# A FORMULA FOR THE VOEVODSKY MOTIVE OF THE MODULI STACK OF VECTOR BUNDLES ON A CURVE

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**Abstract**

We prove a formula for the motive of the stack of vector bundles of fixed rank and degree over a smooth projective curve in Voevodsky's triangulated category of mixed motives with rational coefficients.

Contents



## 1. Introduction

Let $\mathrm{Bun}_{n,d}$ denote the moduli stack of rank $n$, degree $d$ vector bundles on a smooth projective geometrically connected curve $C$ of genus $g$ over a field $k$. In this paper, we prove the following formula for the motive $M(\mathrm{Bun}_{n,d})$ of $\mathrm{Bun}_{n,d}$ in Voevodsky's triangulated category $\mathrm{DM}(k) := \mathrm{DM}(k,\mathbb{Q})$ of mixed motives over $k$ with $\mathbb{Q}$-coefficients (see section 1.4 for the notation used in the statement).

**Theorem 1.1.** *Suppose that $C(k) \neq \emptyset$; then in $\mathrm{DM}(k,\mathbb{Q})$, we have*

$$M(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C,\mathbb{Q}\{i\}),$$

*where $Z(C,\mathbb{Q}\{i\}) := \bigoplus_{j=0}^{\infty} M(\mathrm{Sym}^j(C)) \otimes \mathbb{Q}\{ij\}$ is a motivic zeta value*[1] *and $\mathbb{Q}\{i\} := \mathbb{Q}(i)[2i]$.*

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[1] The map $M \mapsto \bigoplus_{j=0}^{\infty} M(\mathrm{Sym}^j(C)) \otimes M^{\otimes j}$ from $\mathrm{DM}(k,\mathbb{Q})$ to itself is the motivic zeta function $Z(C,-)$ of the curve $C$; the motive $Z(C,\mathbb{Q}\{i\})$ is its value at the given Tate twist.

In particular, this implies a decomposition on Chow groups and $\ell$-adic cohomology and, as explained below, this formula is compatible with previous cohomological descriptions of $\mathrm{Bun}_{n,d}$.

This paper is a continuation of our previous work [26] in which we defined and studied the motive $M(\mathrm{Bun}_{n,d}) \in \mathrm{DM}(k,R)$ for any coefficient ring $R$ (provided the characteristic of $k$ is invertible in $R$ in positive characteristic); more generally, we introduced there the notion of an exhaustive stack and defined motives of smooth exhaustive stacks by generalising a construction of Totaro for quotient stacks [37] (see [26, Definitions 2.15 and 2.17] for details).

1.1. **Overview of our previous results.** In [26, Theorem 3.5], we work with a general coefficient ring $R$ (for which the exponential characteristic is invertible) and give the following description of the motive of the stack $\mathrm{Bun}_{n,d}$ in terms of smooth projective Quot schemes by following a geometric argument for computing the $\ell$-adic cohomology of this stack in [11].

**Theorem 1.2.** *For any effective divisor $D > 0$ on $C$, we have in* $\mathrm{DM}(k,R)$

$$M(\mathrm{Bun}_{n,d}) \simeq \operatorname*{hocolim}_{l\in\mathbb{N}} M(\mathrm{Div}_{n,d}(lD)),$$

*where $\mathrm{Div}_{n,d}(D) = \{\mathcal{E} \subset \mathcal{O}_C(D)^{\oplus n} : \mathrm{rk}(\mathcal{E}) = n, \deg(\mathcal{E}) = d\}$ is a smooth Quot scheme.*

Our approach in [26] to describing the motives $M(\mathrm{Div}_{n,d}(lD))$ is to use Białynicki-Birula decompositions [10] associated to an action of a generic one-parameter subgroup $\mathbb{G}_m \subset \mathrm{GL}_n$ on these Quot schemes, whose fixed loci are disjoint unions of products of symmetric powers of $C$. To use these decompositions to compute the motive of $\mathrm{Bun}_{n,d}$, one needs to understand the behaviour of the transition maps $i_l : \mathrm{Div}_{n,d}(lD) \hookrightarrow \mathrm{Div}_{n,d}((l+1)D)$ in the inductive system in Theorem 1.2 with respect to the motivic Białynicki-Birula decompositions; this is very complicated, as although the closed immersion $i_l$ is $\mathbb{G}_m$-equivariant, the closed subscheme $\mathrm{Div}_{n,d}(lD) \hookrightarrow \mathrm{Div}_{n,d}((l+1)D)$ does not intersect the Białynicki-Birula strata transversally. We conjecture a precise description of these transition maps [26, Conjecture 3.9] and show that the formula for the motive of $\mathrm{Bun}_{n,d}$ appearing in Theorem 1.1 follows from this conjectural description of the transition maps.

1.2. **Summary of the results and methods in this paper.** In this paper, we prove the conjectural formula in [26] under the assumption that $R = \mathbb{Q}$. The main idea is to replace the Quot schemes with Flag-Quot schemes, which are generalisations of Quot schemes that allow flags of sheaves and then to describe the transition maps using these Flag-Quot schemes without using Białynicki-Birula decompositions. The idea to use Flag-Quot schemes was inspired by a result of Laumon in [30] and its application in a paper of Heinloth to study the cohomology of the moduli space of Higgs bundles using Hecke modification stacks [23].

