Motives of stacks of vector bundles on curves & applications

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§1 Overview

C/k smooth projective geom. connected curve.

Moduli of vector bundles on C

Fix discrete invariants for vector bundles: rank n & degree d

Moduli "spaces"
- impose semistability $\rightsquigarrow$ moduli space of semistable rk n deg d vector bundles $M^{ss}_{n,d}$ — projective variety of dim $n^2(g-1)+1$, smooth if $(n,d)=1$
- work with the stack of all vector bundles $\rightsquigarrow$ moduli stack of all rank n degree d vector bundles $\mathrm{Bun}_{n,d}$ — Artin stack of dim $n^2(g-1)$, smooth $\forall\, n,d$

Semistability: $E \to C$ is (semi)stable if $\mu(E') \underset{(=)}{<} \mu(E) := \frac{\deg E}{\mathrm{rk}\, E}$ $\forall\, E' \subset E$

Every vector bundle admits a unique Harder-Narasimhan filtration
$$0 = E^{(0)} \subset E^{(1)} \subset E^{(2)} \subset \dots \subset E^{(l)} = E$$
s.t $E_i := E^{(i)}/E^{(i-1)}$ are semistable with $\mu(E_1) > \mu(E_2) > \dots > \mu(E_l)$

Relationship between these moduli spaces:

$$\begin{array}{ccccc} \mathrm{Bun}_{n,d} & \hookleftarrow & \mathrm{Bun}^{ss}_{n,d} & \longrightarrow & M^{ss}_{n,d} \\ \| & & & & \\ \coprod_{\tau} \mathrm{Bun}^{\tau}_{n,d} & & & & \end{array}$$

Harder-Narasimhan Stratification

HN type

good moduli space (in the sense of Alper)
$\mathbb{G}_m$-gerbe if $(n,d)=1$

Construction of $M^{ss}_{n,d}$: [Seshadri, Mumford, Le Potier, Gieseker, Maruyama, Simpson..]

- Show any semistable bundle is a quotient of a fixed bundle ("boundedness")
  $\rightsquigarrow$ param by Quot scheme on which $\exists$ alg group acting s.t orbits $\longleftrightarrow$ iso classes
- Take a quotient (using Geometric Invariant Theory)

Construction of $\mathrm{Bun}_{n,d}$:

- Show $\exists$ $\mathbb{N}$-indexed family of vector bundles $(F_m)_{m\in\mathbb{N}}$ such that any $E \to C$ of rk n, deg d is a quotient of F_m for $m \gg 0$.
  $\rightsquigarrow$ param. by $\mathbb{N}$-indexed family of Quot schemes $(Q_m)_{m\in\mathbb{N}}$
  $\forall m \in \mathbb{N}$ $\exists$ alg. gp $G_m \curvearrowright Q_m$ s.t. orbits $\leftrightarrow$ iso classes.
- Take the union of quotient stacks: $\mathrm{Bun}_{n,d} = \coprod_m [Q_m/G_m]$

Alg. symplectic version: moduli space of semistable Higgs bundles

$$H^{ss}_{n,d} \underset{\text{dense open}}{\supset} T^* M^{ss}_{n,d}$$

$$T^*_E M^{ss}_{n,d} \underset{\text{def. theory}}{\cong} H^1(C, \mathrm{End}(E))^* \underset{\text{Serre duality}}{\cong} H^0(C, \mathrm{End}(E)\otimes\omega_C) \leftarrow \text{Higgs fields on } E$$

A Higgs bundle is a pair $(E, \Phi : E \to E \otimes \omega_C)$

vector bundle (E) — Higgs field (Φ)

Rmk: Non-abelian Hodge corr. [Corlette, Simpson] & Narasimhan – Seshadri Theorem
Over $k = \mathbb{C}$, $H^{ss}_{n,d}$ (& $M^{ss}_{n,d}$) are homeomorphic to moduli spaces of twisted semisimple (unitary) representations of $\pi_1(C)$.

Goal: Describe the motives of these moduli spaces

motive: in the sense of Grothendieck
(realised by Voevodsky's triangulated category $DM(k, \mathbb{Q})$)

* encodes cohomology groups:

- ($k = \mathbb{C}$) singular cohomology + mixed Hodge structure
- ℓ-adic cohomology + Galois representation

* and algebraic cycles (Chow groups)

Previous results on the geometry of these moduli spaces:

Tautological classes $\begin{array}{c}\mathcal{E}\\ \downarrow\\ \mathrm{Bun}_{n,d}\times C\end{array}$ universal family $\rightsquigarrow c_k(\mathcal{E})\in H^*(\mathrm{Bun}\times C)\cong H^*(\mathrm{Bun})\otimes H^*(C)$

