step 1: (As for GL_n-Higgs bdles) $\mathbb{G}_m \curvearrowright \mathcal{M}_{n,L} \rightsquigarrow$ motivic BB decomposition

$$M(\mathcal{M}_{n,L}) \simeq \bigoplus_{(\underline{m},\underline{e}) \in I} M(Ch^{\alpha_H\text{-ss}}_{\underline{m},\underline{e},L'})\{\mathrm{codim}_{\underline{m},\underline{e}}\}$$

smooth projective varieties

$$Ch^{\alpha_H\text{-ss}}_{\underline{m},\underline{e},L'} := \{F_0 \to \cdots \to F_r \in Ch^{\alpha_H\text{-ss}}_{\underline{m},\underline{e}} : \det(\bigoplus_{i=0}^{r} F_i) \simeq L' := L \otimes \omega_C^{\otimes N}\} \hookrightarrow Ch^{\alpha_H\text{-ss}}_{\underline{m},\underline{e}}$$

$N = \sum_{i=0}^{r} i m_i$

$\mathbb{G}_m$-fixed Higgs bdle $(E, \Phi) \rightsquigarrow$ Chain $F_0 \to \cdots \to F_r$
$E = \bigoplus_{i=0}^{r} E_i$ with $\det(E) \simeq L$ $\qquad F_i = E_i \otimes \omega_C^{\otimes i}$

Total det. of chain fixed (rather than fixing termwise determinants)

$\rightsquigarrow M(\mathcal{M}_{n,L})$ is pure. It suffices to show $M(Ch^{\alpha_H\text{-ss}}_{\underline{m},\underline{e},L})$ is abelian

Step 2 & 3 As for GL_n-Higgs bdles (wall-crossing & HN recursions on stack)

$\rightsquigarrow$ suffices to show $M(Ch^{inj}_{\underline{m},\underline{e},L})$ is abelian
for a constant tuple of ranks

Step 5 $M(Bun_{m,L}) \simeq M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{m-1} Z(C, \mathbb{Q}\{i\})$ generated by $M(C)$

Relative Statement: For a trivial family of curves $C \times S / S$ & $\mathcal{L} \in Pic_{C\times S/S}$

$$M(Bun_{C\times S/S, m, \mathcal{L}}) \simeq M(S) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{m-1} Z(C, \mathbb{Q}\{i\})$$

Step 4 $M(Ch^{inj}_{\underline{m},\underline{e},L})$ is not generated by C but is abelian
Constant ranks $\qquad$ Total determinant $\simeq L$

Take the diagram we had for the GL_n-case above & add determinant maps, then take the fibres at $L \in Pic(C)$:

$(F_0 \hookrightarrow \cdots \hookrightarrow F_r) \longmapsto (F_1/F_0, \ldots, F_r/F_{r-1}, F_r)$

$\rho = \omega \circ \beta$

$(D=(D_1,\ldots,D_r), F) \longmapsto \det(F)^{\otimes r+1} \otimes \mathcal{L}(-D)$, $\quad \mathcal{L}(-D) := \bigotimes_{i=1}^{r} \mathcal{O}(-D_i)^{\otimes i}$

$$
\begin{tikzcd}
\mathcal{C}h^{inj}_{\underline{m},\underline{e}} \arrow[rr, "gr", "\text{smooth}"'] & & \prod_{i=1}^{r} \mathcal{C}oh_{0,l_i} \times Bun_{m_r,e_r} \arrow[rr, "\beta"] & & \prod_{i=1}^{r} C^{(l_i)} \times Bun_{m_r,e_r} \arrow[ddrr, "\omega"] & & \\
& \mathcal{C}h^{inj}_{\underline{m},\underline{e},L} \arrow[ul, hook] \arrow[rr, "gr_L"] & & \rho^{-1}(L) \arrow[ul, hook] \arrow[rr] & & \omega^{-1}(L) \arrow[ul, hook] & \\
& & & & & & L \in \mathrm{Pic}(C) \\
& \widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e},L} \arrow[uu, "\tilde{f}_L"] \arrow[rr] \arrow[dl, hook] & & \tilde{\rho}^{-1}(L) \arrow[uu, "f_L"'] \arrow[rr] \arrow[dl, hook] & & \tilde{\omega}^{-1}(L) \arrow[uu] \arrow[dl, hook] & \\
\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e}} \arrow[uuuu, "\tilde{f}", "\text{small}"'] \arrow[rr] \arrow[rrrr, bend right, "\text{iterated } \mathbb{P}\text{-bdle}"'] & & \prod_{i=1}^{r} \widetilde{\mathcal{C}oh}_{0,l_i} \times Bun_{m_r,e_r} \arrow[uuuu, "f", "\text{small}"'] \arrow[rr] & & \prod_{i=1}^{r} C^{l_i} \times Bun_{m_r,e_r} \arrow[uuuu] \arrow[uurr, "\tilde{\omega}"] & &
\end{tikzcd}
$$

$((c_{ij})_{1\le i\le r,\, 1\le j\le l_i}, F) \longmapsto \det(F)^{\otimes r+1} \otimes \mathcal{L}(-c_{ij})$, $\quad \mathcal{L}(-c_{ij}) := \bigotimes_{i=1}^{r} \mathcal{O}(-\sum_{j=1}^{l_i} c_{ij})^{\otimes i}$

Step 4a. $\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e},L}$ is abelian but not generated by $M(C)$

For this, it suffices to show $\tilde{\omega}^{-1}(L)$ is abelian, as $\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e},L} \to \tilde{\omega}^{-1}(L)$ is an iterated $\mathbb{P}$-bdle

Step 4b. $\tilde{f}_L$ is small & $\prod_{i=1}^{r} S_{l_i}$-torsor on a dense open.

$$\text{Thus } \underbrace{M(\mathcal{C}h^{inj}_{\underline{m},\underline{e},L})}_{\text{abelian}} = \underbrace{M(\widetilde{\mathcal{C}h}^{inj}_{\underline{m},\underline{e},L})}_{\text{abelian}}{}^{\prod_{i=1}^{r} S_{l_i}} \qquad \text{abelian} \Leftarrow \text{abelian}$$

For this, it suffices to show f_L is small, as $\tilde{f}_L$ is the pullback of f_L under the smooth map gr