VBAC
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Motives of moduli spaces of bundles on curves

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Joint work with Lie Fu & Simon Pepin Lehalleur

§1 Overview

Let C/k be a smooth projective curve

Fix: rank n & degree d

Various moduli spaces:

moduli space of semistable vector bundles (resp. with fixed determinant L)

$\mathcal{N} = \mathcal{N}_C(n,d)$ (resp. $\mathcal{N}_L$)

projective variety of dim. $n^2(g-1)+1$ (resp. $(n^2-1)(g-1)$)

smooth if $(n,d)=1$

$\mathcal{M} = \mathcal{M}_C(n,d)$ (resp. $\mathcal{M}_L$)

moduli space of semistable Higgs bundles

quasi-proj. variety $\dim \mathcal{M} = 2\dim\mathcal{N}$ $(T^*\mathcal{N} \subset \mathcal{M})$

smooth if $(n,d)=1$

moduli space of α-semistable parabolic bundles

$D = \{p_1, \dots, p_N\}$

$\mathcal{N}^\alpha = \mathcal{N}^\alpha_{C,D}(n,d,\underline{m})$ (resp. $\mathcal{N}^\alpha_L$)

parabolic points

multiplicities (of flags)

$\dim \mathcal{N}^\alpha = \dim \mathcal{N} + \sum_{i=1}^{N} \dim \mathcal{F}(m_i)$ proj. variety

smooth if α is generic

$\mathcal{M}^\alpha = \mathcal{M}^\alpha_{C,D}(n,d,\underline{m})$ (resp. $\mathcal{M}^\alpha_L$)

m. space of α-semistable parabolic Higgs bundles

quasi-proj. variety $\dim \mathcal{M}^\alpha = 2\dim\mathcal{N}^\alpha$

smooth if α is generic

Goal: Describe the motives of these moduli spaces

motive: in the sense of Grothendieck (via Chow motives)

- encodes cohomology groups:
  - ($k=\mathbb{C}$) singular cohomology + mixed Hodge structure
  - ℓ-adic cohomology + Galois representation
- and algebraic cycles (Chow groups)

§2 Chow motives

Effective Chow motives

$$\mathrm{SmProj}(k) \longrightarrow \mathrm{Corr}(k,\mathbb{Q}) \xrightarrow[\text{completion}]{\text{idempotent}} \mathrm{CHM}^{\mathrm{eff}}(k,\mathbb{Q})$$

ob: X sm. proj. k-variety $\qquad X \qquad (X,p)$, $p \in CH^{d_X}(X\times X)_{\mathbb{Q}}$ idempotent $(p\circ p = p)$

homs: $f: X \to Y \qquad \Gamma_f \in \mathrm{Hom}(X,Y) := CH^{d_Y}(X\times Y)_{\mathbb{Q}} \qquad \mathrm{Hom}((X,p),(Y,q)) := q\circ CH^{d_Y}(X\times Y)\circ p$

$h: \mathrm{SmProj}(k) \longrightarrow \mathrm{CHM}^{\mathrm{eff}}(k,\mathbb{Q})$ symmetric monoidal functor

$\hookrightarrow h(X\times Y) = h(X)\otimes h(Y)$

$$X \longmapsto h(X) := (X, \Delta_X)$$

$$\mathrm{Spec}\,k \longmapsto h(k) =: \mathbb{Q}(0) \quad \text{unit for } \otimes$$

$$\mathbb{P}^1 \longmapsto h(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1) \leftarrow \text{Tate twist}$$

$$\mathbb{P}^n \longmapsto h(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Q}(i)$$

Chow motives: $\mathrm{CHM}^{\mathrm{eff}}(k,\mathbb{Q}) \hookrightarrow \mathrm{CHM}(k,\mathbb{Q}) \leftarrow$ $\otimes$-invert $\mathbb{Q}(1)$

Voevodsky's embedding thm:

$$\begin{tikzcd}
\mathrm{CHM}^{(\mathrm{eff})}(k,\mathbb{Q}) \arrow[r, hook] & \mathrm{DM}^{(\mathrm{eff})}(k,\mathbb{Q}) \\
\mathrm{SmProj}(k) \arrow[u, "h"] \arrow[r, hook] & \mathrm{Var}(k) \arrow[u, "M"]
\end{tikzcd}$$