Taut. classes := Künneth components of $c_k(\mathcal{E})$

<u>Betti cohomology of moduli of vector bundles</u> $(k=\mathbb{C})$ [Atiyah-Bott, Mumford, Zagier...]

i) The tautological classes <u>freely</u> generate $H^*(\mathrm{Bun}_{n,d},\mathbb{Q})$ as a $\mathbb{Q}$-algebra

ii) If $(n,d)=1$, the tautological classes generate $H^*(M^{ss}_{n,d},\mathbb{Q})$. Moreover the tautological relations can also be described.

ii) follows from i) using a "Harder-Narasimhan recursion"

<u>Other invariants</u>

i) [Harder-Narasimhan] recursion for ℓ-adic Betti no.s of $M^{ss}_{n,d}$ when $(n,d)=1$

via Weil conjectures & point counting over finite fields using:

$$|\mathrm{Bun}_{n,d}(\mathbb{F}_q)|_{\text{stacky}}:=\sum_{E}\frac{1}{|\mathrm{Aut}(E)|}=\frac{q^{(n^2-1)(g-1)}}{q-1}\,|\mathrm{Jac}(C)(\mathbb{F}_q)|\prod_{i=2}^{n}\zeta_C(q^{-i})$$

[Harder] stacky point count

ii) [Bifet-Ghione-Letizia] more algebro-geom. approach to ℓ-adic Betti no.s via ind-Quot schemes of matrix divisors.

iii) [Behrend-Dhillon] formula for virtual motivic class of $\mathrm{Bun}_{n,d}$

$$[\mathrm{Bun}_{n,d}]=\mathbb{L}^{(n^2-1)(g-1)}[B\mathbb{G}_m][\mathrm{Jac}(C)]\prod_{i=2}^{n}Z(C,\mathbb{L}^{-i})\ \text{ in }\ \hat{K}_0(\mathrm{Var}_k)$$

dim$^{\text{l}}$ completion of the Grothendieck ring of varieties

where $\mathbb{L}=[\mathbb{A}^1]$ and $Z(C,t):=\sum_{j\geq 0}[C^{(j)}]t^j$

<u>Betti cohomology of moduli of Higgs bundles</u> Assume: $(n,d)=1$

i) [Markman] The tautological classes generate $H^*(\mathcal{H}^{ss}_{n,d},\mathbb{Q})$

ii) [Hitchin, Hausel-Thaddeus] In rank $n=2$, closed formulae for the Betti numbers & tautogical relations.

iii) [Schiffmann] Computes Betti no.s via counting absolutely indecomp. vector bundles on curves over $\mathbb{F}_q$

Our Results:

Thm A [H.-Pepin Lehalleur]

If $C(k) \neq \emptyset$, there is an isomorphism in Voevodsky's triangulated category $DM(k, \mathbb{Q})$ of mixed motives $/k$ with $\mathbb{Q}$-coefficients

$$M(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\})$$

where $Z(C, \mathbb{Q}\{i\}) = \bigoplus_{j \geq 0} M(\mathrm{Sym}^j(C)) \otimes \mathbb{Q}\{ij\}$ & $\mathbb{Q}\{i\} := \mathbb{Q}(i)[2i]$ pure Tate twist

Questions:

- Is there an Atiyah-Bott description of this isomorphism via tautological classes?
- Can we perform a HN recursion to get the motive of $M^{ss}_{n,d}$?

⚠ The HN stratification gives (not necessarily split) Gysin distinguished triangles.

whereas the Gysin l.e.s. in cohomology split by an argument of Atiyah & Bott

With Lie Fu we get some explicit formulae:

Thm B [Fu-H.-Pepin Lehalleur]

i) Positive formulae for $M^{ss}_{n,d}$ in the $\mathbb{L}$-localised Grothendieck group of Chow motives lift to $DM(k, \mathbb{Q})$. ⇝ suffices to eliminate negative signs in HN recursion

ii) Formulae for Chow motive in low ranks: $n=2$ and $n=3$ & coprime d

$$M(M^{ss}_{2,d}) \simeq M(\mathrm{Jac}(C)) \otimes \left(\bigoplus_{i=0}^{g-2} M(C^{(i)}) \otimes (\mathbb{Q}\{i\} \oplus \mathbb{Q}\{3g-3-2i\}) \oplus M(\mathrm{Sym}^{g-1}(C))\{g-1\} \right)$$

$$M(M^{ss}_{3,d}) \simeq M(\mathrm{Jac}(C)) \otimes \left(\bigoplus_{\substack{i,j \geq 0 \\ i+j \leq 2g-2}} M(C^{(i)}) \otimes M(C^{(j)}) \otimes \underbrace{T_{i,j}}_{} \right)$$

explicit sum of pure Tate twists

Thm C [H-Pepin Lehalleur] Assume $(n,d)=1$ and $C(k) \neq \emptyset$.