To prove Theorem 1.1, our starting point is Theorem 1.2, where as we assume that $C$ has a $k$-rational point $x$ we can take the divisor $D := x$ and write $\mathrm{Div}_{n,d}(l) := \mathrm{Div}_{n,d}(lx)$. We replace the Quot schemes $\mathrm{Div}_{n,d}(l)$ with smooth projective Flag-Quot schemes

$$\mathrm{FDiv}_{n,d}(l) = \{\mathcal{E}_{nl-d} \subsetneq \cdots \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_0 = \mathcal{O}_C(lx)^{\oplus n} : \mathrm{rk}(\mathcal{E}_i) = n \text{ and } \deg(\mathcal{E}_i) = nl - i\}.$$

The natural map $\mathrm{FDiv}_{n,d}(l) \to \mathrm{Div}_{n,d}(l)$ sending a chain $\mathcal{E}_\bullet$ to $\mathcal{E}_{nl-d}$ is small (in the sense of Definition 2.4) and is a $S_{nl-d}$-principal bundle over the open subset consisting of subsheaves $\mathcal{E} \subset \mathcal{O}_C(lx)^{\oplus n}$ with torsion quotient that has support consisting of $nl-d$ distinct points. Using these facts, we relate the motives of these two varieties as follows.

**Theorem 1.3.** *There is an induced $S_{nl-d}$-action on $M(\mathrm{FDiv}_{n,d}(l))$ and isomorphisms*

$$M(\mathrm{Div}_{n,d}(l)) \simeq M(\mathrm{FDiv}_{n,d}(l))^{S_{nl-d}} \simeq (M(C \times \mathbb{P}^{n-1})^{\otimes nl-d})^{S_{nl-d}} \simeq \mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1})).$$

An isomorphism $M(\mathrm{Div}_{n,d}(l)) \simeq \mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1}))$ is constructed by del Baño [14, Theorem 4.2] using associated motivic Białynicki-Birula decompositions (see also [26, §3.2]). However, in del Baño's description, we do not understand the transition maps $M(i_l)$.

In fact, we deduce Theorem 1.3 as a special case of a more general result (Theorem 3.8), where we replace $\mathcal{O}_C(lx)^{\oplus n} \to C/k$ with a family of vector bundles $\mathcal{E} \to T \times C/T$ parametrised by a smooth $k$-scheme $T$ and then study the motives of schemes of (iterated) Hecke correspondences as (Flag)-Quot schemes over $T$. This work was inspired by a beautiful description of the cohomology of these schemes due to Heinloth (see the proof of [23, Proposition 11] which uses ideas of Laumon [30, Theorem 3.3.1]). In fact we lift Heinloth's cohomological description of schemes of (iterated) Hecke correspondences to $\mathrm{DM}(k)$. To prove this result, in §2 we study the invariant piece of a motive with a finite group action, which is why we need to work with rational coefficients; the main result is Theorem 2.13, which states that for a small proper map $f : X \twoheadrightarrow Y$ of smooth projective $k$-varieties which is generically a principal $G$-bundle, we have an isomorphism $M(X)^G \cong M(Y)$. In §3, we study the geometry and motives of schemes of (iterated) Hecke correspondences in order to prove Theorem 3.8. Furthermore, we obtain a formula for the motive of the Quot scheme of length $l$ torsion quotients of a rank $n$ locally free sheaf $\mathcal{E}$ on $C$, which is independent of $\mathcal{E}$ (Corollary 3.9); this complements recent analogous results in the Grothendieck ring of varieties [7, 34].

In §4.1, we lift the transition maps $i_l : \mathrm{Div}_{n,d}(l) \to \mathrm{Div}_{n,d}(l+1)$ to the schemes $\mathrm{FDiv}_{n,d}(l)$. It turns out to be much simpler to describe the motivic behaviour of the lifts of the transition maps to Flag-Quot schemes, as those are iterated projective bundles over products of the curve. By symmetrising this description, we deduce the corresponding behaviour for the maps $M(i_l)$ which enables us to prove Theorem 1.1 in §4.2. Finally, in §4.3, we give a second proof of this formula for $M(\mathrm{Bun}_{n,d})$ which follows more closely the ideas in our previous work [26].

1.3. **Applications and future work.** By Poincaré duality, our main theorem implies the following formula for the compactly supported motive $M^c(\mathrm{Bun})$ (see the computation in [26, Theorem 4.1]).

**Corollary 1.4.** *Assume $C(k) \neq \emptyset$; then the compactly supported motive of $\mathrm{Bun}_{n,d}$ is given by*

$$M^c(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}\, C) \otimes M^c(B\mathbb{G}_m)\{(n^2-1)(g-1)\} \otimes \bigotimes_{i=2}^{n} Z(C, \mathbb{Q}\{-i\}).$$

This formula can be viewed as a categorification of the Behrend–Dhillon formula for the virtual class of $\mathrm{Bun}_{n,d}$ in the Grothendieck ring of varieties [9]. More precisely, it allows one to recover many of the corollaries of the formula in [9], such as formulas for Hodge numbers of $\mathrm{Bun}_{n,d}$ and Harder's formula for the stacky point count over a finite field [22] (see the discussion in [26, §4.2]). The formula of [9] has been applied to computations of (virtual) motivic Donaldson–Thomas invariants (for a recent example, see [17]) and our result should lead to categorifications of certain computations in motivic DT-theory.