Properties

- Universal property: any Weil cohomology on $\mathrm{SmProj}(k)$ factors via h
- Chow groups: $X \in \mathrm{SmProj}(k)$: $\quad CH^i(X)_{\mathbb{Q}} \simeq \mathrm{Hom}_{\mathrm{CHM}}(h(X), \mathbb{Q}(i))$
- Projective bundle formula, blow-up formula …

§3 Results on $h(\mathcal{N})$

moduli space of semistable vector bundles on C

Thm A [Fu-H-Pepin Lehalleur] Assume $(n,d)=1$

i) $h(\mathcal{N}_L)$ & $h(\mathcal{N})$ lie in the tensor subcat. $\mathcal{C} = \langle h(C)\rangle^{\otimes} \subset CHM(k,\mathbb{Q})$

ii) $h(\mathcal{N}) \simeq h(\mathcal{N}_L)\otimes h(Jac(C)) \in CHM(k,\mathbb{Q})$

i) adapts an argument of Beauville & Bülles via chern classes of the univ. family

ii) refines the isomorphism of Harder-Narasimhan on ℓ-adic cohomology

Pf of ii) Recall $\Gamma_n = Jac(C)[n] \curvearrowright \mathcal{N}_L$ via $M\cdot E = E\otimes M^{-1}$

and there is an isomorphism

$$\mathcal{N}_L \times^{\Gamma_n} Jac(C) \xrightarrow{\sim} \mathcal{N}$$

Consequently $h(\mathcal{N}) \simeq \left(h(\mathcal{N}_L)\otimes h(Jac(C))\right)^{\Gamma_n}$

We show the Γ_n-action on $h(\mathcal{N}_L)$ & $h(Jac(C))$ is trivial:

(1) Reduce to a field k of char 0.

(2) $h(\mathcal{N}_L)$ is abelian by i).

(3) In char 0, ℓ-adic realisation is conservative on abelian geom. motives & $\Gamma_n \curvearrowright H^*(\mathcal{N}_L,\mathbb{Q}_\ell)$ is trivial. [Harder-Narasimhan]

$Jac(C) \xrightarrow{\cdot n} Jac(C)$ is a Γ_n-torsor

$\Rightarrow h(Jac(C))^{\Gamma_n} \simeq h(Jac(C))$

∎

Thm B [Fu-H.-Pepin Lehalleur]

i) Positive formulae for $\mathcal{N}_L$ in the $\mathbb{L}$-localised Grothendieck group of Chow motives lift to $DM(k,\mathbb{Q})$. ⇝ suffices to eliminate negative signs in HN recursion

ii) Formulae for Chow motive in low ranks: $n=2$ and $n=3$ & coprime d

$$h(\mathcal{N}_L(2,d)) \simeq h(Sym^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} h(C^{(i)})\otimes\left[\mathbb{Q}(i)\oplus\mathbb{Q}(3g-3-2i)\right]$$

$$h(\mathcal{N}_L(3,d)) \simeq \bigoplus_{\substack{i,j\geq 0\\ i+j\leq 2g-2}} h(C^{(i)})\otimes h(C^{(j)})\otimes \underbrace{T_{i,j}}$$

explicit sum of pure Tate twists

i) uses Thm A i) & a conservativity argument

ii) relies on work of Thaddeus & del Baño ($n=2$) & Gomez-Lee ($n=3$)

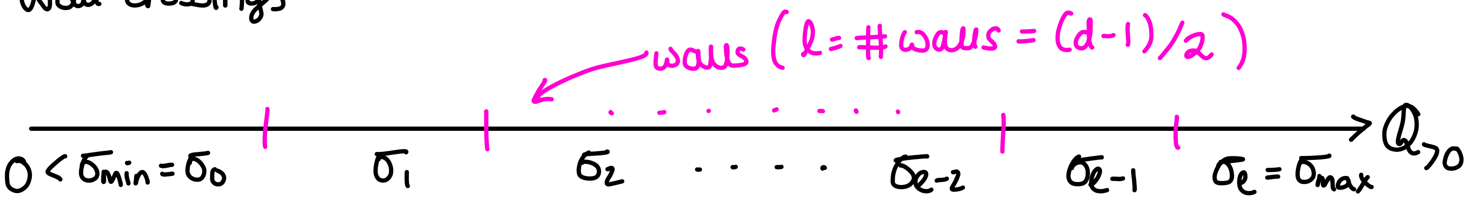
Sketch of ii) for $n=2$: WLOG (by tensoring with a line bdle): $d>>0$