The motive $M(H^{ss}_{n,d}) \in DM(k, \mathbb{Q})$ lies in the thick tensor subcategory $\mathcal{C}$ generated by the motive $M(C)$ of C.

Moreover $M(H^{ss}_{n,d})$ is a direct factor of $M(C^m)$ for $m \gg 0$.

Rmk [Fu - H. - Pepin Lehalleur]

fixed determinant L

The SL_n-Higgs moduli space $H^{ss}_{n,L}$ for a general curve $C/\mathbb{C}$ has motive $M(H^{ss}_{n,L}) \notin \mathcal{C}$.

Thm D [Fu - H. - Pepin Lehalleur] (Formulae for $n=2$ & 3 & coprime d)

$$M(H^{ss}_{2,d}) \simeq M(M^{ss}_{2,d}) \oplus \bigoplus_{j=1}^{g-1} M(\mathrm{Jac}(C)) \otimes M(C^{(2j-1)})\{3g-2j-2\}$$

$$M(H^{ss}_{3,d}) \simeq M(M^{ss}_{3,d}) \oplus \bigoplus_{i \in I} M(J_C) \otimes M(C^{(i)}) \otimes T_i \oplus \bigoplus_{(i,j) \in J} M(J_C) \otimes M(C^{(i)} \times C^{(j)}) \otimes T_{i,j}$$

J_C: $\mathrm{Jac}(C)$

explicit sums of Tate twists

Generalises cohomological results of Hitchin ($n=2$) and Gothen ($n=3$).

* Classical non-abelian Hodge correspondence: [Simpson, Corlette] (Char $k=0$)

$(n,d)=1$

$$H^{ss}_{n,d} = \mathcal{M}^{Dol}_{n,d} \overset{e^\infty}{\simeq} \mathcal{M}^{dR}_{n,d}$$

de Rham moduli space of semistable logarithmic connections on (C,x) of rk n, deg d w/ residue $-\frac{d}{n}$ at x

$$H^{ss}_{n,d} = \mathcal{M}^{Dol}_{n,d} \overset{e^\infty}{\simeq} \mathcal{M}^{Betti}_{n,d}$$

Betti moduli space of twisted semisimple representations $\pi_1(C,x) \longrightarrow GL(n,\mathbb{C})$

Thm E [H - Pepin Lehalleur] (Motivic non-abelian Hodge correspondence)

In Char $k=0$, we have $M(\mathcal{M}^{Dol}_{n,d}) \simeq M(\mathcal{M}^{dR}_{n,d})$ in $DM(k,\mathbb{Z})$

Consequently these have isomorphic Chow rings.

Rmk: $M(\mathcal{M}^{Dol}_{n,d}) \neq M(\mathcal{M}^{Betti}_{n,d})$ ← not a pure motive (non-trivial weight filtration)

$\rightsquigarrow$ P = W conjecture

§2 Motives

Voevodsky: There is a category $DM(k,\mathbb{Q})$ of mixed motives / k with $\mathbb{Q}$-coefficients together with a functor

$$M: \mathrm{Var}_k \longrightarrow DM(k,\mathbb{Q})$$
$$X \longmapsto M(X)$$

realising part of Grothendieck's motivic vision

[$DM(k,\mathbb{Q})$ is a tensor triangulated category constructed from $D^b(Sh(\mathrm{SmVar}_k,\mathbb{Q}))$ by imposing certain axioms.]

Expectation: $DM(k,\mathbb{Q})$ = derived category of an abelian category of mixed motives

Properties

* Künneth isomorphism $M(X\times Y) \simeq M(X)\otimes M(Y)$

* $\mathbb{A}^1$-homotopy invariance: $\underset{\text{vector bundle}}{E\to X} \rightsquigarrow M(E)\simeq M(X)$

* Projective bundle formula: $M(\mathbb{P}(E)) \simeq M(X)\otimes M(\mathbb{P}^{n-1})$ $n=\mathrm{rk}(E)$

$$M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Q}\{i\} \leftarrow$$ Tate twists ⊗-invertible

* Gysin triangles: for $\underset{\text{codim } c}{Z \hookrightarrow X}$ both smooth k-varieties

$$M(X\setminus Z) \longrightarrow M(X) \longrightarrow M(Z)\{c\} \xrightarrow{+1}$$

* Chow gps: X smooth k-variety

$$CH^i(X)_{\mathbb{Q}} \simeq \mathrm{Hom}_{DM}(M(X),\mathbb{Q}\{i\})$$

* Realisation functors:

Betti / de Rham / ℓ-adic cohomology factor via $M: \mathrm{Var}_k \to DM$

Betti + MHS; ℓ-adic + Galois rep.