In the study of cohomological invariants of moduli spaces of bundles, many computations pass via related moduli stacks. A major motivation for this paper was to study the motive of the moduli space of semistable Higgs bundles. In subsequent work [25], we have used Theorem 1.1 to show that the motive of the moduli space of semistable Higgs bundles on $C$ for coprime rank and degree lies in the tensor triangulated subcategory of $\mathrm{DM}(k, \mathbb{Q})$ generated by the motive of $C$.

Another application is a description of the Chow groups of $\mathrm{Bun}_{n,d}$. First, let us clarify what we mean by Chow groups here. One can certainly define a motivic cohomology group by the formula

$$H^{2i}(\mathfrak{X}, \mathbb{Q}\{i\}) := \mathrm{Hom}_{\mathrm{DM}(k)}(M(\mathfrak{X}), \mathbb{Q}\{i\})$$

Vistoli has defined Chow groups with rational coefficients for finite type Artin stacks over $k$ in [38] (extended to integral coefficients in [28, §2.1.11]) and his definition extends directly to the case of Artin stacks locally of finite type by a limit procedure. For a (finite type) quotient stack $\mathfrak{X}$, Vistoli and Kresch's definition coincides with the equivariant Chow groups of Edidin and Graham [16] and Totaro [36] (see [28, Theorem 2.1.12.(iii)]), and for $\mathfrak{X}$ smooth (and with rational coefficients for simplicity) it also coincides with the motivic cohomology group above, where $M(\mathfrak{X})$ is defined as in [37, §8] or equivalently [26, §2] (this is explained in [37, §8]). Since $\mathrm{Bun}_{n,d}$ is a countable filtered union of quotient stacks, one can then show , using ideas from [26, §2], that

$$\mathrm{CH}^i(\mathrm{Bun}_{n,d})_{\mathbb{Q}} = H^{2i}(\mathrm{Bun}_{n,d}, \mathbb{Q}\{i\}) \tag{1}$$

where the left-hand side is Vistoli's Chow group. As a corollary of Theorem 1.1 , we thus obtain decompositions of the rational Chow groups of $\mathrm{Bun}_{n,d}$ (in the sense of Vistoli) in terms of Chow groups of symmetric powers of $C$ and $\mathrm{Jac}(C)$. The Chow groups of symmetric powers of $C$ and $\mathrm{Jac}(C)$ contain natural tautological classes [8] and one can ask under this decomposition how these tautological classes compare with the 'Atiyah–Bott' tautological classes on $\mathrm{Bun}_{n,d}$ given by the Künneth components of the Chern classes $c_i(\mathcal{E})$ of the universal vector bundle $\mathcal{E} \to \mathrm{Bun}_{n,d} \times C$ (which generate the rational cohomology of $\mathrm{Bun}_{n,d}$ [3, 24]).

In fact, by viewing the Chern classes $c_i(\mathcal{E})$ of the universal bundle as morphisms in $\mathrm{DM}(k)$ via (1), one can construct a motivic Atiyah–Bott map

$$M(\mathrm{Bun}_{n,d}) \longrightarrow M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\})$$

and in future work we plan to show that this map is the isomorphism of Theorem 1.1. This would in particular imply that the assumption $C(k) \neq \emptyset$ is unnecessary, since this motivic Atiyah–Bott map is defined without that assumption and the base change functor $\mathrm{DM}(k) \to \mathrm{DM}(k')$ for $k'/k$ a finite field extension is conservative.

We do not know if Theorem 1.1 holds integrally. A guiding principle here should be the description by Atiyah and Bott [3] of the integral cohomology of $\mathrm{Bun}_{n,d}$.

One can also ask about the motive of stack $\mathrm{Bun}_G$ of principal $G$-bundles over $C$ for reductive groups $G$ other than $\mathrm{GL}_n$. For $G = \mathrm{SL}_n$, one can restrict to certain subschemes of the (Flag)-Quot schemes we consider above in order to obtain a formula for $M(\mathrm{Bun}_{\mathrm{SL}_n})$ (see the constructions in [26, §4.3] and also the computation in [9] for $G = \mathrm{SL}_n$). Since our approach involving (Flag)-Quot schemes heavily relies on the close relationship between $\mathrm{GL}_n$-bundles and coherent sheaves, it is not immediately obvious how to extend this to other groups. However, the (Flag)-Quot schemes parametrise vector bundles differing away from a point varying on the curve $C$ and as such are related to the Beilinson-Drinfeld affine Grassmannian for $G = \mathrm{GL}_n$. This could provide a link with the computation by Gaitsgory-Lurie of the $\ell$-adic cohomology of $\mathrm{Bun}_G$ [19].

1.4. **Background on motives.** Voevodsky introduced the categories of mixed motives that now bear his name in [39]. The monograph [32] gives a useful overview of the original development of the theory by Voevodsky (together with Friedlander and Suslin) while the monograph [12] covers more recent developments.