We use rank 2 pairs [Bradlow] & wall-crossings [Thaddeus]

For $\sigma \in \mathbb{Q}_{>0}$ $\exists$ proj. moduli spaces $\mathcal{P}_L^{\sigma}(2,d)$ of σ-ss pairs (E,ϕ)

stability param. ↳ smooth if $\sigma\text{-ss} = \sigma\text{-s}$ $\left.\begin{matrix} \mathrm{rk}=2 \\ \det = L \end{matrix}\right\}$ ↲ ↳ non-zero section

Wall-crossings



$m := d+g-4$

$$\begin{tikzcd}
\mathcal{P}_L^{\sigma_0} \arrow[r, dashed, leftrightarrow, "\substack{\text{type}\\(m-2(\ell-1),\ell-1)}", "\substack{\text{centre}\\C^{(\ell)}}"'] \arrow[d, "\substack{\text{if } d\geq 4g-3\\ \mathbb{P}^{d-2g+1}\text{-bdle}}"'] & \mathcal{P}_L^{\sigma_1} \arrow[r, dashed, leftrightarrow, "\substack{\text{type}\\(m-2(\ell-2),\ell-2)}", "\substack{\text{Centre}\\C^{(\ell-1)}}"'] & \mathcal{P}_L^{\sigma_2} \cdots\cdots & \mathcal{P}_L^{\sigma_{\ell-2}} \arrow[r, dashed, leftrightarrow, "\substack{\text{type}\\(m-2,1)}", "\substack{\text{centre}\\C^{(2)}}"'] & \mathcal{P}_L^{\sigma_{\ell-1}} \arrow[r, dashed, leftrightarrow, "\substack{\text{type}\\(m,0)}", "\substack{\text{centre}\\C}"'] & \mathcal{P}_L^{\sigma_\ell} \arrow[d, "\wr"] \\
\mathcal{N}_L & & & & & \mathbb{P}^{d+g-2}
\end{tikzcd}$$

Def: A birat^{e} map of smooth proj. varieties $X \dashrightarrow X'$ is a standard flip of type (a,b) with smooth proj. centre S if $\exists$ $Z \hookrightarrow X$ & $Z' \hookrightarrow X'$ projective bundles over S s.t:

$$\begin{tikzcd}
 & & Bl_Z(X) \cong Bl_{Z'}(X') \arrow[dl] \arrow[dr] & & \\
Z \arrow[r, hook] \arrow[drr, "\mathbb{P}^a\text{-bdle}"'] & X \arrow[rr, dashed, leftrightarrow] & & X' & Z' \arrow[l, hook'] \arrow[dll, "\mathbb{P}^b\text{-bdle}"] \\
 & & S & &
\end{tikzcd}$$

notation:

$$\begin{tikzcd}
X \arrow[r, dashed, leftrightarrow, "\substack{\text{type}\\(a,b)}", "\substack{\text{centre}\\S}"'] & X'
\end{tikzcd}$$

"flop" if $a=b$

In $\hat{K}_0(CHM(k,\mathbb{Q}))$: $\chi(X) = \chi(X') + \chi(S)\cdot(\chi(\mathbb{P}^a) - \chi(\mathbb{P}^b))$

Thm [Jiang] If $a \geq b$ $h(X) \simeq h(X') \oplus \bigoplus_{j=b+1}^{a} h(S)(j)$ in $CHM(k,\mathbb{Z})$

Working from right to left in the pairs wall-crossing diagram:

$m = d+g-4$

$$\begin{tikzcd}
\mathcal{P}_L^{\sigma_0} \arrow[r, dashed, leftrightarrow, "\text{type } (m-2(\ell-1),\ell-1)", "\text{centre } C^{(\ell)}"'] \arrow[d] & \mathcal{P}_L^{\sigma_1} \arrow[r, dashed, leftrightarrow, "\text{type } (m-2(\ell-2),\ell-2)", "\text{Centre } C^{(\ell-1)}"'] & \mathcal{P}_L^{\sigma_2} \cdots\cdots & \mathcal{P}_L^{\sigma_{\ell-2}} \arrow[r, dashed, leftrightarrow, "\text{type } (m-2,1)", "\text{centre } C^{(2)}"'] & \mathcal{P}_L^{\sigma_{\ell-1}} \arrow[r, dashed, leftrightarrow, "\text{type } (m,0)", "\text{centre } C"'] & \mathcal{P}_L^{\sigma_\ell} \arrow[d, equal] \\
\mathcal{N}_L & & & & & \mathbb{P}^{d+g-2}
\end{tikzcd}$$