§3 Motives of stacks

Several different ways to extend (étale) motives to stacks:

- Via Čech hypercover of an atlas
- Via a Nerve construction

More complicated for Nisnevich motives. The following is inspired by the Borel construction in alg. topology & Totaro's def of motives of quotient stacks

Def: An (Artin) stack $\mathcal{X}$ is exhaustive if it admits an increasing open cover by quasi-compact substacks $\cdots \subset \mathcal{X}_\ell \subset \mathcal{X}_{\ell+1} \subset \cdots \subset \mathcal{X}$, a seq. of vector bundles $(V_\ell \to \mathcal{X}_\ell)_{n \in \mathbb{N}}$ with injections $f_\ell : V_\ell \to V_{\ell+1}|_{\mathcal{X}_\ell}$ & $W_\ell \hookrightarrow V_\ell$ such that

i) $U_\ell := V_\ell \setminus W_\ell$ is a separated finite type k-scheme $\forall \ell$

ii) $\operatorname{codim}(W_\ell, V_\ell) \to \infty$ as $\ell \to \infty$

iii) $f_\ell^{-1}(W_{\ell+1} \times_{\mathcal{X}_{\ell+1}} \mathcal{X}_\ell) \subset W_\ell \quad \forall \ell$

The motive of a smooth exhaustive stack is then defined in $DM(k, \mathbb{Z})$ as

$$M(\mathcal{X}) := \operatorname{hocolim}_\ell M(U_\ell)$$ with transition maps induced by f_ℓ

Rmk: This is indept. of the choices (of $\mathcal{X}_\ell$, $V_\ell \to \mathcal{X}_\ell$ and $W_\ell \hookrightarrow V_\ell$)

↳ Proof uses $\mathbb{A}^1$-homotopy invariance and:

Key Lemma: For an inductive system $U_* \overset{\circ}{\hookrightarrow} X_*$ of open immersions of smooth varieties s.t $\operatorname{codim}(X_\ell \setminus U_\ell, X_\ell) \to \infty$ as $\ell \to \infty$, we have

$$\operatorname{hocolim}_\ell M(U_\ell) \simeq \operatorname{hocolim} M(X_\ell)$$

Example [Totaro] $\mathcal{X} = B\mathbb{G}_m = [*/\mathbb{G}_m]$

$$\begin{array}{ccccc} W_\ell = [\{0\}/\mathbb{G}_m] \hookrightarrow & V_\ell = [\mathbb{A}^\ell/\mathbb{G}_m] & \overset{\circ}{\hookleftarrow} & U_\ell = [\mathbb{A}^\ell - \{0\}/\mathbb{G}_m] = \mathbb{P}^{\ell-1} \\ & \downarrow & & \\ & \mathcal{X}_\ell = \mathcal{X} & & \end{array}$$

$$M(B\mathbb{G}_m) = \operatorname{hocolim}_\ell \underbrace{M(\mathbb{P}^{\ell-1})}_{= \bigoplus_{j=0}^{\ell-1} \mathbb{Q}\{j\}} = \bigoplus_{j \geq 0} \mathbb{Q}\{j\}$$

§4 The motive of Bun

Thm 1 $\mathrm{Bun}_{n,d}$ is an exhaustive stack. For any effective divisor D on C

$$M(\mathrm{Bun}_{n,d}) \simeq \operatorname{hocolim}_{\ell} M(\mathrm{Div}_{n,d}(\ell D)) \quad \text{in } DM(k, \mathbb{Z})$$

where

$$\mathrm{Div}_{n,d}(\ell D) = \{ E \subset \mathcal{O}(\ell D)^{\oplus n} : \mathrm{rk}\, E = n, \deg E = d \}$$

smooth projective Quot scheme of torsion quotients

Proof: Inspired by [Bifet-Ghione-Letizia] using "matrix divisors"

① Filter $\mathrm{Bun}_{n,d}$ by $\mu_{max}(E) := \max\{ \mu(E') := \frac{\deg E'}{\mathrm{rk}\, E'} : E' \subset E \}$

$\mathrm{Bun}_{n,d}^{\leq \mu} = \{ E \in \mathrm{Bun}_{n,d} : \mu_{max}(E) \leq \mu \}$ is open & a quotient stack

HN type is upper semi-cts

boundedness

Any increasing seq $(\mu_\ell)_{\ell \in \mathbb{N}}$ s.t. $\mu_\ell \to \infty$ as $\ell \to \infty$ induces an increasing open cover $\mathrm{Bun}^{\leq \mu_\ell}$ by q-compact substacks.

