

A FORMULA FOR THE VOEVODSKY MOTIVE OF THE MODULI STACK OF VECTOR BUNDLES ON A CURVE

VICTORIA HOSKINS AND SIMON PEPIN LEHALLEUR

Abstract

We prove a formula for the motive of the stack of vector bundles of fixed rank and degree over a smooth projective curve in Voevodsky's triangulated category of mixed motives with rational coefficients.

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1. INTRODUCTION

Let $\mathrm{Bun}_{n,d}$ denote the moduli stack of rank n , degree d vector bundles on a smooth projective geometrically connected curve C of genus g over a field k . In this paper, we prove the following formula for the motive of $\mathrm{Bun}_{n,d}$ in Voevodsky's triangulated category $\mathrm{DM}(k) := \mathrm{DM}(k, \mathbb{Q})$ of mixed motives over k with \mathbb{Q} -coefficients.

Theorem 1.1. *Suppose that $C(k) \neq \emptyset$; then in $\mathrm{DM}(k, \mathbb{Q})$, we have*

$$M(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}),$$

where $Z(C, \mathbb{Q}\{i\}) := \bigoplus_{j=0}^{\infty} M(C^{(j)}) \otimes \mathbb{Q}\{ij\}$ is a motivic Zeta function and $\mathbb{Q}\{i\} := \mathbb{Q}(i)[2i]$.

In particular, this implies a decomposition on Chow groups and ℓ -adic cohomology and, as explained below, this formula is compatible with previous cohomological descriptions of $\mathrm{Bun}_{n,d}$.

This paper is a continuation of our previous work [14] in which we define and study the motive $M(\mathrm{Bun}_{n,d}) \in \mathrm{DM}(k, R)$ for any coefficient ring R (provided the characteristic of k is invertible in R in positive characteristic); more generally, we introduce there the notion of an exhaustive stack and define motives of smooth exhaustive stacks by generalising a construction of Totaro for quotient stacks [18] (see [14, Definitions 2.15 and 2.17] for details).

V.H. is supported by the DFG Excellence Initiative at the Freie Universität Berlin and by the SPP 1786.

1.1. Overview of our previous results. In [14, Theorem 3.5], we work with a general coefficient ring R (for which the exponential characteristic is invertible) and give the following description of the motive of the stack $\mathrm{Bun}_{n,d}$ in terms of smooth projective Quot schemes by following a geometric argument for computing the ℓ -adic cohomology of this stack in [9].

Theorem 1.2. *For any effective divisor $D > 0$ on C , we have in $\mathrm{DM}(k, R)$*

$$M(\mathrm{Bun}_{n,d}) \simeq \mathrm{hocolim}_{l \in \mathbb{N}} M(\mathrm{Div}_{n,d}(lD)),$$

where $\mathrm{Div}_{n,d}(D) = \{\mathcal{E} \subset \mathcal{O}_C(D)^{\oplus n} : \mathrm{rk}(\mathcal{E}) = n, \mathrm{deg}(\mathcal{E}) = d\}$ is a smooth Quot scheme.

Our approach in [14] to describing the motives $M(\mathrm{Div}_{n,d}(lD))$ is to use Białyński-Birula decompositions [8] associated to an action of a generic one-parameter subgroup $\mathbb{G}_m \subset \mathrm{GL}_n$ on these Quot schemes, whose fixed loci are disjoint unions of products of symmetric powers of C . To use these decompositions to compute the motive of $\mathrm{Bun}_{n,d}$, one needs to understand the behaviour of the transition maps $i_l : \mathrm{Div}_{n,d}(lD) \hookrightarrow \mathrm{Div}_{n,d}((l+1)D)$ in the inductive system in Theorem 1.2 with respect to the motivic Białyński-Birula decompositions; this is very complicated, as although the closed immersion i_l is \mathbb{G}_m -equivariant, the closed subscheme $\mathrm{Div}_{n,d}(lD) \hookrightarrow \mathrm{Div}_{n,d}((l+1)D)$ does not intersect the Białyński-Birula strata transversally. We conjecture a precise description of these transition maps [14, Conjecture 3.9] and show that the formula for the motive of $\mathrm{Bun}_{n,d}$ appearing in Theorem 1.1 follows from this conjectural description of the transition maps.

1.2. Summary of the results and methods in this paper. In this paper, we prove the conjectural formula in [14] under the assumption that $R = \mathbb{Q}$. The main idea is to replace the Quot schemes with Flag-Quot schemes, which are generalisations of Quot schemes that allow flags of sheaves and then to describe the transition maps using these Flag-Quot schemes without using Białyński-Birula decompositions. The idea to use Flag-Quot schemes was inspired by a result of Laumon in [16] and its application in a paper of Heinloth to study the cohomology of the moduli space of Higgs bundles using Hecke modification stacks [13].

To prove Theorem 1.1, our starting point is Theorem 1.2, where as we assume that C has a rational point x we can take the divisor $D := x$ and write $\mathrm{Div}_{n,d}(l) := \mathrm{Div}_{n,d}(lx)$. We replace the Quot schemes $\mathrm{Div}_{n,d}(l)$ with smooth projective Flag-Quot schemes

$$\mathrm{FDiv}_{n,d}(l) = \{\mathcal{E}_{nl-d} \subsetneq \cdots \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_0 = \mathcal{O}_C(lx)^{\oplus n} : \mathrm{rk}(\mathcal{E}_i) = n \text{ and } \mathrm{deg} \mathcal{E}_i = nl - i\}.$$

The natural map $\mathrm{FDiv}_{n,d}(l) \rightarrow \mathrm{Div}_{n,d}(l)$ is small and is a S_{nl-d} -principal bundle over the open subset consisting of subsheaves $\mathcal{E} \subset \mathcal{O}_C(lx)^{\oplus n}$ with torsion quotient that has support consisting of $nl-d$ distinct points. Using these facts, we relate the motives of these two varieties as follows.

Theorem 1.3. *There is an induced S_{nl-d} -action on $M(\mathrm{FDiv}_{n,d}(l))$ and isomorphisms*

$$M(\mathrm{Div}_{n,d}(l)) \simeq M(\mathrm{FDiv}_{n,d}(l))^{S_{nl-d}} \simeq (M(C \times \mathbb{P}^{n-1})^{\otimes nl-d})^{S_{nl-d}} \simeq \mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1})).$$

An isomorphism $M(\mathrm{Div}_{n,d}(l)) \simeq \mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1}))$ is constructed by del Baño [11, Theorem 4.2] using associated motivic Białyński-Birula decompositions (see also [14, §3.2]). However, in del Baño's description, we do not understand the transition maps $M(i_l)$.

In fact, we deduce Theorem 1.3 as a special case of a more general result (Theorem 3.8), where we replace $\mathcal{O}_C(lx)^{\oplus n} \rightarrow C/k$ with a family of vector bundles $\mathcal{E} \rightarrow T \times C/T$ parametrised by a smooth k -scheme T and then study the motives of schemes of (iterated) Hecke correspondences as (Flag)-Quot schemes over T . This work was inspired by a beautiful description of the cohomology of these schemes due to Heinloth (see the proof of [13, Proposition 11] which uses ideas of Laumon [16, Theorem 3.3.1]). In fact we lift Heinloth's cohomological description of schemes of (iterated) Hecke correspondences to $\mathrm{DM}(k)$. To prove this result, in §2 we study the invariant piece of a motive with a finite group action, which is why we need to work with rational coefficients; the main result is Theorem 2.11, which states that for a small proper map $f : X \rightarrow Y$ of smooth projective k -varieties which is a principal G -bundle on the locus with finite fibres, we have an isomorphism $M(X)^G \cong M(Y)$. In §3, we study the geometry and motives of schemes of (iterated) Hecke correspondences in order to prove Theorem 3.8. Furthermore,

we obtain a formula for the motive of the Quot scheme of length l torsion quotients of a rank n locally free sheaf \mathcal{E} on C , which is independent of \mathcal{E} (Corollary 3.9); this complements recent analogous results in the Grothendieck ring of varieties [6, 17].

In §4.1, we lift the transition maps $i_l : \text{Div}_{n,d}(l) \rightarrow \text{Div}_{n,d}(l+1)$ to the schemes $\text{FDiv}_{n,d}(l)$. It turns out to be much simpler to describe the motivic behaviour of the lifts of the transition maps to Flag-Quot schemes, as those are iterated projective bundles over products of the curve. By symmetrising this description, we deduce the corresponding behaviour for the maps $M(i_l)$ which enables us to prove Theorem 1.1 in §4.2. Finally, in §4.3, we give a second proof of this formula for $M(\text{Bun}_{n,d})$ which follows more closely the ideas in our previous work [14].

It remains an interesting open question as to whether Theorem 1.1 holds integrally. One may expect this to be the case, as Atiyah and Bott [2] gave an integral description of the cohomology of $\text{Bun}_{n,d}$. In fact, in future work we plan to remove the assumption that C has a rational point, by giving a more canonical construction of the isomorphism in Theorem 1.1 inspired by [2].

By Poincaré duality, we obtain a formula for the compactly supported motive $M^c(\text{Bun})$, which compares nicely with previous results, such as the Behrend–Dhillon formula for the virtual class of $\text{Bun}_{n,d}$ in the Grothendieck ring of varieties [7] and Harder’s formula for the stacky point count over a finite field [12] (see the discussion in [14, §4.2]).

Corollary 1.4. *Assume $C(k) \neq \emptyset$; then the compactly supported motive of $\text{Bun}_{n,d}$ is given by*

$$M^c(\text{Bun}_{n,d}) \simeq M(\text{Jac } C) \otimes M^c(\text{BG}_m)\{(n^2 - 1)(g - 1)\} \otimes \bigotimes_{i=2}^n Z(C, \mathbb{Q}\{-i\}).$$

1.3. Background on motives. Let us briefly recall some basic properties about $\text{DM}(k) := \text{DM}(k, \mathbb{Q})$. It is a monoidal \mathbb{Q} -linear triangulated category. For a separated scheme X of finite type over k , we can associate a motive $M(X) \in \text{DM}(k)$, which is covariantly functorial in X and behaves like a homology theory. The motive $M(\text{Spec } k) := \mathbb{Q}\{0\}$ is the unit for the monoidal structure, and there are Tate motives $\mathbb{Q}\{n\} := \mathbb{Q}(n)[2n] \in \text{DM}(k)$ for all $n \in \mathbb{Z}$. For any motive M and $n \in \mathbb{Z}$, we write $M\{n\} := M \otimes \mathbb{Q}\{n\}$.