Let us briefly recall some basic properties about $\mathrm{DM}(k) := \mathrm{DM}(k, \mathbb{Q})$. It is a monoidal $\mathbb{Q}$-linear triangulated category. For a separated scheme $X$ of finite type over $k$, we can associate a motive $M(X) \in \mathrm{DM}(k)$, which is covariantly functorial in $X$ and behaves like a homology theory. The motive $M(\operatorname{Spec} k) := \mathbb{Q}\{0\}$ is the unit for the monoidal structure, and there are Tate motives $\mathbb{Q}\{n\} := \mathbb{Q}(n)[2n] \in \mathrm{DM}(k)$ for all $n \in \mathbb{Z}$. For any motive $M$ and $n \in \mathbb{Z}$, we write $M\{n\} := M \otimes \mathbb{Q}\{n\}$.

In $\mathrm{DM}(k)$, there are Künneth isomorphisms, $\mathbb{A}^1$-homotopy invariance, Gysin distinguished triangles, projective bundle formulae and Poincaré duality isomorphisms, as well as realisation functors (to compare with Betti, de Rham and $\ell$-adic cohomology) and descriptions of Chow groups as homomorphism groups in $\mathrm{DM}(k)$. For a precise statement of these results, besides the references above, we point the reader to the summary in [26, §2].

In this paper, unlike in [26], we need to use categories of relative motives over varying base schemes, and the associated "six operations" formalism. We only need a small portion of the machinery, which we summarise here; for more details, see [5, §3] and [12]. Given a base scheme $S$, which in this paper will always be of finite type and separated over the field $k$, there is a monoidal $\mathbb{Q}$-linear triangulated category $\mathrm{DM}(S)$, which we take to be the category $\mathrm{DA}^{\text{ét}}(S, \mathbb{Q})$ of [5] and [6, §3]. The monoidal unit of $\mathrm{DM}(S)$ is denoted by $\mathbb{Q}_S$ (in particular, $\mathbb{Q}_k := \mathbb{Q}\{0\} \in \mathrm{DM}(k)$). Given a morphism $f : S \to T$ between two such base schemes (so that $f$ is automatically separated and of finite type), there are two adjunctions

$$f^* : \mathrm{DM}(T) \leftrightarrows \mathrm{DM}(S) : f_* \qquad \text{and} \qquad f_! : \mathrm{DM}(S) \leftrightarrows \mathrm{DM}(T) : f^!$$

which satisfy the same formal properties as the corresponding adjunctions $(Lf^*, Rf_*)$ and $(Rf_!, f^!)$ in the setting of derived categories of $\ell$-adic sheaves. In particular, we have natural isomorphisms $f_* \simeq f_!$ for $f$ proper, and $f^* \simeq f^!$ for $f$ étale. We also have proper base change (in the general form of [5, Theorem 3.9]) and a purity isomorphism $f^! \simeq f^*(-)\{d\}$ for $f$ smooth of relative dimension $d$.

Many constructions in $\mathrm{DM}(k)$ have an alternative description in terms of the six operations formalism: for a $k$-scheme $X$ with structure map $\pi_X$, we have

$$M(X) \simeq \pi_{X!}\pi_X^!\mathbb{Q}_k \qquad \text{and} \qquad M^c(X) \simeq \pi_{X*}\pi_X^!\mathbb{Q}_k.$$

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**Conventions.** We fix a base field $k$. An algebraic variety over $k$ is a separated finite type $k$-scheme.

## 2. Small maps and induced actions on motives

2.1. **Codimension.** We briefly recall the definition and basic property of codimensions of subsets of schemes. Let $X$ be a scheme. If $Z$ is an irreducible (in particular non-empty) closed subset of $X$, we define $\mathrm{codim}_X(Z) \in \mathbb{N} \cup \{\infty\}$ to be the supremum of the lengths of chains of irreducible closed subsets in $X$ starting at $Z$. If $Z$ is a general closed subset of $X$, we define $\mathrm{codim}_X(Z)$ to be the infimum[2] of $\mathrm{codim}_X(Z')$ for $Z' \subseteq Z$ irreducible closed subscheme of $X$. Finally, if $Z$ is a locally closed subset of $X$, we define $\mathrm{codim}_X(Z)$ to be $\mathrm{codim}_X(\overline{Z})$.

There is a good notion of locally closed substacks of algebraic stacks [35, 04YM]. Let $\mathcal{X}$ be an algebraic stack and $\mathcal{Z}$ be a locally closed substack. By the case of immersions of schemes and descent, one sees that there exists a unique closed substack $\overline{\mathcal{Z}}$ of $\mathcal{X}$ such that, for any smooth presentation $X \to \mathcal{X}$ of $\mathcal{X}$, we have $\overline{\mathcal{Z}} \times_{\mathcal{X}} X = \overline{\mathcal{Z} \times_{\mathcal{X}} X}$. We introduce the notion of codimension for locally closed substacks of (locally noetherian) algebraic stacks (in the sense of [35, 04YM]), following closely [33, Definition 6.1].