② Construct vector bundles using matrix divisor

$D \geqslant 0 \rightsquigarrow \mu_\ell := \ell \deg(D) - 2g + 1 - \frac{1}{n^2}$

$\forall E \in \mathrm{Bun}^{\leq \mu_\ell} : \quad H^1(E^\vee \otimes \mathcal{O}(\ell D)^{\oplus n}) = 0$

$\Rightarrow \exists$ vector bundle $V_\ell = Rp_*(\mathcal{E}_{univ}^\vee \otimes q^* \mathcal{O}(\ell D)^{\oplus n})$

$$\begin{array}{ccccc} V_\ell & & & & \\ \downarrow & & & & \\ \mathrm{Bun}^{\leq \mu_\ell} & \xleftarrow{p} & \mathrm{Bun}^{\leq \mu_\ell} \times C & \xrightarrow{q} & C \end{array}$$

$$\begin{array}{c} \mathrm{Hom}(E, \mathcal{O}(\ell D)^{\oplus n}) \\ \downarrow \\ E \end{array}$$

③ Construct $U_\ell \hookrightarrow V_\ell$ as the locus of injective homomorphisms

$$\begin{array}{ccccc} U_\ell = V_\ell^{inj} & \hookrightarrow & V_\ell & \hookleftarrow & W_\ell = V_\ell^{non\text{-}inj} \\ \| & & \downarrow & & \\ \mathrm{Div}_{n,d}(\ell D)^{\leq \mu_\ell} & & \mathrm{Bun}^{\leq \mu_\ell} & & \end{array}$$

[B-G-L] Compute codimensions

$$M(\mathrm{Bun}_{n,d}) \simeq \operatorname{hocolim}_{\ell} M(\mathrm{Div}_{n,d}(\ell D)^{\leq \mu_\ell}) \simeq \operatorname{hocolim}_{\ell} M(\mathrm{Div}_{n,d}(\ell D))$$

key lemma

∎

Unfortunately describing the transition maps is complicated!

Solution: rigidify further using an ind-flag Quot scheme

Assume: $x \in C(k) \neq \emptyset$ & consider the (flag)-Quot schemes:

$$\mathrm{Div}(\ell) := \mathrm{Div}_{n,d}(\ell x) = \{ E \subset \mathcal{O}(\ell x)^{\oplus n} : \mathrm{rk}\, E = n, \deg E = d \}$$

$$\uparrow$$

$$\mathrm{FDiv}(\ell) = \{ E = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n\ell - d} = \mathcal{O}(\ell x)^{\oplus n} : \mathrm{rk}\, E_i = n, \deg E_i = d + i \}$$

Smooth projective flag-Quot scheme

small map & principal $S_{n\ell-d}$-torsor on locus where torsion quotient has support consisting of $n\ell - d$ distinct points.

Inspired by work of Laumon & Heinloth on the cohomology of small maps of Hecke modification schemes we prove:

<u>Thm 2</u> There is a $S_{n\ell-d}$-action on $M(\mathrm{FDiv}(\ell))$ and

$$\begin{array}{ccc} M(\mathrm{Div}(\ell)) & \underset{②}{\simeq} & M(\mathrm{FDiv}(\ell))^{S_{n\ell-d}} \\ \wr\wr & & \wr\wr ① \\ M(\mathrm{Sym}^{n\ell-d}(C \times \mathbb{P}^{n-1})) & \simeq & (M(C \times \mathbb{P}^{n-1})^{\otimes n\ell-d})^{S_{n\ell-d}} \end{array} \quad \text{in } DM(k, \mathbb{Q})$$

Outline of the proof

① follows as

$$\mathrm{FDiv}(\ell) \xrightarrow{\mathrm{supp}} C^{n\ell-d}$$
$$E_\bullet \longmapsto \mathrm{supp}(E_i / E_{i-1})$$

is an $(n\ell - d)$-iterated $\mathbb{P}^{n-1}$-bundle (an elementary Hecke modification of F at $p \in C(k) \Leftrightarrow F_p \twoheadrightarrow k$ up to iso)

By the proj bundle formula $M(\mathrm{FDiv}(\ell)) \simeq M(C^{n\ell-d}) \otimes M(\mathbb{P}^{n-1})^{n\ell-d}$.

② follows from:

<u>Thm 3</u> For a small proper map $f : X \to Y$ of smooth varieties whose restriction to the open locus $f| : X^\circ \to Y^\circ$ with finite fibres is a G-torsor, the G-action on $M(X^\circ)$ extends to $M(X)$ & $M(X)^G \simeq M(Y)$ in $DM(k, \mathbb{Q})$

To define the G-invariant part, we need $\mathbb{Q}$-coefficients.