In $\text{DM}(k)$, there are Künneth isomorphisms, \mathbb{A}^1 -homotopy invariance, Gysin distinguished triangles, projective bundle formulae and Poincaré duality isomorphisms, as well as realisation functors (to compare with Betti, de Rham and ℓ -adic cohomology) and descriptions of Chow groups as homomorphism groups in $\text{DM}(k)$. For a precise statement of these results, we refer the reader to the summary in [14, §2].

In this paper, unlike in [14], we need to use categories of relative motives over varying base schemes, and the associated “six operations” formalism. We only need a small portion of the machinery, which we summarise here; for more details, see [4, §3]. Given a base scheme S , which in this paper will always be of finite type and separated over the field k , there is a monoidal \mathbb{Q} -linear triangulated category $\text{DM}(S)$, which we take to be the category $\text{DA}^{\text{ét}}(S, \mathbb{Q})$ of [4] and [5, §3]. The monoidal unit of $\text{DM}(S)$ is denoted by \mathbb{Q}_S (in particular, $\mathbb{Q}_k := \mathbb{Q}\{0\} \in \text{DM}(k)$). Given a morphism $f : S \rightarrow T$ between two such base schemes (so that f is automatically separated and of finite type), there are two adjunctions

$$f^* : \text{DM}(T) \rightleftarrows \text{DM}(S) : f_* \quad \text{and} \quad f_! : \text{DM}(S) \rightleftarrows \text{DM}(T) : f^!$$

which satisfies the same formal properties as the corresponding adjunctions (f^*, Rf_*) and $(Rf_!, f^!)$ in the setting of derived categories of ℓ -adic sheaves. In particular, we have natural isomorphisms $f_* \simeq f_!$ for f proper, and $f^* \simeq f^!$ for f étale. We also have proper base change (in the general form of [4, Theorem 3.9]) and a purity isomorphism $f^! \simeq f^*(-)\{d\}$ for f smooth of relative dimension d .

Many constructions in $\text{DM}(k)$ have an alternative description in terms of the six operations formalism: for a k -scheme X with structure map π_X , we have

$$M(X) \simeq \pi_{X!} \pi_X^! \mathbb{Q}_k \quad \text{and} \quad M^c(X) \simeq \pi_{X*} \pi_X^! \mathbb{Q}_k.$$

Acknowledgements. We thank Elden Elmanto, Michael Gröchenig, Jochen Heinloth, Frances Kirwan and Marc Levine for useful discussions.

2. SMALL MAPS AND INDUCED ACTIONS ON MOTIVES

2.1. Properties of small maps. Let us recall the following definition.

Definition 2.1. Let X and Y be algebraic varieties¹ over k , and let $f : X \rightarrow Y$ be a proper morphism. For $\delta \in \mathbb{N}$, define

$$Y_{f,\delta} := \{y \in Y \mid \dim(f^{-1}(y)) = \delta\}.$$

This is a locally closed subscheme of Y , and so its codimension in Y makes sense. We say f is

- (i) semismall if $\text{codim}_Y(Y_{f,\delta}) \geq 2\delta$ for all $\delta \geq 0$.
- (ii) small if f is semismall and $\text{codim}_Y(Y_{f,\delta}) > 2\delta$ if for all $\delta > 0$.

Remark 2.2. This formulation in terms of codimension also makes sense when X, Y are algebraic stacks and $f : X \rightarrow Y$ is a proper representable morphism.

Lemma 2.3. *Proper (semi)small morphisms are stable by flat base change.*

Proof. Let $f : X \rightarrow Y$ be a proper morphism and $g : Z \rightarrow Y$ be a flat morphism. Write $\tilde{f} : X \times_Y Z \rightarrow Z$. With the notations of Definition 2.1, for all $\delta \in \mathbb{N}$ we have $Z_{\tilde{f},\delta} = g^{-1}(Y_{f,\delta})$. Since g is flat, we deduce that

$$\text{codim}_Z(Z_{\tilde{f},\delta}) = \text{codim}_Z(g^{-1}(Y_{f,\delta})) \geq \text{codim}_Y(Y_{f,\delta})$$

which implies the result. \square

Remark 2.4. This property also holds for proper representable (semi-)small morphisms between algebraic stacks, with the same proof.

The key property of (semi-)small morphisms for this paper is the following lemma.

Lemma 2.5. [10, Proposition 2.1.1, Remark 2.1.2] *Let $f : X \rightarrow Y$ be a proper morphism of varieties. For $\delta \in \mathbb{N}$, let $Y_{f,\delta}$ be as in Definition 2.1 and $X_{f,\delta} := f^{-1}(Y_{f,\delta})$.*

- (i) *The morphism f is semismall if and only if*

$$\dim(X \times_Y X) \leq \dim(Y).$$

- (ii) *If f is semismall and surjective, then $\dim(X \times_Y X) = \dim(X)$.*
- (iii) *If f is small and surjective, then the irreducible components of dimension $\dim(X)$ of $X \times_Y X$ are the closures of the irreducible components of $X_{f,0} \times_{Y_{f,0}} X_{f,0}$ and in particular dominate Y .*

2.2. Endomorphisms of motives of small maps. Given a morphism of schemes $f : X \rightarrow Y$, we denote by $\text{Aut}_Y(X)$ the group of automorphisms of X as a Y -scheme. For a k -scheme X and an integer $i \in \mathbb{N}$, we denote by $Z_i(X)$ the group of i -dimensional cycles with rational coefficients on X , and $\text{CH}_i(X)$ the i -th Chow group, i.e., the quotient of $Z_i(X)$ by rational equivalence.

Proposition 2.6. *Let $f : X \rightarrow Y$ be a proper morphism with X smooth equidimensional of dimension $d \in \mathbb{N}$. Then there exist an isomorphism*

$$\phi_f : \text{CH}_d(X \times_Y X) \simeq \text{End}_{\text{DM}(Y)}(f_*\mathbb{Q}_X)$$

such that, if $e : U \hookrightarrow Y$ is a étale morphism and $\tilde{e} : V \hookrightarrow X$ is its base change along f and $\tilde{f} : V \rightarrow U$ the base change of f along e , we have a commutative diagram

$$\begin{array}{ccc} \text{CH}_d(X \times_Y X) & \xrightarrow{\phi_f} & \text{End}_{\text{DM}(Y)}(f_*\mathbb{Q}_X) \\ \downarrow (\tilde{e} \times \tilde{e})^* & & \downarrow e^* \\ \text{CH}_d(V \times_U V) & \xrightarrow{\phi_{\tilde{f}}} & \text{End}_{\text{DM}(U)}(\tilde{f}_*\mathbb{Q}_V). \end{array}$$

¹Here as in the rest of the paper, variety means finite type separated over k , not necessarily irreducible.

Proof. Write $p_1, p_2 : X \times_Y X \rightarrow X$ for the two projections. For a k -scheme Z , write $\pi_Z : Z \rightarrow \text{Spec}(k)$ for its structure map. We start with the isomorphism

$$\text{CH}_d(X \times_Y X) \simeq \text{Hom}_{\text{DM}(k)}(\mathbb{Q}\{d\}, M^c(X \times_Y X)) \simeq \text{Hom}_{\text{DM}(X \times_Y X)}(\mathbb{Q}_{X \times_Y X}, \pi_{X \times_Y X}^! \mathbb{Q}\{-d\})$$

where we have used the description of Chow groups for general varieties in $\text{DM}(k)$, the formula for M^c in terms of the six operations and the adjunction $(\pi_{X \times_Y X}^*, \pi_{X \times_Y X*})$. We then write

$$\begin{aligned} \text{Hom}_{\text{DM}(X \times_Y X)}(\mathbb{Q}_{X \times_Y X}, \pi_{X \times_Y X}^! \mathbb{Q}_k\{-d\}) &\simeq \text{Hom}_{\text{DM}(X \times_Y X)}(\mathbb{Q}_{X \times_Y X}, p_1^! \pi_X^! \mathbb{Q}_k\{-d\}) \\ &\simeq \text{Hom}_{\text{DM}(X \times_Y X)}(\mathbb{Q}_{X \times_Y X}, p_1^! \mathbb{Q}_X) \\ &\simeq \text{Hom}_{\text{DM}(X)}(\mathbb{Q}_X, p_{1*} p_1^! \mathbb{Q}_X) \\ &\simeq \text{Hom}_{\text{DM}(X)}(\mathbb{Q}_X, f^! f_* \mathbb{Q}_X) \\ &\simeq \text{End}_{\text{DM}(Y)}(f_* \mathbb{Q}_X) \end{aligned}$$

where the first isomorphism follows from $\pi_{X \times_Y X} = \pi_X \circ p_1$, the second follows from relative purity for the smooth morphism π_X , the third is the adjunction (p_1^*, p_{1*}) , the fourth is proper base change and the fifth uses the adjunction $(f_!, f^!)$ and the properness of f .

The isomorphism ϕ_f is defined as the composition of the sequence of isomorphisms above. Its compatibility with pullback by an étale morphism e is a matter of carefully going through the construction and using the natural isomorphism $e^! \simeq e^*$ and proper base change. \square

Remark 2.7. Since the target of ϕ_f is clearly a \mathbb{Q} -algebra, the proposition endows $\text{CH}_d(X \times_Y X)$ with a \mathbb{Q} -algebra structure. The multiplication can be described using refined Gysin morphisms, but we will not need this.

Proposition 2.8. *Let $f : X \rightarrow Y$ be a surjective proper small morphism with X and Y smooth varieties. Let $f^\circ : X^\circ \rightarrow Y^\circ$ be the restriction of f to the locus with finite fibers, and $j : Y^\circ \rightarrow Y$ the corresponding open immersion. Then the natural map*

$$j^* : \text{End}_{\text{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \rightarrow \text{End}_{\text{DM}(Y^\circ)}(f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ})$$

is an isomorphism of rings.