- Initially we have flips of type (a,b) with $a \geqslant b$

  → good news: $\mathfrak{h}$ increases

  $\mathfrak{h}(\mathcal{P}^{\sigma_{\ell-j}}) = \mathfrak{h}(\mathcal{P}^{\sigma_\ell}) \oplus$ contributions from the centres of each flip

  ↳ for small j $\left(j \leq \frac{d+g-1}{3}\right)$

  ie $\mathcal{P}^{\sigma_{\ell-j}} \overset{\text{type } (m-2(j-1),j-1)}{\underset{a \geqslant b}{\dashleftarrow\dashrightarrow}} \mathcal{P}^{\sigma_{\ell-j+1}}$ $m = d+g-4$

- However for larger j we get flips of type (a,b) with $a \leq b$

  → bad news: we need to "cancel" part of the motive

<u>Solution</u>: - work in the ($\mathbb{L}$-completed) Grothendieck group $\hat{K}_0(CHM(k,\mathbb{Q}))$

  [del Baño]: Computation in $\hat{K}_0(CHM(k,\mathbb{Q}))$

- eliminate minus signs: find a positive expression for $\chi(\mathcal{N}_L)$ and apply Thm B i). ☒

§4 Results on $\mathfrak{h}(\mathcal{M})$

moduli space of ss Higgs bundles

⚠ quasi-proj. variety

<u>Thm C</u> [H.-Pepin Lehalleur] Assume $(n,d)=1$

i) The motive of $\mathcal{M}$ is pure. Thus $\mathfrak{h}(\mathcal{M}) \in CHM(k,\mathbb{Q})$

ii) Assume $C(k) \neq \phi$. Then $\mathfrak{h}(\mathcal{M}) \in \mathcal{C} = \langle \mathfrak{h}(C) \rangle^{\otimes}$.

The starting point for the proof is the Białynicki-Birula decomposition of $\mathcal{M}$ associated to Hitchin's scaling action $\mathbb{G}_m \curvearrowright \mathcal{M}$ $t \cdot [E,\Phi] = [E, t\Phi]$

- fixed locus is projective

  $\mathcal{M}^{\mathbb{G}_m} = \mathcal{N} \amalg$ moduli spaces of chains $\rightsquigarrow E = \oplus E_i$

  $E_0 \to E_1 \otimes \omega_C \to E_2 \otimes \omega_C^{\otimes 2} \to \cdots$

- the flow as $t \to 0$ exists $\forall\, [E,\Phi] \in \mathcal{M}$

} "semi-proj." $\mathbb{G}_m$-action [Hausel]

[García-Prada – Heinloth – Schmitt] study this decomposition & the classes of chain moduli spaces in the Grothendieck ring of varieties via variation of stability.

Rmk [Fu - H. - Pepin Lehalleur] The SL_n-Higgs moduli space $\mathcal{M}_L$ for a general curve $C/\mathbb{C}$ has motive $h(\mathcal{M}_L) \notin \mathcal{C} = \langle h(C) \rangle^{\otimes}$

In particular $h(\mathcal{M}) \ncong h(\mathcal{M}_L) \otimes h(\mathrm{Jac}(C))$

$\Gamma_n \curvearrowright h(\mathcal{M}_L)$ non-trivially

Already seen in rank $n=2$ [Hitchin]

Thm D [Fu - H. - Pepin Lehalleur] (Formulae for $n = 2$ & 3 & coprime d)

$$h(\mathcal{M}(2,d)) \simeq h(\mathcal{N}(2,d)) \oplus \bigoplus_{j=1}^{g-1} h(\mathrm{Jac}(C)) \otimes h(C^{(2j-1)})(3g-2j-2)$$

holds with $\mathbb{Z}$-coeffs

$$h(\mathcal{M}(3,d)) \simeq h(\mathcal{N}(3,d)) \oplus \bigoplus_{i \in I} h(J_C) \otimes h(C^{(i)}) \otimes T_i \oplus \bigoplus_{(i,j) \in J} h(J_C) \otimes h(C^{(i)} \times C^{(j)}) \otimes T_{i,j}$$

with $\mathbb{Q}$-coeffs

$\mathrm{Jac}(C)$

explicit sums of Tate twists

Generalises cohomological results of Hitchin ($n=2$) and Gothen ($n=3$).