**Definition 2.1.** Let $\mathcal{X}$ be a locally noetherian algebraic stack and $\mathcal{Z}$ be a locally closed substack of $\mathcal{X}$. The codimension of $\mathcal{Z}$ in $\mathcal{X}$ is defined as

$$\mathrm{codim}_{\mathcal{X}}(\mathcal{Z}) := \mathrm{codim}_X(\mathcal{Z} \times_{\mathcal{X}} X)$$

where $X \to \mathcal{X}$ is any smooth presentation of $\mathcal{X}$.

**Lemma 2.2.** *This is well-defined (i.e. independent of the choice of the smooth presentation) and finite if $Z$ is non-empty.*

*Proof.* This statement is [33, Proposition 6.2] in the case when $\mathcal{Z}$ is a closed substack of $\mathcal{X}$. Let $\mathcal{Z}$ be a locally closed substack. We then have

$$\mathrm{codim}_{\mathcal{X}}(\mathcal{Z}) := \mathrm{codim}_X(\mathcal{Z} \times_{\mathcal{X}} X) := \mathrm{codim}_X(\overline{\mathcal{Z} \times_{\mathcal{X}} X}) = \mathrm{codim}_X(\overline{\mathcal{Z}} \times_{\mathcal{X}} X)$$

so that the case of locally closed substacks follows from the case for closed substacks. $\square$

We need the following properties of codimension for algebraic stacks.

**Lemma 2.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of locally noetherian algebraic stacks. Let $\mathcal{Z} \subset \mathcal{Y}$ be a locally closed substack. Then*

$$\mathrm{codim}_{\mathcal{X}}(f^{-1}(\mathcal{Z})) \geq \mathrm{codim}_{\mathcal{Y}}(\mathcal{Z}).$$

*If additionally $f$ is surjective, then this is an equality.*

[2] In particular, with this definition, we always have $\mathrm{codim}_X(\emptyset) = \infty$, even for $X = \emptyset$. This will imply that the identity map of the empty scheme is small, and in particular that Lemma 2.6 below is true as stated.

*Proof.* Let us first treat the case when $\mathcal{X} = X$ and $\mathcal{Y} = Y$ are schemes. If moreover $f$ is surjective, the statement of the lemma is precisely [21, Corollaire 6.1.4]. In general, since $f$ is a flat morphism of locally noetherian schemes, $f$ is open. For every locally closed subset $Z$ in $Y$, we have

$$\operatorname{codim}_{f(X)}(Z \cap f(X)) := \operatorname{codim}_{f(X)}(\overline{Z \cap f(X)}) \geq \operatorname{codim}_{f(X)}(\overline{Z} \cap f(X)) \geq \operatorname{codim}_Y(\overline{Z}) =: \operatorname{codim}_Y(Z)$$

where the first inequality comes from $\overline{Z \cap f(X)} \subseteq \overline{Z} \cap f(X)$ and the second is an application of [20, Proposition 14.2.3].

We now do the case of algebraic stacks. Let $X \to \mathcal{X}$ and $Y \to \mathcal{Y}$ be smooth presentations such that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

with $g$ (necessarily) flat. Let $\mathcal{Z}$ be a locally closed substack, and put $Z := \mathcal{Z} \times_{\mathcal{Y}} Y$ and $\overline{Z} = \overline{\mathcal{Z} \times_{\mathcal{Y}} Y} = \overline{\mathcal{Z}} \times_{\mathcal{Y}} Y$. We have

$$\begin{aligned} \operatorname{codim}_{\mathcal{X}}(f^{-1}(\mathcal{Z})) &:= \operatorname{codim}_X(p^{-1}f^{-1}(\mathcal{Z})) \\ &= \operatorname{codim}_X(g^{-1}q^{-1}(\mathcal{Z})) \\ &\geq \operatorname{codim}_Y(q^{-1}(\mathcal{Z})) \\ &=: \operatorname{codim}_{\mathcal{Y}}(\mathcal{Z}) \end{aligned}$$

where we apply the case of flat morphisms of schemes to the flat morphism $g$. Moreover $f$ is surjective if and only if $g$ is surjective, and in this case we have equality. □

2.2. **Properties of small maps.** In this section, algebraic stacks are assumed locally of finite type over $k$. Let us recall the following definition.

**Definition 2.4.** Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks. Let $f$ be a proper representable morphism $f : \mathcal{X} \to \mathcal{Y}$. For $\delta \in \mathbb{N}$, define

$$\mathcal{Y}_{f,\delta} := \{y \in \mathcal{Y} | \dim(f^{-1}(y)) = \delta\}.$$

Since $f$ is proper, by [35, 0D4I] and the definition of locally closed substacks, we see that $\mathcal{Y}_{f,\delta}$ is locally closed in $\mathcal{Y}$. We say that $f$ is

(i) semismall if $\operatorname{codim}_{\mathcal{Y}}(\mathcal{Y}_{f,\delta}) \geq 2\delta$ for all $\delta \geq 0$.
(ii) small if $f$ is semismall and $\operatorname{codim}_{\mathcal{Y}}(\mathcal{Y}_{f,\delta}) > 2\delta$ for all $\delta > 0$.

We are mostly interested in the case of small maps between algebraic varieties over $k$, but to apply a smallness result of [30] in Section 3.2 it will be convenient to have the notion of a small morphism of algebraic stacks at our disposal.