Proof. First, let us explain how j^* is defined. Write $\tilde{j} : X^\circ \rightarrow X$. Then we have

$$j^* f_! f^! \mathbb{Q}_Y \simeq f_!^\circ \tilde{j}^* f^! \mathbb{Q}_Y \simeq f_!^\circ \tilde{j}^! f^! \mathbb{Q}_Y \simeq f_!^\circ (f^\circ)^\circ j^! \mathbb{Q}_Y \simeq f_!^\circ (f^\circ)^\circ j^* \mathbb{Q}_Y \simeq f_!^\circ (f^\circ)^\circ \mathbb{Q}_{Y^\circ}$$

where we have used proper base change, compatibility of $(-)^!$ with composition and the fact that $e^! \simeq e^*$ for e étale. Then j^* is defined as

$$\text{End}_{\text{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \xrightarrow{j^*} \text{End}_{\text{DM}(Y)}(j^* f_! f^! \mathbb{Q}_Y) \simeq \text{End}_{\text{DM}(Y^\circ)}(f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ}).$$

The map j^* is clearly compatible with addition and composition, hence is a homomorphism of rings. It remains to show that it is bijective.

Since X and Y are both smooth of dimension d over k , we can use purity isomorphisms to obtain an isomorphism

$$(1) \quad f^! \mathbb{Q}_Y \simeq f^! \pi_Y^! \mathbb{Q}_k\{-d\} \simeq \pi_X^! \mathbb{Q}_k\{-d\} \simeq \mathbb{Q}_X.$$

We deduce that $f_* \mathbb{Q}_X \simeq f_* f^! \mathbb{Q}_Y \simeq f_! f^! \mathbb{Q}_Y$, and similarly that $f_*^\circ \mathbb{Q}_{X^\circ} \simeq f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ}$. These two isomorphisms are compatible with restriction along j . Combining this observation with Proposition 2.6, we have the commutative diagram with horizontal isomorphisms

$$\begin{array}{ccccc} \text{CH}_d(X \times_Y X) & \xrightarrow{\sim \phi_f} & \text{End}_{\text{DM}(Y)}(f_* \mathbb{Q}_X) & \xrightarrow{\sim} & \text{End}_{\text{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \\ (j \times j)^* \downarrow & & j^* \downarrow & & j^* \downarrow \\ \text{CH}_d(X^\circ \times_{Y^\circ} X^\circ) & \xrightarrow{\sim \phi_{f^\circ}} & \text{End}_{\text{DM}(Y^\circ)}(f_*^\circ \mathbb{Q}_{X^\circ}) & \xrightarrow{\sim} & \text{End}_{\text{DM}(Y^\circ)}(f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ}). \end{array}$$

On a variety of dimension d , we have $\text{CH}_d = Z_d$, i.e., rational equivalence is trivial on top-dimensional cycles. By Lemma 2.5 (ii), this implies $\text{CH}_d(X \times_Y X) \simeq Z_d(X \times_Y X)$ and also

$\mathrm{CH}_d(X \times_Y X) \simeq Z_d(X^\circ \times_{Y^\circ} X^\circ)$. By Lemma 2.5 (iii), the restriction morphism $Z_d(X \times_Y X) \rightarrow Z_d(X^\circ \times_{Y^\circ} X^\circ)$ is a bijection. We deduce that the left vertical map in the diagram above is a bijection, and conclude that the right vertical map is a bijection. \square

Lemma 2.9. *Let $f : X \rightarrow Y$ be a finite type separated morphism with Y smooth. Then there exist an morphism of \mathbb{Q} -algebras*

$$\psi_f : \mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \rightarrow \mathrm{End}_{\mathrm{DM}(k)}(M(X))$$

such that, for $e : U \hookrightarrow Y$ an étale morphism, $\tilde{e} : V \hookrightarrow X$ its base change along f and $\tilde{f} : V \rightarrow U$ the base change of f along e , we have a commutative diagram

$$\begin{array}{ccc} \mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) & \xrightarrow{\psi_f} & \mathrm{End}_{\mathrm{DM}(k)}(M(X)) \\ \downarrow e^* & & \downarrow e^* \\ \mathrm{End}_{\mathrm{DM}(U)}(\tilde{f}_! \tilde{f}^! \mathbb{Q}_U) & \xrightarrow{\psi_{\tilde{f}}} & \mathrm{End}_{\mathrm{DM}(k)}(M(V)). \end{array}$$

Proof. Recall that, for Z a smooth variety of dimension e over k , we have a canonical purity isomorphism $\pi_Z^! \mathbb{Q}_k \simeq \mathbb{Q}_Z\{e\}$. By working with each connected component of Y separately, we can assume that Y is equidimensional of dimension d . We deduce that

$$M(X) := \pi_{X!} \pi_X^! \mathbb{Q}_k \simeq \pi_{Y!} f_! f^! \pi_Y^! \mathbb{Q}_k \simeq \pi_{Y!} f_! f^! \mathbb{Q}_Y\{d\}$$

by using the purity isomorphism for the smooth morphism π_Y . We define ψ_f as the composition

$$\mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \xrightarrow{\pi_{Y!}(-)\{d\}} \mathrm{End}_{\mathrm{DM}(k)}(\pi_{Y!} f_! f^! \mathbb{Q}_Y\{d\}) \simeq \mathrm{End}_{\mathrm{DM}(k)} M(X).$$

The compatibility with pullbacks by étale morphisms follows again easily from the natural isomorphism $e^! \simeq e^*$ for an étale morphism e . \square

2.3. Group actions on motives of small maps. Let S be a scheme, $M \in \mathrm{DM}(S)$ a motive and G a group. An action of G on M is a morphism of groups $a : G \rightarrow \mathrm{Aut}_{\mathrm{DM}(S)}(M)$. In particular, given a morphism $f : X \rightarrow Y$, we have an action $\mathrm{Aut}_Y(X) \rightarrow \mathrm{Aut}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$.

Assuming further that G is finite, let

$$\Pi_a := \frac{1}{|G|} \sum_{g \in G} a(g) \in \mathrm{End}_{\mathrm{DM}(S)}(M)$$

which makes sense since $\mathrm{DM}(S)$ is \mathbb{Q} -linear. Then Π_a is idempotent, and since $\mathrm{DM}(S)$ is idempotent-complete we define the invariant motive $M^G \in \mathrm{DM}(S)$ as the image of Π_a .

Example 2.10. An important example for this paper are motives of symmetric products. For a quasi-projective variety X over k and $n \in \mathbb{N}$, we have a morphism $f : X^n \rightarrow \mathrm{Sym}^n(X)$. The symmetric group S_n acts on X^n over $\mathrm{Sym}^n(X)$, so that we get an induced action on $M(X^n)$ such that $M(f) : M(X^n) \rightarrow M(\mathrm{Sym}^n(X))$ factors via $M(X^n)^{S_n} \rightarrow M(\mathrm{Sym}^n(X))$. Since S_n acts transitively on the geometric fibers of f , this second morphism is an isomorphism $M(X^n)^{S_n} \simeq M(\mathrm{Sym}^n(X))$ by [3, Corollaire 2.1.166].

The main result of this section is a generalisation of the previous example where we do not have a global action on X and f is not necessarily finite but only small.

Theorem 2.11. *Let $f : X \rightarrow Y$ be a small surjective proper morphism between smooth connected varieties. Assume that the restriction $f^\circ : X^\circ \rightarrow Y^\circ$ to the locus with finite fibers is a principal G -bundle. Then the action of G on $M(X^\circ)$ extends to an action on $M(X)$ which induces an isomorphism $M(X)^G \simeq M(Y)$; we have a commutative diagram*

$$\begin{array}{ccccc} M(X^\circ) & \longrightarrow & M(X^\circ)^G & \xrightarrow{\sim} & M(Y^\circ) \\ \downarrow & & \downarrow & & \downarrow \\ M(X) & \longrightarrow & M(X)^G & \xrightarrow{\sim} & M(Y). \end{array}$$

Proof. By working separately with each connected component, we can assume Y is connected, and in particular equidimensional. Write $d = \dim(X) = \dim(Y)$. Since $f^\circ : X^\circ \rightarrow Y^\circ$ is a principal G -bundle, we have a morphism of groups $G \rightarrow \text{Aut}_{Y^\circ}(X^\circ)$. We deduce a morphism of groups $G \rightarrow \text{Aut}_{\text{DM}(Y^\circ)}(f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ})$. By Proposition 2.8, this yields a morphism of groups $G \rightarrow \text{Aut}_{\text{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$. We compose with the morphism ψ_f of Lemma 2.9 and get a morphism of groups $G \rightarrow \text{Aut}_{\text{DM}(k)}(M(X))$, which is the required action.

Let us check that the morphism $M(f) : M(X) \rightarrow M(Y)$ factors through $M(X)^G$. Given its construction, it suffices to show that the counit morphism $f_! f^! \mathbb{Q}_Y \rightarrow \mathbb{Q}_Y$ factors through $(f_! f^! \mathbb{Q}_Y)^G$. For this, it suffices to show that, for any $g \in G$, the composition $f_! f^! \mathbb{Q}_Y \xrightarrow{g} f_! f^! \mathbb{Q}_Y \rightarrow \mathbb{Q}_Y$ coincides with the counit of the adjunction $(f_!, f^!)$. By the same adjunction, this amounts to comparing two maps $f^! \mathbb{Q}_Y \rightarrow f^! \mathbb{Q}_Y$. By equation (1), we have $f^! \mathbb{Q}_Y \simeq \mathbb{Q}_X$. By [5, Proposition 11.1], and using the fact that X° is dense in X , we have

$$\text{Hom}_{\text{DM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X) \simeq \mathbb{Q}^{\pi_0(X)} \hookrightarrow \mathbb{Q}^{\pi_0(X^\circ)} \simeq \text{Hom}_{\text{DM}(X^\circ)}(\mathbb{Q}_{X^\circ}, \mathbb{Q}_{X^\circ})$$

hence we can check the required equality after restriction to X° ; that is, we must show that for any $g \in G$, the composition $f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ} \xrightarrow{g} f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ} \rightarrow \mathbb{Q}_{Y^\circ}$ coincides with the counit of the adjunction $(f_!^\circ, f^{\circ!})$. This is clear since G acts through $\text{Aut}_{Y^\circ}(X^\circ)$.

By construction, to show that the induced map $M(X)^G \rightarrow M(Y)$ is an isomorphism, it suffices to show that the morphism $(f_! f^! \mathbb{Q}_Y)^G \rightarrow \mathbb{Q}_Y$ is an isomorphism. Let $\Pi_G \in \text{End}_{\text{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$ the projector onto $(f_! f^! \mathbb{Q}_Y)^G$. Since X and Y are smooth of the same dimension d , the purity isomorphisms yield an isomorphism $f^! \mathbb{Q}_Y \simeq \mathbb{Q}_X$ (equation (1)). Moreover, this isomorphism is compatible with restriction to Y° , in the sense that after applying $j^! = \tilde{j}^*$ for $\tilde{j} : X^\circ \rightarrow X$, it coincides with the simpler isomorphism $f^{\circ!} \mathbb{Q}_{Y^\circ} \simeq f^{\circ*} \mathbb{Q}_{X^\circ} \simeq \mathbb{Q}_{X^\circ}$ (using that f° is étale).