Sketch of the proof: We study the BB decomp. associated to

$\mathbb{G}_m \curvearrowright \mathcal{M} \rightsquigarrow \mathcal{M}^{\mathbb{G}_m} = \mathcal{N} \amalg \coprod_{(\underline{m},\underline{e})} \mathrm{Ch}_{\underline{m},\underline{e}}^{\alpha_H\text{-ss}}$

Hitchin's scaling action

α_H = Higgs stability param. for chains

tuple of ranks & degrees

Motivic BB decomp. (with $\mathbb{Z}$-coeffs) $\quad h(\mathcal{M}) = h(\mathcal{N}) \oplus \bigoplus_{(\underline{m},\underline{e})} h(\mathrm{Ch}_{\underline{m},\underline{e}}^{\alpha_H\text{-ss}})(\mathrm{codim}_{\underline{m},\underline{e}})$

In low ranks we can analyse the types $(\underline{m},\underline{e})$ of chains appearing & describe the motives of the corresponding chain moduli spaces.

Rank 2

(Hitchin)

- $\underline{m} = (2)$ & $\underline{e} = (d)$ ⇝ $\mathcal{N}(2,d)$
- $\underline{m} = (1,1)$ & $\underline{e} = (e_1, e_2)$ $\quad e_2 = d - e_1 + 2g - 2$

stability & $\phi \neq 0$ ⇒ finitely many possible values of e_1

$L_0 \xrightarrow[\phi]{} L_1 \otimes \omega_C$ ⇝ param. by $\mathrm{Pic}^{e_1}(C) \times C^{(e_2 - e_1)}$

Tate twist = codim of BB stratum

(calculate using fact that downward flow is Lagrangian)

Rank 3

(Gothen)

- $\underline{m} = (3)$ & $\underline{e} = (d)$ ⇝ $\mathcal{N}(3,d)$

finitely many possibilities for $\underline{e}$ in each case

- $\underline{m} = (1,1,1)$ & $\underline{e} = (e_1, e_2, e_3)$ $\quad e_1 + e_2 + e_3 = d + 6(g-1)$

  $L_0 \to L_1 \otimes \omega_C \longrightarrow L_2 \otimes \omega_C^{\otimes 2}$

  ⇝ param. by $\mathrm{Pic}^{e_0}(C) \times C^{(e_2 - e_1)} \times C^{(e_3 - e_2)}$

- $\underline{m} = (1,2)$ & $\underline{e} = (e_1, e_2)$

  $L \to \underset{\substack{\uparrow \\ \text{rk } 2}}{E} \otimes \omega_C$ ⇝ rk 2 pair $(F = E \otimes \omega_C \otimes L^{-1}, \underset{\text{section}}{\phi})$

  ⇝ param. by $\mathrm{Pic}^{e_1}(C) \times \mathcal{P}^{\sigma\text{-}ss}(2,f)$ ← moduli space of σ-semistable pairs of rk 2 degree f

  Motives computed by pairs wall-crossings (explicit flips) as in the proof of Thm B.

- $\underline{m} = (2,1)$ & $\underline{e} = (e_1, e_2)$ (dual to $\underline{m} = (1,2)$)

  $F \twoheadrightarrow L \otimes \omega_C$ ⇝ param by $\mathrm{Pic}^{e_2}(C) \times \mathcal{P}^{\sigma\text{-}ss}(2,f)$ ☒

Rmk: The BB decomposition for $\mathcal{M}_L(2,d)$ gives

$$\mathcal{M}_L(2,d) = \mathcal{N}_L(2,d) \amalg \coprod_{j=1}^{g-1} \widetilde{C}^{(2j-1)}$$

[Hitchin]

where

$$\begin{tikzcd} \widetilde{C}^{(j)} \arrow[r, "\text{deg } 2^{2g}", "\text{étale cover}"'] \arrow[d] & C^{(j)} \arrow[d] \\ \mathrm{Jac}(C) \arrow[r, "\cdot 2"] & \mathrm{Jac}(C) \end{tikzcd}$$

⇝ $h(\mathcal{M}_L(2,d)) \in \langle h(\widetilde{C}) \rangle^{\otimes}$

For a general complex curve C:

$$\langle h(C) \rangle^{\otimes} \subsetneq \langle h(\widetilde{C}) \rangle^{\otimes}$$

§5 Results on $h(\mathcal{N}^\alpha)$ and $h(\mathcal{M}^\alpha)$

moduli spaces of α-ss parabolic (Higgs) bundles

Fix $D=\{p_1,\dots,p_N\}$ parabolic points on C.