<u>Proof of Thm A</u> (formula for $M(\mathrm{Bun}_{n,d})$) for simplicity $d=0$

We have commutative diagrams for all ℓ

$$\begin{tikzcd}
\mathrm{Div}(\ell) \arrow[d, hook, "i_\ell"] & \mathrm{FDiv}(\ell) \arrow[l] \arrow[r, "\mathrm{supp}"'] \arrow[d, hook, "j_\ell"] & C^{n\ell} \arrow[d, "x^n \times \mathrm{Id}"] & (C \times \mathbb{P}^{n-1})^{n\ell} \arrow[l] \arrow[d, "a_\ell = (x,p)^n \times \mathrm{Id}"] \\
\mathrm{Div}(\ell+1) & \mathrm{FDiv}(\ell+1) \arrow[l] \arrow[r, "\mathrm{supp}"'] & C^{n(\ell+1)} & (C \times \mathbb{P}^{n-1})^{n(\ell+1)} \arrow[l]
\end{tikzcd}$$

choose any $p \in \mathbb{P}^{n-1}$

j_ℓ are determined by a choice of flag $\mathcal{O}_C^{\oplus n} = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_n = \mathcal{O}_C(x)^{\oplus n}$

a_ℓ are determined by a choice of $p \in \mathbb{P}^{n-1}$

↪ all choices give the same maps of motives.

This induces commutative diagrams in $DM(k,\mathbb{Q})$ $\forall \ell$

$$\begin{tikzcd}
M(\mathrm{FDiv}(\ell)) \arrow[d, "M(j_\ell)"'] \arrow[r, phantom, "\simeq"] & M_{C,n}^{\otimes n\ell} \arrow[d, "M(a_\ell)"] \\
M(\mathrm{FDiv}(\ell+1)) \arrow[r, phantom, "\simeq"] & M_{C,n}^{\otimes n(\ell+1)}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
M(\mathrm{Div}(\ell)) \arrow[d, "M(i_\ell)"'] \arrow[r, phantom, "\simeq"] & \mathrm{Sym}^{n\ell}(M_{C,n}) \arrow[d] \arrow[r, phantom, "\simeq"] & \bigoplus_{i=0}^{n\ell} \mathrm{Sym}^i(\overline{M}_{C,n}) \arrow[d, hook] \\
M(\mathrm{Div}(\ell+1)) \arrow[r, phantom, "\simeq"] & \mathrm{Sym}^{n(\ell+1)}(M_{C,n}) \arrow[r, phantom, "\simeq"] & \bigoplus_{i=0}^{n(\ell+1)} \mathrm{Sym}^i(\overline{M}_{C,n})
\end{tikzcd}$$

where $M_{C,n} := M(C \times \mathbb{P}^{n-1}) = \mathbb{Q}\{0\} \oplus \overline{M}_{C,n}$ ← reduced motive

$$M(C) = \mathbb{Q}\{0\} \oplus \overline{M}(C) = \mathbb{Q}\{0\} \oplus M_1(\mathrm{Jac}(C)) \oplus \mathbb{Q}\{1\}$$

Thus $\overline{M}_{C,n} = \overline{M}(C) \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\} = M_1(\mathrm{Jac}(C)) \oplus \mathbb{Q}\{1\} \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\}$ (*)

Hence $M(\mathrm{Bun}_{n,d}) \simeq \operatorname{hocolim}_\ell M(\mathrm{Div}(\ell)) \simeq \operatorname{hocolim}_\ell \bigoplus_{i=0}^{n\ell} \mathrm{Sym}^i(\overline{M}_{C,n})$

$$\| $$

$$\mathrm{Sym}^\bullet(\overline{M}_{C,n}) := \bigoplus_{i \geq 0} \mathrm{Sym}^i(\overline{M}_{C,n})$$

$\mathrm{Sym}^\bullet(\bigoplus_{i=1}^{r} M_i) \simeq \bigotimes_{i=1}^{r} \mathrm{Sym}^\bullet(M_i)$ ⟶ $\wr\!\wr$ (*)

$$\underbrace{\mathrm{Sym}^\bullet(M_1(\mathrm{Jac}(C)))}_{M(\mathrm{Jac}(C))} \otimes \underbrace{\mathrm{Sym}^\bullet(\mathbb{Q}\{1\})}_{M(B\mathbb{G}_m)} \otimes \bigotimes_{i=1}^{n-1} \underbrace{\mathrm{Sym}^\bullet(M(C)\{i\})}_{Z(C,\mathbb{Q}\{i\})}$$

This completes the proof of Thm A. ∎

§5 The motive of $H^{ss}_{n,d}$

Thm C: $M(H^{ss}_{n,d}) \in \mathcal{C} := \langle M(C) \rangle^{\otimes} \subset DM(k,\mathbb{Q})$

Geometric strategy: Hitchin, García-Prada-Heinloth-Schmitt

Step 1: Hitchin's scaling action & (motivic) Białynicki-Birula decomp.