Consider the composition

$$\Pi' : f_! f^! \mathbb{Q}_Y \xrightarrow{\eta_!} \mathbb{Q}_Y \xrightarrow{\frac{1}{|G|}} \mathbb{Q}_Y \xrightarrow{\epsilon_*} f_* \mathbb{Q}_X \simeq f_! f^! \mathbb{Q}_Y$$

where ϵ_* is the unit for the adjunction (f^*, f_*) and $\eta_!$ is the counit for the adjunction $(f_!, f^!)$. By [3, Lemme 2.1.165], we see that $j^* \Pi'$ is a projector which coincides with $j^* \Pi_G$. By the injectivity of j^* (Proposition 2.8), this implies that $\Pi' = \Pi_G$, thus Π' is a projector, and to conclude it remains to identify the image of Π' with the morphism $f_! f^! \mathbb{Q}_Y \rightarrow \mathbb{Q}_Y$.

For this, it is clearly enough to show that the composition

$$\mathbb{Q}_Y \xrightarrow{\epsilon_*} f_* \mathbb{Q}_X \simeq f_! f^! \mathbb{Q}_Y \xrightarrow{\eta_!} \mathbb{Q}_Y$$

coincides with the multiplication by $|G|$. Since Y and Y° are connected, by [5, Proposition 11.1] we have

$$\text{Hom}_{\text{DM}(Y)}(\mathbb{Q}_Y, \mathbb{Q}_Y) \simeq \mathbb{Q} \simeq \text{Hom}_{\text{DM}(Y^\circ)}(\mathbb{Q}_{Y^\circ}, \mathbb{Q}_{Y^\circ})$$

hence it is enough to show this after restriction to Y° . The corresponding composition is

$$\mathbb{Q}_{Y^\circ} \xrightarrow{\epsilon_*} f_*^\circ \mathbb{Q}_{X^\circ} \simeq f_!^\circ f^{\circ!} \mathbb{Q}_{Y^\circ} \xrightarrow{\eta_!} \mathbb{Q}_{Y^\circ}$$

which coincides with multiplication by $|G|$ by [3, Lemme 2.1.165]. \square

Remark 2.12. Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{g} & Y' \end{array}$$

with f and f' satisfying the assumptions of Theorem 2.11 with groups G, G' . If g does not send the locus Y^0 into $(Y^0)'$, it is not clear how to formulate conditions which make the morphism $M(X) \rightarrow M(X')$ equivariant with respect to some given homomorphism $G \rightarrow G'$. However in the application in §4, we have an alternative description of the actions which make a certain equivariance property clear (see Proposition 4.2).

3. MOTIVES OF SCHEMES OF HECKE CORRESPONDENCES

In this section, we introduce some generalisations of the schemes of matrix divisors $\text{Div}_{n,d}(D)$ and the flag-generalisation $\text{FDiv}_{n,d}(D)$ and study their motives. The main result in this section is inspired by work of Laumon [16] and Heinloth [13].

3.1. Definitions and basic properties. For a family \mathcal{E} of vector bundles on C parametrised by a k -scheme T , we write $\text{rk}(\mathcal{E}) = n$ and $\text{deg}(\mathcal{E}) = d$ if the fibrewise rank and degree of this family are n and d respectively.

Definition 3.1. For $l \in \mathbb{N}$ and a family \mathcal{E} of rank n degree d vector bundles over C parametrised by k -scheme T , we define two T -schemes $\mathcal{H}_{\mathcal{E}/T}^l$ and $\mathcal{FH}_{\mathcal{E}/T}^l$ as follows: over $g : S \rightarrow T$, the points of these schemes are given by

$$\mathcal{H}_{\mathcal{E}/T}^l(S) := \left\{ \phi : \mathcal{F} \hookrightarrow (g \times \text{id}_C)^* \mathcal{E} : \begin{array}{l} \mathcal{F} \rightarrow S \times C \text{ family of vector bundles on } C \\ \text{rk}(\mathcal{F}) = n, \text{deg}(\mathcal{F}) = d - l, \text{rk}(\phi) = n \end{array} \right\}$$

and

$$\mathcal{FH}_{\mathcal{E}/T}^l(S) := \left\{ \mathcal{F}_l \hookrightarrow \mathcal{F}_{l-1} \cdots \hookrightarrow \mathcal{F}_0 := (g \times \text{id}_C)^* \mathcal{E} : \begin{array}{l} \mathcal{F}_i \rightarrow S \times C \text{ family of vector bundles} \\ \text{rk}(\mathcal{F}_i) = n, \text{deg}(\mathcal{F}_i) = d - i \\ \text{rk}(\mathcal{F}_i \rightarrow \mathcal{F}_{i-1}) = n \text{ for } i = 1, \dots, l \end{array} \right\}.$$

We refer to $\mathcal{H}_{\mathcal{E}/T}^l$ as the T -scheme of length l Hecke correspondences of \mathcal{E} and the $\mathcal{FH}_{\mathcal{E}/T}^l$ as the T -scheme of l -iterated Hecke correspondences of \mathcal{E} .

Let us first explain why these are both schemes over T . The scheme of length l Hecke correspondences $\mathcal{H}_{\mathcal{E}/T}^l$ is the Quot scheme over T

$$\mathcal{H}_{\mathcal{E}/T}^l = \text{Quot}_{T \times C/T}^{(0,l)}(\mathcal{E})$$

parametrising quotients families of \mathcal{E} of rank 0 and degree l , which is a projective T -scheme. Similarly $\mathcal{FH}_{\mathcal{E}/T}^l$ is a generalisation of Quot schemes to allow flags of arbitrary length, called a Flag-Quot or Drap scheme (see [15, Appendix 2A]); thus $\mathcal{FH}_{\mathcal{E}/T}^l$ is also projective over T . In fact, as we are considering torsion quotients of a smooth projective curve, both $\mathcal{H}_{\mathcal{E}/T}^l$ and $\mathcal{FH}_{\mathcal{E}/T}^l$ are smooth T -schemes (see [15, Propositions 2.2.8 and 2.A.12]). In particular, if T/k is smooth (resp. projective), then both these schemes are smooth (resp.) projective over k .

Example 3.2. Let $T = \text{Spec}(k)$ and $\mathcal{E} = \mathcal{O}_C(D)^{\oplus n}$ for a divisor D on C ; then

$$\mathcal{H}_{\mathcal{E}/T}^{n \text{deg}(D) - d} = \text{Div}_{n,d}(D) \quad \text{and} \quad \mathcal{FH}_{\mathcal{E}/T}^{n \text{deg}(D) - d} = \text{FDiv}_{n,d}(D),$$

which are both smooth and projective.

We introduce some notation and properties of these Hecke schemes in the following remark.

Remark 3.3. Let \mathcal{E} be a family of rank n degree d vector bundles over C parametrised by T .

- (i) For $l = 0$, we note that $\mathcal{FH}_{\mathcal{E}/T}^0 = \mathcal{H}_{\mathcal{E}/T}^0 = T$ and for $l = 1$, we have

$$\mathcal{FH}_{\mathcal{E}/T}^1 = \mathcal{H}_{\mathcal{E}/T}^1 \cong \mathbb{P}(\mathcal{E}) \rightarrow T \times C,$$

where this projection is given by taking the support of the family of degree 1 torsion sheaves. Indeed, an elementary modification of a vector bundle $E \rightarrow C$ at $x \in C$ is equivalent to a surjection $E_x \rightarrow \kappa(x)$ (up to scalar multiplication).

- (ii) Since $\mathcal{FH}_{\mathcal{E}/T}^l$ is a Flag-Quot scheme there is a universal flag of vector bundles

$$\mathcal{U}_l^1 \hookrightarrow \mathcal{U}_{l-1}^1 \hookrightarrow \cdots \hookrightarrow \mathcal{U}_1^1 \hookrightarrow \mathcal{U}_0^1 := p_2^* \mathcal{E}$$

over $\mathcal{FH}_{\mathcal{E}/T}^l \times_T (T \times C) \cong \mathcal{FH}_{\mathcal{E}/T}^l \times C$. In fact, Flag-Quot schemes, and in particular schemes of iterated Hecke correspondences, are constructed as iterated relative Quot schemes. More precisely, we have

$$\pi_l : \mathcal{FH}_{\mathcal{E}/T}^l \cong \mathcal{H}_{\mathcal{U}_{l-1}^1 / \mathcal{FH}_{\mathcal{E}/T}^{l-1}}^1 \cong \mathbb{P}(\mathcal{U}_{l-1}^{l-1}) \rightarrow \mathcal{FH}_{\mathcal{E}/T}^{l-1} \times_T (T \times C) \cong \mathcal{FH}_{\mathcal{E}/T}^{l-1} \times C.$$

where $\pi_l(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) := (\mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0, \text{supp}(\mathcal{F}_{l-1}/\mathcal{F}_l))$

(iii) There is a map $P_l : \mathcal{FH}_{\mathcal{E}/T}^l \rightarrow T \times C^l$ obtained by composing the maps π_j for $1 \leq j \leq l$

$$P_l : \mathcal{FH}_{\mathcal{E}/T}^l \rightarrow \mathcal{FH}_{\mathcal{E}/T}^{l-1} \times_T (T \times C) \rightarrow \mathcal{FH}_{\mathcal{E}/T}^{l-2} \times_T (T \times C)^{\times T^2} \cdots \rightarrow T \times_T (T \times C)^{\times T^l}.$$

Explicitly, we have $P_l(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) = (\text{supp}(\mathcal{F}_0/\mathcal{F}_1), \dots, \text{supp}(\mathcal{F}_{l-1}/\mathcal{F}_l))$.

(iv) For $1 \leq j \leq l$, we let $\text{pr}_j^l : \mathcal{FH}_{\mathcal{E}/T}^l \rightarrow T \times C$ denote the composition of P_l with the projection onto the j th copy of $T \times C$; that is,

$$\text{pr}_j^l(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) = \text{supp}(\mathcal{F}_{j-1}/\mathcal{F}_j).$$

(v) Let $p_l : \mathcal{FH}_{\mathcal{E}/T}^l \rightarrow \mathcal{FH}_{\mathcal{E}/T}^{l-1}$ denote the composition of π_l with the projection to the first factor; then for $1 \leq j \leq l-1$, we have $(p_l \times \text{id}_C)^* \mathcal{U}_j^{l-1} = \mathcal{U}_j^l$.