Def: A <u>(quasi)-parabolic bundle</u> on (C,D) is $E_* = (E, E_{i,j})$

vector bdle ↗ ↖ flag in E_{p_i} $\forall p_i \in D$

$$E_{p_i} = E_{i,1} \supsetneq E_{i,2} \supsetneq \cdots \supsetneq E_{i,l_i} \supsetneq E_{i,l_i+1} = 0$$

l_i = length of the flag

$m_{i,j} := \dim(E_{i,j}/E_{i,j+1}) > 0$ flag multiplicities

- A <u>(quasi)-parabolic Higgs bundle</u> on (C,D) is $(E_*, \Phi: E \to E \otimes \omega_C(D))$

(quasi)-parabolic vector bundle

strongly parabolic Higgs field
↳ ie $\Phi(E_{i,j}) \subset E_{i,j+1} \otimes \omega_C(D)$

Def: E_* is (<u>semi)stable</u> w.r.t. $\alpha = (\alpha_{i,j})$ [$\in \mathbb{R}_{>0}$] if $\forall E' \subset E$

$$\mu_\alpha(E') \underset{(-)}{<} \mu_\alpha(E) = \frac{\deg(E) + \sum_{p_i \in D} \sum_{j=1}^{l_i} \alpha_{i,j} m_{i,j}}{\mathrm{rk}(E)}$$

↑ $m'_{i,j} := \dim(E'_{p_i} \cap E_{i,j} / E'_{p_i} \cap E_{i,j+1}) \geq 0$

<u>Moduli spaces</u>: [Metha-Seshadri, Yokogawa]

$\mathcal{N}^\alpha_{C,D}(n,d,\underline{m})$ is the moduli space of α-ss parabolic vector bundles of rank n, degree d with multiplicities $\underline{m} = (m_{i,j})$.

↳ proj. variety

smooth if α-ss $= \alpha$-s

$$\dim \mathcal{N}^\alpha_{C,D}(n,d,\underline{m}) = \underbrace{n^2(g-1)+1}_{\dim(\mathcal{N}_C(n,d))} + \sum_{p_i \in D} \underbrace{\sum_{j>k} m_{i,j} m_{i,k}}_{\dim \text{ flag variety } Fl(m_i)}$$

$\mathcal{M}^\alpha_{C,D}(n,d,\underline{m})$ is the moduli space of α-ss parabolic Higgs bundles of rank n, degree d with multiplicities $\underline{m} = (m_{i,j})$

↳ quasi-proj variety

smooth if α-ss $= \alpha$-s

$\exists\, \mathbb{G}_m \curvearrowright \mathcal{M}^\alpha$ $\Rightarrow$ $h(\mathcal{M}^\alpha) \in CHM(k,\mathbb{Q})$

semi-projective

motivic BB decomp.

$$\dim(\mathcal{M}^\alpha) = 2\dim(\mathcal{N}^\alpha)$$

Variation of stability for parabolic vector bundles

[Boden-Hu, Boden-Yokogawa, Thaddeus]

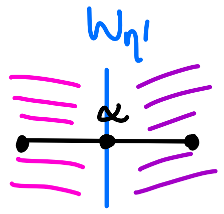
Fix invariant $\eta=(n,d,\underline{m})$ $\quad \exists$ walls $W_{\eta'} \subset \mathcal{A}=\{(\alpha_{i,j})\}\cong\mathbb{R}^{\sum_{i=1}^{N} l_i}$

invariant of subbundle $\nearrow$ $\quad W_{\eta'} = \{\alpha : \mu_\alpha(\eta')=\mu_\alpha(\eta)\}$

Assume $\underline{m}=\underline{1}$ (full flags) $\quad$ Note: $W_{\eta'}=W_{\eta''}$ $(\eta''=\eta-\eta')$

$\uparrow$ invariant of quotient

At a single wall:



$W^-_{\eta'}\ni\alpha_-$ $\qquad$ $\alpha_+\in W^+_{\eta'}=\{\alpha : \mu_\alpha(\eta')-\mu_\alpha(\eta)>0\}$

there is a standard flip

$$\begin{tikzcd}
 & \mathcal{N}^{\alpha_-} \arrow[rr, dashed, leftrightarrow, "\text{type }(n_-,n_+)", "\text{centre}"'] & & \mathcal{N}^{\alpha_+} & \\
\mathbb{P}(\mathrm{Ext}^1(\mathcal{E}''_*,\mathcal{E}'_*)) \arrow[ur, hook] \arrow[drr, "-\chi_{par}((\eta'',\alpha'')^\vee\otimes(\eta',\alpha'))-1=n_-"'] & & \mathcal{N}^{\alpha'}(\eta')\times\mathcal{N}^{\alpha''}(\eta'') & & \mathbb{P}(\mathrm{Ext}^1(\mathcal{E}'_*,\mathcal{E}''_*)) \arrow[ul, hook'] \arrow[dll, "n_+=-\chi_{par}((\eta',\alpha')^\vee\otimes(\eta'',\alpha''))-1"] \\
 & & \mathcal{N}^{\alpha'}(\eta')\times\mathcal{N}^{\alpha''}(\eta'')\cong\mathcal{N}^{\alpha\text{-sss}} \arrow[r, hook] & \mathcal{N}^{\alpha} &
\end{tikzcd}$$

<u>Cor</u> [Fu-H.-Pepin Lehalleur]

i) $$\mathfrak{h}(\mathcal{N}^{\alpha_-})\oplus\bigoplus_{n_+<j\le n_-}\mathfrak{h}(\mathcal{N}^{\alpha\text{-sss}})(j)\simeq\mathfrak{h}(\mathcal{N}^{\alpha_+})\oplus\bigoplus_{n_-<j\le n_+}\mathfrak{h}(\mathcal{N}^{\alpha\text{-sss}})(j)$$

$\nwarrow$ at least one of these sums is empty $\nearrow$

ii) For $i\in\mathbb{N}$, provided $g\gg0$: $CH^i(\mathcal{N}^\alpha)\simeq CH^i(\mathcal{N}^\beta)$ for α,β generic

(depending only on η & i) $\qquad$ & same for CH_i

<u>Rmk</u> (flag degenerations) [Boden-Yokogawa] $\quad$ Assume $(n,d)=1$

For suff. small (generic) α: $\quad$ α-ss of $E_*=(E,E_{i,j})\Leftrightarrow$ ss of E.

Thus there is a forgetful map $\mathcal{N}^\alpha\longrightarrow\mathcal{N}$ which is a flag bundle.

Consequently: $\quad \mathfrak{h}(\mathcal{N}^\alpha)=\mathfrak{h}(\mathcal{N})\otimes\bigotimes_{p_i\in D}\mathfrak{h}(\mathcal{F}l(m_i)) \leftarrow \mathfrak{h}(\mathcal{F}l)$ direct sum of Tate twists

for α suff. small $\nearrow$ $\qquad$ flag var.

Explicit formulae in rank 2: N parabolic pts $D=\{p_1,\dots,p_N\}$ with full flags.

WLOG $\begin{cases}\alpha_{i,1}=0 \quad \forall i\\ \alpha_{i,2}\in(0,1)\end{cases}$ (notion of ss preserved by certain shifts)

Space of weights: $\mathcal{A}_N=(0,1)^N$

Symmetries: ① Hecke modifications at $D'\subset D$

$$\mathcal{H}_{D'}: \mathcal{N}^{\alpha}(\eta)\xrightarrow{\sim}\mathcal{N}^{\alpha_{D'}}(2,d-|D'|,\underline{m})$$

If $|D'|=2e$: $\mathcal{O}(e)\otimes\mathcal{H}_{D'}:\mathcal{N}^{\alpha}(\eta)\xrightarrow{\sim}\mathcal{N}^{\alpha_{D'}}(\eta)$

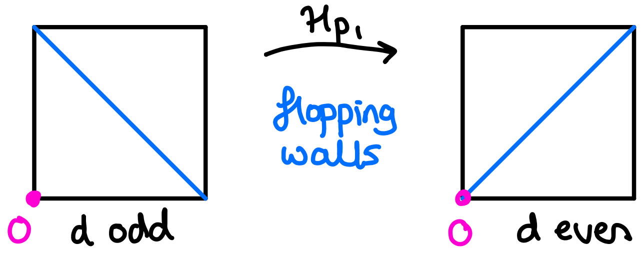
Hecke modifications at pairs $(p_i,p_{i+1}) \rightsquigarrow (\mathbb{Z}/2\mathbb{Z})^{N-1}\curvearrowright\mathcal{A}_N$