Idea: Use the $\mathbb{G}_m$-action on $H^{ss}_{n,d}$ scaling the Higgs field

$$t \cdot [E,\Phi] := [E, t\Phi]$$

to stratify $H^{ss}_{n,d}$ into simpler pieces

Key properties of this action [Hitchin, Simpson]

* fixed locus is projective
* flow as $t \to 0$ exists for all points

"semi-projective $\mathbb{G}_m$-action" [Hausel]

(Motivic) Białynicki-Birula decompositions

[Białynicki-Birula] Any semi-projective $\mathbb{G}_m$-action on a smooth quasi-proj. variety X induces a BB-decomposition: $X = \coprod_{i=1}^{m} X_i^+$ where

$$X^{\mathbb{G}_m} = \coprod_{i=1}^{m} X_i \quad \text{and} \quad X_i^+ = \{x \in X : \lim_{t\to 0} t\cdot x \in X_i\} \longrightarrow X_i$$

$\mathbb{A}^{r_i}$-fibration

[Brosnan, Karpenko] There is a motivic BB decomposition

$$M(X) \simeq \bigoplus_{i=1}^{m} M(X_i)\{\operatorname{codim} X_i^+\} \quad \text{in } DM(k,\mathbb{Z})$$

Fixed points of Hitchin's scaling action:

Either $\Phi = 0$ or $\exists\ \mathbb{G}_m \subset \operatorname{Aut}(E) \rightsquigarrow E = \bigoplus_i E_i$ weight decomposition

$\downarrow$ $M^{ss}_{n,d}$

$\downarrow$ "chain" $E_0 \to E_1 \otimes \omega_C \to E_2 \otimes \omega_C^{\otimes 2} \to \dots$

Moduli of chains [Alvarez-Consul, García-Prada, Schmitt...]

A chain $F_\bullet = (F_0 \to F_1 \to \cdots \to F_r)$ is semistable wrt $\alpha = (\alpha_0, \dots, \alpha_r) \in \mathbb{R}^{r+1}$

if $\mu_\alpha(F_\bullet') \leq \mu_\alpha(F_\bullet) := \dfrac{\sum_i \deg F_i + \alpha_i \operatorname{rk} F_i}{\sum_i \operatorname{rk} F_i}$ for all $F_\bullet' \subset F_\bullet$

i) Every chain has a ! α-Harder-Narasimhan filtration

$\rightsquigarrow$ α-HN Stratification on the stack of all chains:

$$\mathcal{C}h_{\underline{n},\underline{d}} = \coprod_\tau \mathcal{C}h^{\alpha,\tau}_{\underline{n},\underline{d}}$$

with ranks $\underline{n}$ & degrees $\underline{d}$

ii) $\exists$ projective moduli spaces $Ch^{\alpha-ss}_{\underline{n},\underline{d}}$ of α-ss chains

smooth if α-ss = α-s and $\alpha \in \Delta_r \subset \mathbb{R}^{r+1}$

$\Delta_r = \{\alpha : \alpha_i - \alpha_{i+1} \geq 2g-2\}$

iii) $\exists$ Higgs stability parameter

$$\alpha_H = (r(2g-2), \dots, (2g-2), 0) \in \partial\Delta_r$$

s.t $\left(H^{ss}_{n,d}\right)^{\mathbb{G}_m} = M^{ss}_{n,d} \sqcup \coprod_{(\underline{m},\underline{e})} Ch^{\alpha_H-ss}_{\underline{m},\underline{e}}$

Motivic BB decomp for $H^{ss}_{n,d}$:

$$M(H^{ss}_{n,d}) \simeq \bigoplus_{(\underline{m},\underline{e}) \in I} M(Ch^{\alpha_H-ss}_{\underline{m},\underline{e}})\{\operatorname{codim}_{\underline{m},\underline{e}}\}$$

$\rightsquigarrow$ suffices to show $M(Ch^{\alpha_H-ss}_{\underline{m},\underline{e}}) \in \mathcal{C}$