Lemma 3.4. *Let \mathcal{E} be a family of rank n degree d vector bundles over C parametrised by a scheme T ; then the scheme $\mathcal{FH}_{\mathcal{E}/T}^l$ is an l -iterated \mathbb{P}^{n-1} -bundle over $T \times C^l$. More precisely, we have the following sequence of projective bundles*

$$\mathcal{FH}_{\mathcal{E}/T}^l \cong \mathbb{P}(\mathcal{U}_{l-1}^{l-1}) \rightarrow \mathcal{FH}_{\mathcal{E}/T}^{l-1} \times C \cong \mathbb{P}(\mathcal{U}_{l-2}^{l-2}) \times C \rightarrow \cdots \rightarrow \mathcal{FH}_{\mathcal{E}/T}^1 \times C^{l-1} \cong \mathbb{P}(\mathcal{E}) \times C^{l-1} \rightarrow T \times C^l.$$

Proof. This follows by induction from Remark 3.3 (i) and (ii). \square

By repeatedly applying the projective bundle formula, we obtain the following corollary.

Corollary 3.5. *Let \mathcal{E} be family of rank n degree d vector bundles over C parametrised by a scheme T . Then*

$$M(\mathcal{FH}_{\mathcal{E}/T}^l) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}.$$

In fact, we will need to explicitly identify this isomorphism. For a rank n vector bundle \mathcal{V} over a scheme X , the projective bundle $\pi : \mathbb{P}(\mathcal{V}) \rightarrow X$ is equipped with a line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. The first chern class of this line bundle defines a map $c_1(\mathcal{L}) : M(\mathbb{P}(\mathcal{V})) \rightarrow \mathbb{Q}\{1\}$ and for $i \geq 0$ it induces maps

$$c_1(\mathcal{L})^{\otimes i} : M(\mathbb{P}(\mathcal{V})) \xrightarrow{M(\Delta)} M(\mathbb{P}(\mathcal{V}))^{\otimes i} \xrightarrow{c_1(\mathcal{L})^{\otimes i}} \mathbb{Q}\{i\}$$

which together define a map $[c_1(\mathcal{L})] := \bigoplus_{i=0}^{n-1} c_1(\mathcal{L})^{\otimes i} : M(\mathbb{P}(\mathcal{V})) \rightarrow \bigoplus_{i=0}^{n-1} \mathbb{Q}\{i\} \simeq M(\mathbb{P}^{n-1})$. Then the projective bundle formula isomorphism can be explicitly written as the composition

$$\text{PB}(\mathcal{L}) : M(\mathbb{P}(\mathcal{V})) \xrightarrow{M(\Delta)} M(\mathbb{P}(\mathcal{V}))^{\otimes 2} \xrightarrow{M(\pi) \otimes [c_1(\mathcal{L})]} M(X) \otimes M(\mathbb{P}^{n-1}).$$

Remark 3.6. On $\mathcal{FH}^l := \mathcal{FH}_{\mathcal{E}/T}^l$, we can inductively define l line bundles $\mathcal{L}_1^l, \dots, \mathcal{L}_l^l$ by

- (i) $\mathcal{L}_l^l := \mathcal{O}(1) \rightarrow \mathbb{P}(\mathcal{U}_{l-1}^{l-1})$,
- (ii) $\mathcal{L}_j^l := p_l^* \mathcal{L}_j^{l-1}$ for $1 \leq j \leq l-1$, where $p_l : \mathcal{FH}^l \rightarrow \mathcal{FH}^{l-1}$.

These l line bundles on \mathcal{FH}^l induce a morphism

$$\text{PB}(\mathcal{L}_\bullet^l) : M(\mathcal{FH}_{\mathcal{E}/T}^l) \xrightarrow{M(\Delta)} M(\mathcal{FH}_{\mathcal{E}/T}^l)^{\otimes l+1} \xrightarrow{M(P_l) \otimes [c_1(\mathcal{L}_\bullet^l)]} M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l},$$

where $[c_1(\mathcal{L}_\bullet^l)] = \bigoplus_{i=1}^l [c_1(\mathcal{L}_i^l)]$. Furthermore, on \mathcal{FH}^l we have two universal objects:

- (i) a surjection $\pi_l^* \mathcal{U}_{l-1}^{l-1} \twoheadrightarrow \mathcal{L}_l^l$ over \mathcal{FH}^l (as $\mathcal{FH}^l \cong \mathbb{P}(\mathcal{U}_{l-1}^{l-1})$ by Remark 3.6),
- (ii) a short exact sequence $0 \rightarrow \mathcal{U}_l^l \rightarrow \mathcal{U}_{l-1}^l \rightarrow \mathcal{T}_l^l \rightarrow 0$ over $\mathcal{FH}^l \times C$.

Since $\mathcal{U}_{l-1}^l = (p_l \times \text{id}_C)^* \mathcal{U}_{l-1}^{l-1}$, the relationship between the line bundle $\mathcal{L}_l^l \rightarrow \mathcal{FH}^l$ and the family of degree 1 torsion sheaves \mathcal{T}_l^l on C parametrised by \mathcal{FH}^l is

$$\mathcal{L}_l^l \cong ((\text{id}_{\mathcal{FH}^l} \times \pi_l) \circ \Delta_{\mathcal{FH}^l})^* \mathcal{T}_l^l,$$

for $(\text{id}_{\mathcal{FH}^l} \times \pi_l) \circ \Delta_{\mathcal{FH}^l} : \mathcal{FH}^l \xrightarrow{\Delta_{\mathcal{FH}^l}} \mathcal{FH}^l \times_{\mathcal{FH}^{l-1}} \mathcal{FH}^l \xrightarrow{\text{id} \times \pi_l} \mathcal{FH}^l \times_{\mathcal{FH}^{l-1}} (\mathcal{FH}^{l-1} \times C) \simeq \mathcal{FH}^l \times C$. In fact, for $1 \leq j \leq l$, we can define maps

$$r_j^l = (\text{id}_{\mathcal{FH}^l} \times \text{pr}_j^l) \circ \Delta_{\mathcal{FH}^l} : \mathcal{FH}^l \rightarrow \mathcal{FH}^l \times_T \mathcal{FH}^l \rightarrow \mathcal{FH}^l \times_T (T \times C) \cong \mathcal{FH}^l \times C$$

such that $r_l^l = (\text{id}_{\mathcal{F}\mathcal{H}^l} \times \pi_l) \circ \Delta_{\mathcal{F}\mathcal{H}^l}$. For $j < l$, the family of degree 1 torsion sheaves $\mathcal{T}_j^l := \mathcal{U}_{j-1}^l / \mathcal{U}_j^l$ on C parametrised by $\mathcal{F}\mathcal{H}^l$ is obtained as a pullback of \mathcal{T}_j^{l-1} via the map $p_l \times \text{id}_C$. Hence, for $1 \leq j \leq l$, we have isomorphisms relating the line bundles and families of torsion sheaves

$$(2) \quad \mathcal{L}_j^l \cong (r_j^l)^* \mathcal{T}_j^l.$$

We can now give a precise description of the isomorphism in Corollary 3.5.

Lemma 3.7. *The tuple $\mathcal{L}_\bullet^l = (\mathcal{L}_1^l, \dots, \mathcal{L}_l^l)$ of line bundles on $\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l$ induces a morphism*

$$\text{PB}(\mathcal{L}_\bullet^l) : M(\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l) \xrightarrow{M(\Delta)} M(\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l)^{\otimes l+1} \xrightarrow{M(P_l) \otimes [c_1(\mathcal{L}_\bullet^l)]} M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l},$$

which coincides with the composition

$$M(\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l) \xrightarrow{\text{PB}(\mathcal{L}_1^l)} M(\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^{l-1}) \otimes M(C \times \mathbb{P}^{n-1}) \xrightarrow{\text{PB}(\mathcal{L}_{l-1}^{l-1}) \otimes M(\text{id})} \dots \longrightarrow M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l}$$

and thus is an isomorphism.

Proof. For this one uses that Chern classes are compatible with pullbacks, so that $c_1(\mathcal{L}_j^{l-1}) \circ M(p_l) = c_1(\mathcal{L}_j^l)$ for $1 \leq j \leq l-1$, as $p_l^*(\mathcal{L}_j^{l-1}) = \mathcal{L}_j^l$. Then one uses that P_l is defined as the composition of the maps π_i for $i \leq l$ together with the fact that for any morphism $f : X \rightarrow Y_1 \times Y_2$, we have the following commutative diagram

$$\begin{array}{ccc} M(X) & \xrightarrow{M(\Delta)} & M(X) \otimes M(X) \\ M(f) \downarrow & & \downarrow M(f_1) \otimes M(f_2) \\ M(Y_1 \times Y_2) & \xrightarrow{\simeq} & M(Y_1) \otimes M(Y_2), \end{array}$$

where $f_i := \text{pr}_i \circ f : X \rightarrow Y_i$ and the lower map in this square is the Künneth isomorphism. \square

3.2. The motive of the scheme of Hecke correspondences. There is a forgetful map

$$f : \mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l \rightarrow \mathcal{H}_{\mathcal{E}/T}^l$$

that we will use to relate the motive of $\mathcal{H}_{\mathcal{E}/T}^l$ to that of $\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l$, which we computed above. In fact, we plan to use the above section to compare these motives, as the map f is small. To prove that f is a small map, we will describe it as the pullback of a small map along a flat morphism by generalising an argument of Heinloth [13, Proposition 11].