② $S_N\curvearrowright\mathcal{A}_N$ ⇝ does not give isomorphic moduli spaces

However: preserves type (n_-,n_+) of flip at wall-crossings

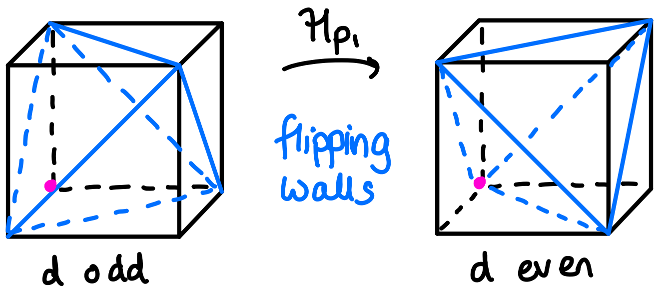
Pictures of wall-crossings for low N:

- $N=2$



$h(\mathcal{N}^{\alpha})=h(\mathcal{N})\otimes h(\mathbb{P}^1)^{\otimes 2}$

- $N=3$



outer chambers are permuted by Hecke modifications at pairs of parabolic points $\{p_i,p_j\}$

$\alpha_{ext}\in$ Exterior of tetrahedron: $h(\mathcal{N}^{\alpha_{ext}})=h(\mathcal{N})\otimes h(\mathbb{P}^1)^{\otimes 3}$

$\alpha_{in}\in$ Interior of tetrahedron: $h(\mathcal{N}^{\alpha_{in}})=h(\mathcal{N}^{\alpha_{ext}})\oplus h(\mathrm{Jac}(C))^{\otimes 2}(g)$

In general: i) moving from the exterior to the centre of $\mathcal{A}_N=(0,1)^N$ increases h

ii) A wall is a flopping wall ⟺ it contains the centre. Only happens if N is even

iii) We can compute $h(\mathcal{N}^{\alpha})$ for even d via Hecke modifications

Thm E [Fu-H.-Pepin Lehalleur]

For $n=2$ with full flags at N parabolic pts:

$$h(\mathcal{N}^{\alpha}) \simeq h(\mathcal{N}) \otimes h(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} h(\mathrm{Jac}(C))^{\otimes 2}(g+j)^{\oplus b_j(\alpha)}$$

Holds with $\mathbb{Z}$-coefficients

exponents can be explicitly computed

* We know $h(\mathcal{N})$ with $\mathbb{Q}$-coeffs by Thms A & B.

Generalises cohomological results of Bauer (over $\mathbb{P}^1$) & Holla.

Variation of stability for parabolic Higgs bundles

Thm F [Fu-H.-Pepin Lehalleur]

Fix (C,D) and $\eta=(n,d,\underline{1})$. full flags Then for a generic weight α
$h(\mathcal{M}^{\alpha}_{C,D}(\eta)) \in CHM^{\mathrm{eff}}(k,\mathbb{Z})$ is indept of α

Sketch of proof:

roughly: "alg. symplectic versions of standard flips"

(i) [Thaddeus] $\mathcal{M}^{\alpha}$ & $\mathcal{M}^{\beta}$ are related by Mukai flops

(ii) Mukai flops between smooth varieties preserve Chow groups & preserve motives in $DM(k,\mathbb{Z})$

(Extending a result of [Lee-Lin-Wang] for Mukai flops of smooth projective varieties) $\square$

Over $k=\mathbb{C}$, on the level of Betti cohomology, this result was seen in

- rank $n=2$ by Boden-Yokogawa (Nakajima showed the spaces are diffeo)
- rank $n=3$ by García-Prada - Gothen - Muñoz

Thm G [Fu-H.-Pepin Lehalleur]

For rank $n=2$ we have

$$h(\mathcal{M}^{\alpha}) = h(\mathcal{N}) \otimes h(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \le \ell \le N \\ \frac{\ell+1-N}{2} \le j \le g-1}} h(J_C) \otimes h(C^{(2j+N-\ell-1)})(3g-2j+\ell-2)^{\oplus \binom{N}{\ell}}$$

with $\mathbb{Z}$-coeffs.

References

This talk was based on joint work with Lie FU & Simon PEPIN LEHALLEUR

[arXiv:2011.14872] "Motives of moduli spaces of bundles on curves via variation of stability & flips"

[arXiv:2102.07546] "Motives of moduli spaces of rank 3 vector bundles and Higgs bundles on a curve"

See also the references therein.