**Lemma 2.5.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper representable morphism of algebraic stacks locally of finite type over $k$.*

*(1) If $f$ is finite, then $f$ is small.*
*(2) If $f$ is proper and semi-small, then $\mathcal{Y}_{f,0}$ is open dense in $\mathcal{Y}$ and $f_{|f^{-1}(\mathcal{Y}_{f,0})}$ is finite.*

*Proof.* The first statement is clear. The second follows from semi-continuity of the fibre dimension for proper morphisms [35, Tag 0D4I] applied after base change to a smooth presentation of $\mathcal{Y}$. □

**Lemma 2.6.** *(Semi)small morphisms of algebraic stacks are stable by flat base change.*

*Proof.* Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper (semi)small morphism and $g : \mathcal{Z} \to \mathcal{Y}$ be a flat morphism. Write $\tilde{f} : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$. With the notation of Definition 2.4, for all $\delta \in \mathbb{N}$ we have $\mathcal{Z}_{\tilde{f},\delta} = g^{-1}(\mathcal{Y}_{f,\delta})$. Since $g$ is flat, we can apply Lemma 2.3

$$\operatorname{codim}_{\mathcal{Z}}(\mathcal{Z}_{\tilde{f},\delta}) = \operatorname{codim}_{\mathcal{Z}}(g^{-1}(\mathcal{Y}_{f,\delta})) \geq \operatorname{codim}_{\mathcal{Y}}(\mathcal{Y}_{f,\delta})$$

which implies the result. □

The key property of (semi)small morphisms of algebraic varieties for this paper is the following. Unlike the previous lemma, we do not need it in the case of algebraic stacks and we stick to the case of varieties.

**Lemma 2.7.** [13, Proposition 2.1.1, Remark 2.1.2] *Let $f : X \to Y$ be a proper morphism of algebraic varieties over $k$. For $\delta \in \mathbb{N}$, let $Y_{f,\delta}$ be as in Definition 2.4 and $X_{f,\delta} := f^{-1}(Y_{f,\delta})$.*

(i) *The morphism $f$ is semismall if and only if*

$$\dim(X \times_Y X) \leq \dim(Y).$$

(ii) *If $f$ is semismall and surjective, then* $\dim(X \times_Y X) = \dim(X)$.
(iii) *If $f$ is small and surjective, then the irreducible components of dimension* $\dim(X)$ *of $X \times_Y X$ are the closures of the irreducible components of $X_{f,0} \times_{Y_{f,0}} X_{f,0}$ and in particular dominate $Y$.*

2.3. **Endomorphisms of motives of small maps.** Given a morphism of schemes $f : X \to Y$, we denote by $\operatorname{Aut}_Y(X)$ the group of automorphisms of $X$ as a $Y$-scheme. For a $k$-scheme $X$ and an integer $i \in \mathbb{N}$, we denote by $Z_i(X)$ the group of $i$-dimensional cycles with rational coefficients on $X$, and $\mathrm{CH}_i(X)$ the $i$-th Chow group, i.e., the quotient of $Z_i(X)$ by rational equivalence.

For the application to the next section and to the main theorem, we only need the results of Proposition 2.8 and Lemma 2.11 in the special case where the source and target are smooth varieties. In this special case, the proofs can be simplified to a large extent, in particular to remove any mention of exceptional operations. However, most interesting small morphisms do not satisfy this assumption, and the finer results that we prove here, allowing the target to be singular, could be useful in other situations (for instance small resolutions of singularities).

**Proposition 2.8.** *Let $f : X \to Y$ be a proper morphism with $X$ smooth equidimensional of dimension $d \in \mathbb{N}$. Then there exists an isomorphism*

$$\phi_f : \mathrm{CH}_d(X \times_Y X) \simeq \operatorname{End}_{\mathrm{DM}(Y)}(f_*\mathbb{Q}_X)$$

*such that, if $e : U \to Y$ is a étale morphism and $\tilde{e} : V \to X$ is its base change along $f$ and $\tilde{f} : V \to U$ the base change of $f$ along $e$, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{CH}_d(X \times_Y X) \arrow[r, "\phi_f"'] \arrow[d, "(\tilde{e}\times\tilde{e})^*"] & \operatorname{End}_{\mathrm{DM}(Y)}(f_*\mathbb{Q}_X) \arrow[d, "e^*"] \\
\mathrm{CH}_d(V \times_U V) \arrow[r, "\phi_{\tilde{f}}"'] & \operatorname{End}_{\mathrm{DM}(U)}(\tilde{f}_*\mathbb{Q}_V).
\end{tikzcd}$$

*Proof.* Write $p_1, p_2 : X \times_Y X \to X$ for the two projections. For a $k$-scheme $Z$, write $\pi_Z : Z \to \operatorname{Spec}(k)$ for its structure map. We start with the isomorphism

$$\mathrm{CH}_d(X \times_Y X) \simeq \operatorname{Hom}_{\mathrm{DM}(k)}(\mathbb{Q}\{d\}, M^c(X \times_Y X)) \simeq \operatorname{Hom}_{\mathrm{DM}(X\times_Y X)}(\mathbb{Q}_{X\times_Y X}, \pi^!_{X\times_Y X}\mathbb{Q}\{-d\})$$

where we have used the description of Chow groups for general varieties in $\mathrm{DM}(k)$, the formula for $M^c$ in terms of the six operations and the adjunction $(\pi^*_{X\times_Y X}, \pi_{X\times_Y X*})$. We then write