Let $\text{Coh}_{0,l}$ denote the stack of rank 0 degree l coherent sheaves on C and let $\widetilde{\text{Coh}}_{0,l}$ denote the stack which associates to a scheme S the groupoid

$$\widetilde{\text{Coh}}_{0,l}(S) = \langle \mathcal{T}_1 \hookrightarrow \mathcal{T}_2 \hookrightarrow \dots \hookrightarrow \mathcal{T}_l : \mathcal{T}_i \in \text{Coh}_{0,i}(S) \rangle.$$

The forgetful map $f' : \widetilde{\text{Coh}}_{0,l} \rightarrow \text{Coh}_{0,l}$ fits into the following commutative diagram

$$(3) \quad \begin{array}{ccccc} \mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l & \xrightarrow{\text{gr}} & T \times \widetilde{\text{Coh}}_{0,l} & \longrightarrow & T \times C^l \\ \downarrow f & & \downarrow \text{id}_T \times f' & & \downarrow \\ \mathcal{H}_{\mathcal{E}/T}^l & \xrightarrow{\text{gr}} & T \times \text{Coh}_{0,l} & \longrightarrow & T \times C^{(l)} \end{array}$$

such that the left square in this diagram is Cartesian. Furthermore, by [16, Theorem 3.3.1], the map f' is small and generically a S_l -covering. By Lemma 2.3, $\text{id}_T \times f'$ is small and generically a S_l covering. Since the morphism gr is smooth and thus flat (see the proof of [13, Proposition 11]), we deduce by Lemma 2.3 that f is small and generically a S_l -covering. By Theorem 2.11, there is an induced S_l -action on $M(\mathcal{F}\mathcal{H}_{\mathcal{E}/T}^l)$ and we can now prove the following result.

Theorem 3.8. *Let \mathcal{E} be family of rank n degree d vector bundles over C parametrised by a smooth k -scheme T . Then via the isomorphism $M(\mathcal{FH}_{\mathcal{E}/T}^l) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}$ of Corollary 3.5, the S_l -action permutes the l -copies of $M(C \times \mathbb{P}^{n-1})$. Moreover, we have*

$$M(\mathcal{H}_{\mathcal{E}/T}^l) \cong M(T) \otimes M(\mathrm{Sym}^l(C \times \mathbb{P}^{n-1})).$$

Proof. We note that as T is smooth, both $\mathcal{H}_{\mathcal{E}/T}^l$ and $\mathcal{FH}_{\mathcal{E}/T}^l$ are smooth over k . By Lemma 3.7, there is an isomorphism

$$M(\mathcal{FH}_{\mathcal{E}/T}^l) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}$$

induced by l line bundles $\mathcal{L}_1^l, \dots, \mathcal{L}_l^l$ on $\mathcal{FH}_{\mathcal{E}/T}^l$ (which are the pullbacks of the ample bundles on each projective bundle) and the projection $P_l : \mathcal{FH}_{\mathcal{E}/T}^l \rightarrow T \times C^l$. The S_l -action on $M(\mathcal{FH}_{\mathcal{E}/T}^l)$ from Theorem 2.11 is induced by the S_l -action on the open subset $\mathcal{FH}_{\mathcal{E}/T}^{l,\circ} = p^{-1}(\mathcal{H}_{\mathcal{E}/T}^{l,\circ})$, where $\mathcal{H}_{\mathcal{E}/T}^{l,\circ}$ parametrises length l Hecke correspondences whose degree l torsion quotient has support consisting of l distinct points. The S_l -action on $\mathcal{FH}_{\mathcal{E}/T}^{l,\circ}$ corresponds to permuting the l universal degree 1 torsion quotients $\mathcal{T}_1^l, \dots, \mathcal{T}_l^l$. By Remark 3.6, this corresponds to permuting the l line bundles \mathcal{L}_i^l on $\mathcal{FH}_{\mathcal{E}/T}^l$ (see equation (2)). Therefore, the induced S_l -action on $M(\mathcal{FH}_{\mathcal{E}/T}^l)$ permutes the l -copies of $M(C \times \mathbb{P}^{n-1})$. As f is a small proper surjective map of smooth varieties, Theorem 2.11 yields an isomorphism

$$M(\mathcal{FH}_{\mathcal{E}/T}^l)^{S_l} \cong M(\mathcal{H}_{\mathcal{E}/T}^l).$$

Finally, by Example 2.10 we have $\mathrm{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1}) \simeq M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1}))$. \square

In particular, if we apply this to $T = \mathrm{Spec} k$ and $\mathcal{E} = \mathcal{O}_C(D)^{\oplus n}$ for a divisor D on C , we obtain Theorem 1.3 as a special case of this result. Furthermore, the motive of the Quot scheme of length l torsion quotients of a locally free sheaf \mathcal{E} over $T \times C/T$ only depends on the rank of \mathcal{E} ; we explicitly state this as a corollary for $T = \mathrm{Spec} k$, as there are similar recent results concerning the class of such Quot schemes in the Grothendieck ring of varieties [6, 17].

Corollary 3.9. *Let \mathcal{E} be rank n locally free sheaf on C and $l \in \mathbb{N}$. Then the motive of the Quot scheme $\mathrm{Quot}_{C/k}^{(0,l)}(\mathcal{E})$ parametrising length l torsion quotients of \mathcal{E} is*

$$M(\mathrm{Quot}_{C/k}^{(0,l)}(\mathcal{E})) \cong M(\mathrm{Sym}^l(C \times \mathbb{P}^{n-1})).$$

In particular, this motive only depends on the rank n of \mathcal{E} .

4. THE FORMULA FOR THE MOTIVE OF THE STACK OF VECTOR BUNDLES

4.1. The transition maps in the inductive system. Throughout this section we fix $x \in C(k)$ and let $s_x : \mathrm{Spec} k \rightarrow C$ be the inclusion of x . The inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(x)$ defines an inductive sequence of morphisms $i_l : \mathrm{Div}_{n,d}(l) \rightarrow \mathrm{Div}_{n,d}(l+1)$ indexed by $l \in \mathbb{N}$. In this section, we will lift the maps $i_l : \mathrm{Div}_{n,d}(l) \rightarrow \mathrm{Div}_{n,d}(l+1)$ to the schemes of iterated Hecke correspondences and compute the induced maps of motives. We recall that

$$\mathrm{Div}_{n,d}(l) = \mathcal{H}_{\mathcal{O}_C(lx)^{\oplus n}/\mathrm{Spec} k}^{nl-d} \quad \text{and} \quad \mathrm{FDiv}_{n,d}(l) = \mathcal{FH}_{\mathcal{O}_C(lx)^{\oplus n}/\mathrm{Spec} k}^{nl-d}$$

and we will drop the subscripts for Hecke schemes throughout the rest of this section.

The inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(x)$ induces an inclusion $\mathcal{O}_C^{\oplus n} \hookrightarrow \mathcal{O}_C(x)^{\oplus n}$. Any full flag

$$\mathcal{F}_\bullet = (\mathcal{O}_C^{\oplus n} = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = \mathcal{O}_C(x)^{\oplus n})$$

determines, for $l \in \mathbb{N}$, a morphism $A_l(\mathcal{F}_\bullet) : \mathrm{FDiv}_{n,d}(l) \rightarrow \mathrm{FDiv}_{n,d}(l+1)$ lifting the morphism $\mathrm{Div}_{n,d}(l) \rightarrow \mathrm{Div}_{n,d}(l+1)$. Recall that we have maps $P_{nl-d} : \mathrm{FDiv}_{n,d}(l) = \mathcal{FH}^{nl-d} \rightarrow C^{nl-d}$

defined in Remark 3.3. The morphism $A_l(\mathcal{F}_\bullet)$ sits in a commutative diagram

$$(4) \quad \begin{array}{ccc} \mathrm{FDiv}_{n,d}(l) & \xrightarrow{A_l(\mathcal{F}_\bullet)} & \mathrm{FDiv}_{n,d}(l+1) \\ P_{nl-d} \downarrow & & \downarrow P_{n(l+1)-d} \\ C^{nl-d} & \xrightarrow{c_l} & C^{n(l+1)-d}. \end{array}$$

where $c_l := s_x^n \times \mathrm{id}_{C^{nl-d}}$. Recall that $\mathrm{pr}_j^{nl-d} : \mathrm{FDiv}_{n,d}(l) \rightarrow C^{nl-d} \rightarrow C$ denotes the composition of P_{nl-d} with the projection onto the j th factor. We have

$$(5) \quad \mathrm{pr}_j^{n(l+1)-d} \circ A_l(\mathcal{F}_\bullet) = \begin{cases} t_x & \text{if } 1 \leq j \leq n \\ \mathrm{pr}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d, \end{cases}$$

where $t_x : \mathrm{FDiv}_{n,d}(l) \rightarrow \mathrm{Spec} k \rightarrow C$ is the composition of the structure map with s_x .

Similarly, a tuple $p := (p_1, \dots, p_n) \in (\mathbb{P}^{n-1})^n$ induces $b_l(p) : (\mathbb{P}^{n-1})^{nl-d} \rightarrow (\mathbb{P}^{n-1})^{n(l+1)-d}$ which is the identity on the last $nl-d$ factors. We define

$$a_l(p) := c_l \times b_l(p) : (C \times \mathbb{P}^{n-1})^{nl-d} \rightarrow (C \times \mathbb{P}^{n-1})^{n(l+1)-d}.$$

Lemma 4.1. *Every choice of flag \mathcal{F}_\bullet induces the same map of motives*

$$M(A_l) := M(A_l(\mathcal{F}_\bullet)) : M(\mathrm{FDiv}_{n,d}(l)) \rightarrow M(\mathrm{FDiv}_{n,d}(l+1))$$

and every choice of tuple $p \in (\mathbb{P}^{n-1})^n$ induces the same map of motives

$$M(b_l) = M(b_l(p)) : M(\mathbb{P}^{n-1})^{\otimes nl-d} \rightarrow M(\mathbb{P}^{n-1})^{\otimes n(l+1)-d}.$$

Proof. A flag \mathcal{F}_\bullet as above is specified by a full flag in k^n , which is parametrised by the flag variety GL_n/B , which is \mathbb{A}^1 -chain connected and so all flags induce the same map of motives. The second statement follows similarly as projective spaces are also \mathbb{A}^1 -chain connected. \square

As we are only interested in studying these maps motivically, we will drop the choice of flag \mathcal{F}_\bullet and tuple p from the notation and simply write A_l , b_l and a_l for these morphisms.

By Lemma 3.7, there is an S_{nl-d} -equivariant isomorphism

$$\mathrm{PB}(\mathcal{L}_\bullet^{nl-d}) : M(\mathrm{FDiv}_{n,d}(l)) = M(\mathcal{FH}^{nl-d}) \rightarrow M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$$

determined by line bundles \mathcal{L}_j^{nl-d} on \mathcal{FH}^{nl-d} for $1 \leq j \leq nl-d$. Moreover, we have homomorphisms $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$ such that the maps $c_l : C^{nl-d} \rightarrow C^{n(l+1)-d}$ are equivariant.