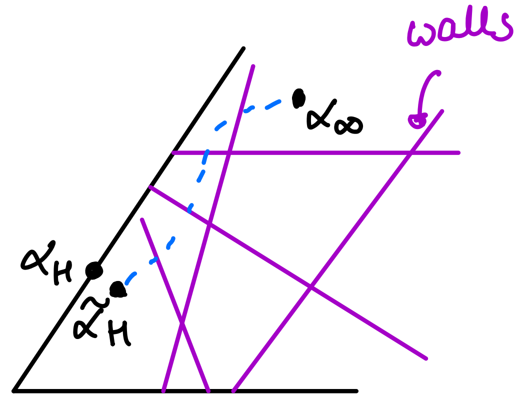
Rmk: The proof of the motivic non-abelian Hodge corr. (Thm E) relies heavily on BB-decompositions on the Deligne-Simpson Hodge moduli space $M^{Hodge}_{n,d} \to \mathbb{A}^1$ of logarithmic λ-connections

Step 2: $\mathcal{C}h^{\alpha_H-ss}_{\underline{m},\underline{e}} = \mathcal{C}h^{\alpha_H-s}_{\underline{m},\underline{e}} \xrightarrow[\mathbb{G}_m\text{-gerbe}]{\text{trivial}} Ch^{\alpha_H-ss}_{\underline{m},\underline{e}}$ $\left(M(B\mathbb{G}_m) = \bigoplus_{j \geq 0} \mathbb{Q}\{j\}\right)$

stack

$\rightsquigarrow$ suffices to show $M(\mathcal{Ch}^{\alpha_H\text{-ss}}_{\underline{m},\underline{e}}) \in \mathcal{C}$ (or technically its ind-completion)

Step 3: Wall-crossing & Harder-Narasimhan recursion



<u>Thm</u> [García-Prada – Heinloth – Schmitt]

$\exists$ path $\{\alpha_t\}_{t\geq 0}$ in Δ°_r from $\tilde{\alpha}_H$ to $\alpha_\infty = \alpha_t$ for $t \gg 0$ s.t

i) If $\underline{m}$ is non-constant, $\mathcal{Ch}^{\alpha_\infty\text{-ss}}_{\underline{m},\underline{e}} = \phi$

ii) If $\underline{m}$ is constant, $\mathcal{Ch}^{\alpha_\infty\text{-ss}}_{\underline{m},\underline{e}} \subset \mathcal{Ch}^{inj}_{\underline{m},\underline{e}}$ ← injective chain homomorphisms

↖ union of α_∞-HN strata

At each wall-crossing, the semistable loci are related by finitely many Gysin triangles involving higher HN strata.

By a HN recursion (for chains)

$\rightsquigarrow$ suffices to show $M(\mathcal{Ch}^{inj}_{\underline{m},\underline{e}}) \in \mathcal{C}$ (or rather its ind-completion)

for $\underline{m}$ constant ↗

Step 4: Explicit formula for $M(\mathcal{Ch}^{inj}_{\underline{m},\underline{e}})$ with $\mathbb{Q}$-coeffs

$\underline{m} = (m, \ldots, m)$

using stacks of Hecke modifications and motivic descriptions of certain small maps

<u>Thm 4</u>: For $\underline{m} = (m, \ldots, m)$ constant, the forgetful maps

$$\mathcal{Ch}^{inj}_{\underline{m}_{\geq 0}, \underline{e}_{\geq 0}} \longrightarrow \mathcal{Ch}^{inj}_{\underline{m}_{\geq 1}, \underline{e}_{\geq 1}} \rightarrow \cdots\cdots \rightarrow \mathcal{Ch}^{inj}_{\underline{m}_{\geq r}, \underline{e}_{\geq r}} = \mathrm{Bun}_{m, e_r}$$

$$(E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_r) \mapsto (E_1 \hookrightarrow \cdots \hookrightarrow E_r) \mapsto \cdots\cdots \mapsto E_r$$

are Hecke modification stacks.

Moreover,

$$M(\mathcal{Ch}^{inj}_{\underline{m},\underline{e}}) \simeq \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{e_i - e_{i-1}}(C \times \mathbb{P}^{n-1})) \otimes M(\mathrm{Bun}_{m, e_r})$$

Proof: Generalisation of Thm 2. (Motives of $\mathrm{Div}(\ell)$ & $\mathrm{FDiv}(\ell)$)

$\rightsquigarrow$ suffices to show $M(\mathrm{Bun}_{m,e}) \in \mathcal{C}$

Step 5: Explicit formula for $M(\mathrm{Bun}_{m,e})$: Thm A

$$M(\mathrm{Bun}_{m,e}) \simeq M(\mathrm{Jac}\, C) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{m-1} Z(C, \mathbb{Q}\{i\})$$

In particular, $M(\mathrm{Bun}_{m,e}) \in \mathcal{C} = \langle M(C) \rangle^{\otimes}$. (technically its ind-completion)

$\boxed{Z}$