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}(X\times_Y X)}(\mathbb{Q}_{X\times_Y X}, \pi^!_{X\times_Y X}\mathbb{Q}_k\{-d\}) &\simeq \mathrm{Hom}_{\mathrm{DM}(X\times_Y X)}(\mathbb{Q}_{X\times_Y X}, p_1^!\pi_X^!\mathbb{Q}_k\{-d\})\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(X\times_Y X)}(\mathbb{Q}_{X\times_Y X}, p_1^!\mathbb{Q}_X)\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Q}_X, p_{2*}p_1^!\mathbb{Q}_X)\\ &\simeq \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Q}_X, f^!f_*\mathbb{Q}_X)\\ &\simeq \mathrm{End}_{\mathrm{DM}(Y)}(f_*\mathbb{Q}_X)\end{aligned}$$

where the first isomorphism follows from $\pi_{X\times_Y X} = \pi_X \circ p_1$, the second follows from relative purity for the smooth morphism $\pi_X$, the third is the adjunction $(p_2^*, p_{2*})$, the fourth is proper base change and the fifth uses the adjunction $(f_!, f^!)$ and the properness of $f$.

The isomorphism $\phi_f$ is defined as the composition of the sequence of isomorphisms above. Its compatibility with pullback by an étale morphism $e$ is a matter of carefully going through the construction and using the natural isomorphism $e^! \simeq e^*$ and proper base change. □

**Remark 2.9.** Since the target of $\phi_f$ is clearly a $\mathbb{Q}$-algebra, the proposition endows $\mathrm{CH}_d(X\times_Y X)$ with a $\mathbb{Q}$-algebra structure. The multiplication can be described using composition of relative correspondences, but we will not need this.

**Proposition 2.10.** *Let $f : X \to Y$ be a surjective proper small morphism with $X$ and $Y$ smooth varieties. Let $f^\circ : X^\circ \to Y^\circ$ be the restriction of $f$ to the locus with finite fibers, and $j : Y^\circ \to Y$ the corresponding open immersion. Then the natural map*

$$j^* : \mathrm{End}_{\mathrm{DM}(Y)}(f_!f^!\mathbb{Q}_Y) \to \mathrm{End}_{\mathrm{DM}(Y^\circ)}(f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ})$$

*is an isomorphism of rings.*

*Proof.* First, let us explain how $j^*$ is defined. Write $\tilde{\jmath} : X^\circ \to X$. Then we have

$$j^*f_!f^!\mathbb{Q}_Y \simeq f^\circ_!\tilde{\jmath}^*f^!\mathbb{Q}_Y \simeq f^\circ_!\tilde{\jmath}^!f^!\mathbb{Q}_Y \simeq f^\circ_!(f^\circ)^!j^!\mathbb{Q}_Y \simeq f^\circ_!(f^\circ)^!j^*\mathbb{Q}_Y \simeq f^\circ_!(f^\circ)^!\mathbb{Q}_{Y^\circ}$$

where we have used proper base change, compatibility of $(-)^!$ with composition and the fact that $e^! \simeq e^*$ for $e$ étale. Then $j^*$ is defined as

$$\mathrm{End}_{\mathrm{DM}(Y)}(f_!f^!\mathbb{Q}_Y) \xrightarrow{j^*} \mathrm{End}_{\mathrm{DM}(Y)}(j^*f_!f^!\mathbb{Q}_Y) \simeq \mathrm{End}_{\mathrm{DM}(Y^\circ)}(f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ}).$$

The map $j^*$ is clearly compatible with addition and composition, hence is a homomorphism of rings. It remains to show that it is bijective.

Since $X$ and $Y$ are both smooth of dimension $d$ over $k$, we can use purity isomorphisms to obtain an isomorphism

$$f^!\mathbb{Q}_Y \simeq f^!\pi_Y^!\mathbb{Q}_k\{-d\} \simeq \pi_X^!\mathbb{Q}_k\{-d\} \simeq \mathbb{Q}_X. \tag{2}$$

We deduce that

$$f_*\mathbb{Q}_X \simeq f_*f^!\mathbb{Q}_Y \simeq f_!f^!\mathbb{Q}_Y,$$

and similarly that

$$f^\circ_*\mathbb{Q}_{X^\circ} \simeq f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ}.$$

These two isomorphisms are compatible with restriction along $j$. Combining this observation with Proposition 2.8, we have the commutative diagram with horizontal isomorphisms

$$\begin{tikzcd} \mathrm{CH}_d(X\times_Y X) \arrow[r, "\sim", "\phi_f"'] \arrow[d, "(j\times j)^*"'] & \mathrm{End}_{\mathrm{DM}(Y)}(f_*\mathbb{Q}_X) \arrow[r, "\sim"] \arrow[d, "j^*"'] & \mathrm{End}_{\mathrm{DM}(Y)}(f_!f^!\mathbb{Q}_Y) \arrow[d, "j^*"'] \\ \mathrm{CH}_d(X^\circ\times_{Y^\circ} X^\circ) \arrow[r, "\sim", "\phi_{f^\circ}"'] & \mathrm{End}_{\mathrm{DM}(Y^\circ)}(f^\circ_*\mathbb{Q}_{X^\circ}) \arrow[r, "\sim"] & \mathrm{End}_{\mathrm{DM}(Y^\circ)}(f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ}). \end{tikzcd}$$