Proposition 4.2. *For each l , we have a commutative diagram*

$$(6) \quad \begin{array}{ccc} M(\mathrm{FDiv}_{n,d}(l)) & \xrightarrow{M(A_l)} & M(\mathrm{FDiv}_{n,d}(l+1)) \\ \mathrm{PB}(\mathcal{L}_\bullet^{nl-d}) \downarrow \wr & & \wr \downarrow \mathrm{PB}(\mathcal{L}_\bullet^{n(l+1)-d}) \\ M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} & \xrightarrow{M(a_l)} & M(C \times \mathbb{P}^{n-1})^{\otimes n(l+1)-d} \end{array}$$

such that the horizontal maps are equivariant with respect to $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$.

Proof. We claim that the pullbacks via A_l (for any flag \mathcal{F}_\bullet) of the line bundles $\mathcal{L}_j^{n(l+1)-d}$ satisfy

$$(7) \quad A_l^* \mathcal{L}_j^{n(l+1)-d} = \begin{cases} \mathcal{O}_{\mathcal{FH}^{nl-d}} & \text{if } 1 \leq j \leq n \\ \mathcal{L}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d. \end{cases}$$

We recall that we have $n(l+1)-d$ families of degree 1 torsion sheaves on C parametrised by $\mathrm{FDiv}_{n,d}(l+1) = \mathcal{FH}^{n(l+1)-d}$ given by the successive quotients of the universal flag of vector bundles on $\mathcal{FH}^{n(l+1)-d} \times C$; these families of torsion sheaves are denoted by

$$\mathcal{T}_j^{n(l+1)-d} := \mathcal{U}_{j-1}^{n(l+1)-d} / \mathcal{U}_j^{n(l+1)-d} \quad \text{for } 1 \leq j \leq n(l+1)-d.$$

The pullbacks of these families of torsion sheaves along A_l (for any flag \mathcal{F}_\bullet) are as follows:

$$(8) \quad (A_l \times \mathrm{id}_C)^* \mathcal{T}_j^{n(l+1)-d} = \begin{cases} p_C^* k_x & \text{if } 1 \leq j \leq n \\ \mathcal{T}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d, \end{cases}$$

where $p_C : \mathcal{FH}^{n(l+1)-d} \times C \rightarrow C$ denote the projection and k_x is the skyscraper sheaf at x . Consequently, Claim (7) follows from equations (2), (5) and (8).

Similarly, if we let \mathcal{M}_j^{nl-d} denote the line bundle on $(C \times \mathbb{P}^{n-1})^{nl-d}$ obtained by pulling back $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ via the j th projection, we have

$$a_l^* \mathcal{M}_j^{n(l+1)-d} = \begin{cases} \mathcal{O}_{(C \times \mathbb{P}^{n-1})^{nl-d}} & \text{if } 1 \leq j \leq n \\ \mathcal{M}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d. \end{cases}$$

Since the action of the symmetric groups on these motives corresponds to permuting the order of these line bundles, we see that $M(A_l)$ and $M(a_l)$ are both equivariant with respect to φ_l .

Finally let us prove the commutativity of the square (6). For this we require the explicit formula for the iterated projective bundle isomorphisms given in Lemma 3.7:

$$\text{PB}(\mathcal{L}_{\bullet}^{nl-d}) = (M(P_{nl-d}) \otimes [c_1(\mathcal{L}_{\bullet}^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}),$$

where $[c_1(\mathcal{L}_{\bullet}^{nl-d})] : M(\mathcal{FH}^{nl-d}) \rightarrow M(\mathbb{P}^{n-1})^{nl-d}$ is the map induced by powers of the first chern classes of the line bundles \mathcal{L}_j^{nl-d} for $1 \leq j \leq nl-d$. If we insert n copies of the structure sheaf on \mathcal{FH}^{nl-d} into this family, we obtain a map

$$[c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_{\bullet}^{nl-d})] : M(\mathcal{FH}^{nl-d}) \rightarrow M(\mathbb{P}^{n-1})^{n(l+1)-d}.$$

In fact, since $c_1(\mathcal{O})$ is the zero map, we see that $[c_1(\mathcal{O})] : M(\mathcal{FH}^{nl-d}) \rightarrow M(\mathbb{P}^{n-1})$ is the composition of the structure map $M(\mathcal{FH}^{nl-d}) \rightarrow \mathbb{Q}\{0\}$ with the inclusion $\mathbb{Q}\{0\} \hookrightarrow M(\mathbb{P}^{n-1})$ of any point in \mathbb{P}^{n-1} . Therefore, we can write the lower diagonal composition in (6) as

$$M(a_l) \circ \text{PB}(\mathcal{L}_{\bullet}^{nl-d}) = (M(c_l \circ P_{nl-d}) \otimes [c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_{\bullet}^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}).$$

Then by (7), we have

$$[c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_{\bullet}^{nl-d})] = [c_1(\mathcal{L}_{\bullet}^{n(l+1)-d})] \circ M(A_l)$$

and as diagram (4) commutes, we deduce that

$$\text{PB}(\mathcal{L}_{\bullet}^{n(l+1)-d}) \circ M(A_l) = (M(c_l \circ P_{nl-d}) \otimes [c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_{\bullet}^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}),$$

which completes the proof that the square (6) commutes. \square

Since $a_l : (C \times \mathbb{P}^{n-1})^{nl-d} \rightarrow (C \times \mathbb{P}^{n-1})^{n(l+1)-d}$ is equivariant with respect to $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$, we obtain an induced map between the associated symmetric products

$$\begin{array}{ccc} (C \times \mathbb{P}^{n-1})^{nl-d} & \xrightarrow{a_l} & (C \times \mathbb{P}^{n-1})^{n(l+1)-d} \\ \downarrow & & \downarrow \\ \text{Sym}^{nl-d}(C \times \mathbb{P}^{n-1}) & \xrightarrow{\text{Sym}(a_l)} & \text{Sym}^{n(l+1)-d}(C \times \mathbb{P}^{n-1}). \end{array}$$

By Theorem 1.3, there is an isomorphism

$$e_l : M(\text{Div}_{n,d}(l)) \cong M(\text{FDiv}_{n,d}(l))^{S_{nl-d}} \cong \text{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1})$$

where the second isomorphism is induced by the S_{nl-d} -equivariant isomorphism $\text{PB}(\mathcal{L}_{\bullet}^{nl-d})$.

Corollary 4.3. *The following diagram commutes*

$$\begin{array}{ccc} M(\text{Div}_{n,d}(l)) & \xrightarrow{M(i_l)} & M(\text{Div}_{n,d}(l+1)) \\ e_l \downarrow \wr & & \downarrow \wr e_{l+1} \\ \text{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1}) & \xrightarrow{M(\text{Sym}(a_l))} & \text{Sym}^{n(l+1)-d} M(C \times \mathbb{P}^{n-1}). \end{array}$$

Proof. By the equivariance property of $M(A_l)$ observed in Proposition 4.2 and the fact that A_l lifts i_l , the isomorphisms of Theorem 1.3 fit in a commutative diagram

$$\begin{array}{ccc} M(\mathrm{FDiv}_{n,d}(l))^{S_{nl-d}} & \xrightarrow{M(A_l)} & M(\mathrm{FDiv}_{n,d}(l+1))^{S_{n(l+1)-d}} \\ \downarrow \wr & & \downarrow \wr \\ M(\mathrm{Div}_{n,d}(l)) & \xrightarrow{M(i_l)} & M(\mathrm{Div}_{n,d}(l+1)) \end{array}$$

The corollary then follows from combining this diagram with the diagram of Proposition 4.2. \square

4.2. A proof of the formula. The rational point $x \in C(k)$ gives rise to a decomposition $M(C) = \mathbb{Q}\{0\} \oplus \overline{M}(C)$, where $\overline{M}(C) = M_1(\mathrm{Jac}(C)) \oplus \mathbb{Q}\{1\}$, see [1, Proposition 4.2.5]. The motive of $\mathrm{Jac}(C)$ can be recovered from the motive $M_1(\mathrm{Jac}(C))$ using [1, Proposition 4.3.5]:

$$M(\mathrm{Jac}(C)) = \bigoplus_{i=0}^{2g} \mathrm{Sym}^i(M_1(\mathrm{Jac}(C))) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M_1(\mathrm{Jac}(C))).$$

We can then write

$$M(C \times \mathbb{P}^{n-1}) = M(C) \otimes \left(\bigoplus_{i=0}^{n-1} \mathbb{Q}\{i\} \right) = \mathbb{Q}\{0\} \oplus \overline{M}(C) \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\}.$$

Let $M_{C,n} := \overline{M}(C) \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\}$; then (for example, by [1, Lemma B.3.1])

$$\mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1})) = \mathrm{Sym}^{nl-d}(\mathbb{Q}\{0\} \oplus M_{C,n}) = \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}).$$

Lemma 4.4. *There is a commutative diagram*

$$\begin{array}{ccc} M(\mathrm{Div}_{n,d}(l)) & \xrightarrow{M(i_l)} & M(\mathrm{Div}_{n,d}(l+1)) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}) & \longrightarrow & \bigoplus_{i=0}^{n(l+1)-d} \mathrm{Sym}^i(M_{C,n}) \end{array}$$

where the lower map is the obvious inclusion.

Proof. Let us start with the description of the transition map given in Corollary 4.3. We see that the map $a_l : (C \times \mathbb{P}^{n-1})^{nl-d} \rightarrow (C \times \mathbb{P}^{n-1})^{n(l+1)-d}$ can be described motivically as

$$M(a_l) : M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \cong \mathbb{Q}\{0\}^{\otimes n} \otimes M(C \times \mathbb{P}^{n-1})^{\otimes (nl-d)} \xrightarrow{\iota^{\otimes n} \otimes M(\mathrm{id})} M(C \times \mathbb{P}^{n-1})^{\otimes (n(l+1)-d)}$$

where $\iota : \mathbb{Q}\{0\} \rightarrow M(C \times \mathbb{P}^{n-1}) = \mathbb{Q}\{0\} \oplus M_{C,n}$ is the natural inclusion of this direct factor. It thus follows that the symmetrised map $M(\mathrm{Sym}(a_l))$ is the claimed inclusion. \square

Theorem 4.5. *If $C(k) \neq \emptyset$, then the motive of $\mathrm{Bun}_{n,d}$ satisfies*

$$M(\mathrm{Bun}_{n,d}) \simeq \mathrm{hocolim}_l \left(\bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}) \right) \simeq \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M_{C,n}).$$

More precisely, we have

$$M(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}).$$

Proof. The first claim follows from Lemma 4.4 and Theorem 1.2. For the second claim, we introduce the notation $\mathrm{Sym}^*(M) := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M)$ for any motive M ; then

- (i) $\mathrm{Sym}^*(M_1 \oplus M_2) = \mathrm{Sym}^*(M_1) \otimes \mathrm{Sym}^*(M_2)$ (by [1, Lemma B.3.1]),
- (ii) $Z(C, \mathbb{Q}\{i\}) = \mathrm{Sym}^*(M(C)\{i\})$ (by definition of the motivic Zeta function),

- (iii) $\mathrm{Sym}^*(\mathbb{Q}\{1\}) = M(B\mathbb{G}_m)$ (see [14, Example 2.21] based on [18, Lemma 8.7]),
- (iv) $\mathrm{Sym}^*(M_1(\mathrm{Jac}(C))) = M(\mathrm{Jac}(C))$ (by [1, Proposition 4.3.5]),

and the formula follows from these observations. \square

4.3. An alternative proof using previous results. We will give a second proof of this formula for $M(\mathrm{Bun}_{n,d})$, also based on Corollary 4.3 but which follows more closely our previous work [14]. The idea is to describe the unsymmetrised transition maps $M(a_l)$ by decomposing the motives $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ using $M(\mathbb{P}^{n-1}) = \bigoplus_{i=0}^{n-1} \mathbb{Q}\{i\}$.

Remark 4.6. By returning to the decomposition $M(\mathbb{P}^{n-1}) = \bigoplus_{i=0}^{n-1} \mathbb{Q}\{i\}$, we can describe the maps $M(a_l)$ explicitly. Indeed we have a decomposition $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ indexed by ordered tuples $I = (i_1, \dots, i_{nl-d}) \in \mathcal{I}_l := \{0, \dots, n-1\}^{\times nl-d}$ of the form

$$M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} = \bigoplus_{I \in \mathcal{I}_l} \bigotimes_{j=1}^{nl-d} M(C)\{i_j\} = \bigoplus_{I \in \mathcal{I}_l} M(C^{nl+d})\{|I|\},$$

where $|I| = \sum_{j=1}^{nl-d} i_j$.

There is a map $h_l : \mathcal{I}_l \rightarrow \mathcal{I}_{l+1}$ given by $I \mapsto (0, \dots, 0, I)$ (inserting n zeros) such that the map $M(a_l) : M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \rightarrow M(C \times \mathbb{P}^{n-1})^{\otimes n(l+1)-d}$ sends the direct summand indexed by $I \in \mathcal{I}_l$ to the direct summand indexed by the tuple $h_l(I) \in \mathcal{I}_{l+1}$ via the map

$$(9) \quad M(c_l)\{|I|\} : M(C^{nl+d})\{|I|\} \rightarrow M(C^{n(l+1)+d})\{|I|\} = M(C^{n(l+1)+d})\{|(0, \dots, 0, I)|\}.$$

The S_{nl-d} -action on $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ permutes these direct summands via the obvious action of S_{nl-d} on \mathcal{I}_l . The invariant part is the motive of $\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})$ which has an associated decomposition. The index set for this decomposition is

$$\mathcal{B}_l := \left\{ m = (m_0, \dots, m_{n-1}) \in \mathbb{N}^n : \sum_{i=0}^{n-1} m_i = nl-d \right\}.$$

Moreover, for $I \in \mathcal{I}_l$, we let $\tau_l(I)_r = \#\{i_j : i_j = r\}$, then $\tau_l(I) = (\tau_l(I)_0, \dots, \tau_l(I)_{n-1}) \in \mathcal{B}_l$ and the map $\tau_l : \mathcal{I}_l \rightarrow \mathcal{B}_l$ is S_{nl-d} -invariant with $|I| = \sum_{i=0}^{n-1} i\tau_l(I)_i$. By grouping together the factors with the same values of i_j , there is a map

$$(10) \quad C^{nl+d} \rightarrow \prod_{i=0}^{n-1} \mathrm{Sym}^{\tau_l(I)_i}(C)$$

which is the quotient of the natural action of $\mathrm{Stab}(I) \cong \prod_{i=0}^{n-1} S_{\tau_l(I)_i}$.

Lemma 4.7. *For each l , we have a decomposition*

$$M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})) = \bigoplus_{m \in \mathcal{B}_l} \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\}$$

such that the following statements hold.

- (i) For each $m \in \mathcal{B}_l$, we have a commutative diagram

$$\begin{array}{ccc} M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} & \longrightarrow & M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})) \\ \downarrow & & \downarrow \\ \bigoplus_{I \in \tau_l^{-1}(m)} M(C^{nl-d})\{|I|\} & \longrightarrow & \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\} \end{array}$$

where the lower maps are induced by the maps (10).

(ii) *The transition maps $M(\text{Sym}(a_l))$ decompose as maps*

$$\kappa_{m,m'} : \bigotimes_{i=0}^{n-1} \text{Sym}^{m_i}(M(C))\{im_i\} \rightarrow \bigotimes_{i=0}^{n-1} \text{Sym}^{m'_i}(M(C))\{im'_i\}$$

for $m \in \mathcal{B}_l$ and $m' \in \mathcal{B}_{l+1}$ with $\kappa_{m,m'} = 0$ unless $m' = m + (n, 0, \dots, 0)$, in which case this map is induced by the morphism of varieties

$$\prod_{i=0}^{n-1} \text{Sym}^{m_i}(C) \rightarrow \prod_{i=0}^{n-1} \text{Sym}^{m'_i}(C)$$

which is the map $\text{Sym}(s_x^n \times \text{id}_{C^{m_i}})$ on the 0th factor and the identity on all other factors.

Proof. We will give the decomposition and the proof of (i) simultaneously, by collecting the direct summands in the decomposition of $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ which are preserved by the S_{nl-d} -action and taking their invariant parts. For this, we recall that there is a S_{nl-d} -action on \mathcal{I}_l and the map $\tau_l : \mathcal{I}_l \rightarrow \mathcal{B}_l$ is S_{nl-d} -invariant and the fibres consist of single orbits. For $I \in \mathcal{I}_l$ with $m = \tau_l(I)$, we note that the quotient of the associated action of $\text{Stab}(I) = \prod_{i=0}^{n-1} S_{m_i}$ on C^{nl-d} is isomorphic to $\prod_{i=0}^{n-1} \text{Sym}^{m_i}(C)$. Therefore, the motive appearing in the left lower corner of the diagram in statement (i) is a direct summand of $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ that is preserved by the S_{nl-d} -action and its S_{nl-d} -invariant piece is precisely the motive appearing in the lower right corner. This proves the first statement and the decomposition.

To describe the behaviour of the symmetrised transition maps with respect to this decomposition, we recall that the unsymmetrised transition maps send the direct summand indexed by $I \in \mathcal{I}_l$ to $h_l(I) = (0, \dots, 0, I) \in \mathcal{I}_{l+1}$. The unsymmetrised transition maps on these direct summands are described by (9) and so it remains to describe the induced map on the invariant parts for the actions of the symmetric groups. Since $h_l : \mathcal{I}_l \rightarrow \mathcal{I}_{l+1}$ is equivariant for the actions of the symmetric groups via the homomorphism $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$, it descends to map

$$\bar{h} : \mathcal{B}_l \rightarrow \mathcal{B}_{l+1} \quad \text{where} \quad \bar{h}(m) = m + (n, 0, \dots, 0).$$

Thus, $\kappa_{m,m'}$ is zero unless $m' = \bar{h}(m)$. For $m' = \bar{h}(m)$, $I \in \tau_l^{-1}(m)$ and $I' \in \tau_{l+1}^{-1}(m')$ note that

$$|I| = |I'| = \sum_{i=0}^{n-1} im_i = \sum_{i=0}^{n-1} im'_i$$

and

$$\text{Stab}(I) = \prod_{i=0}^{n-1} S_{m_i} \quad \text{and} \quad \text{Stab}(I') = \prod_{i=0}^{n-1} S_{m'_i} = S_{m_0+n} \times \prod_{i=1}^{n-1} S_{m_i}.$$

In particular, the map $c_l = s_x^n \times \text{id} : C^{nl-d} \rightarrow C^{n(l+1)-d}$ is equivariant for the induced actions of $\text{Stab}(I)$ and $\text{Stab}(I')$ and there is a map between the quotients

$$\begin{array}{ccc} C^{nl-d} & \xrightarrow{c_l} & C^{n(l+1)-d} \\ \downarrow & & \downarrow \\ \prod_{i=0}^{n-1} \text{Sym}^{m_i}(C) & \longrightarrow & \prod_{i=0}^{n-1} \text{Sym}^{m'_i}(C) \end{array}$$

which is $\text{Sym}(s_x^n \times \text{id}_{C^{m_i}})$ on the 0th factor and the identity on the other factors. Combined with (i), this concludes the proof of (ii). \square

Corollary 4.8. *The transition maps $M(i_l) : M(\text{Div}_{n,d}(l)) \rightarrow M(\text{Div}_{n,d}(l+1))$ fit in the following commutative diagram*

$$\begin{array}{ccc}
 M(\text{Div}_{n,d}(l)) & \xrightarrow{M(i_l)} & M(\text{Div}_{n,d}(l+1)) \\
 \downarrow \wr & & \downarrow \wr \\
 \bigoplus_{m \in \mathcal{B}_l} \bigotimes_{i=0}^{n-1} \text{Sym}^{m_i}(M(C))\{im_i\} & \xrightarrow{\bigoplus_{m,m'} \kappa_{m,m'}} & \bigoplus_{m' \in \mathcal{B}_{l+1}} \bigotimes_{i=0}^{n-1} \text{Sym}^{m'_i}(M(C))\{im'_i\}.
 \end{array}$$

where the maps $\kappa_{m,m'}$ are as in Lemma 4.7.

Proof. This follows from Lemma 4.7 and Corollary 4.3. □

This looks very similar to [14, Conjecture 3.9], except we do not know whether the vertical maps in this commutative diagram coincide with the maps given by the Białynicki-Birula decompositions used in the formulation of this conjecture. Nevertheless, with the description of the transition maps in Corollary 4.8, one can apply the proof of [14, Theorem 3.18] to obtain an alternative proof of the formula for the motive of $\text{Bun}_{n,d}$ appearing in Theorem 1.1.

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Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

hoskins@math.fu-berlin.de, simon.pepin.lehalleur@gmail.com