On a variety of dimension $d$, we have $\mathrm{CH}_d = Z_d$, i.e., rational equivalence is trivial on top-dimensional cycles. By Lemma 2.7 (ii), this implies $\mathrm{CH}_d(X\times_Y X) \simeq Z_d(X\times_Y X)$ and also $\mathrm{CH}_d(X\times_Y X) \simeq Z_d(X^\circ\times_{Y^\circ} X^\circ)$. By Lemma 2.7 (iii), the restriction morphism $Z_d(X\times_Y X) \to$

$Z_d(X^\circ \times_{Y^\circ} X^\circ)$ is a bijection. We deduce that the left vertical map in the diagram above is a bijection, and conclude that the right vertical map is a bijection. □

**Lemma 2.11.** *Let $f : X \to Y$ be a finite type separated morphism with $Y$ smooth. Then there exists a morphism of $\mathbb{Q}$-algebras*

$$\psi_f : \mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \to \mathrm{End}_{\mathrm{DM}(k)}(M(X))$$

*such that, for $e : U \to Y$ an étale morphism, $\tilde{e} : V \to X$ its base change along $f$ and $\tilde{f} : V \to U$ the base change of $f$ along $e$, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \arrow[r, "\psi_f"'] \arrow[d, "e^*"] & \mathrm{End}_{\mathrm{DM}(k)}(M(X)) \arrow[d, "e^*"] \\
\mathrm{End}_{\mathrm{DM}(U)}(\tilde{f}_! \tilde{f}^! \mathbb{Q}_U) \arrow[r, "\psi_{\tilde{f}}"'] & \mathrm{End}_{\mathrm{DM}(k)}(M(V)).
\end{tikzcd}$$

*Proof.* Recall that, for $Z$ a smooth variety of dimension $e$ over $k$, we have a canonical purity isomorphism $\pi_Z^! \mathbb{Q}_k \simeq \mathbb{Q}_Z\{e\}$. By working with each connected component of $Y$ separately, we can assume that $Y$ is equidimensional of dimension $d$. We deduce that

$$M(X) := \pi_{X!} \pi_X^! \mathbb{Q}_k \simeq \pi_{Y!} f_! f^! \pi_Y^! \mathbb{Q}_k \simeq \pi_{Y!} f_! f^! \mathbb{Q}_Y\{d\}$$

by using the purity isomorphism for the smooth morphism $\pi_Y$. We define $\psi_f$ as the composition

$$\mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y) \xrightarrow{\pi_{Y!}(-)\{d\}} \mathrm{End}_{\mathrm{DM}(k)}(\pi_{Y!} f_! f^! \mathbb{Q}_Y\{d\}) \simeq \mathrm{End}_{\mathrm{DM}(k)} M(X).$$

The compatibility with pullbacks by étale morphisms follows again easily from the natural isomorphism $e^! \simeq e^*$ for an étale morphism $e$. □

2.4. **Group actions on motives of small maps.** Let $S$ be a scheme, $M \in \mathrm{DM}(S)$ a motive and $G$ a group. An action of $G$ on $M$ is a morphism of groups $a : G \to \mathrm{Aut}_{\mathrm{DM}(S)}(M)$. In particular, given a morphism $f : X \to Y$, we have an action $\mathrm{Aut}_Y(X) \to \mathrm{Aut}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$.

Assuming further that $G$ is finite, let

$$\Pi_{G,a} := \frac{1}{|G|} \sum_{g \in G} a(g) \in \mathrm{End}_{\mathrm{DM}(S)}(M)$$

which makes sense since $\mathrm{DM}(S)$ is $\mathbb{Q}$-linear. When the action is clear, we write simply $\Pi_G$. Then $\Pi_{G,a}$ is idempotent, and since $\mathrm{DM}(S)$ is idempotent-complete we define the invariant motive $M^G \in \mathrm{DM}(S)$ as the image of $\Pi_{G,a}$. This elementary construction coincides with the homotopy fixed points $M^{hG}$ of the $G$-action [12, §3.3.21].

For any motive $M \in \mathrm{DM}(S)$, we define $\mathrm{Sym}^n(M) := (M^{\otimes n})^{S_n}$ to be the invariant motive for the permutation action of the symmetric group on $M^{\otimes n}$.

**Example 2.12.** An important example for this paper is motives of symmetric products. For a quasi-projective variety $X$ over $k$ and $n \in \mathbb{N}$, we have a morphism $f : X^n \to \mathrm{Sym}^n(X)$. The symmetric group $S_n$ acts on $X^n$ over $\mathrm{Sym}^n(X)$, so that we get an induced action on $M(X^n)$ such that $M(f) : M(X^n) \to M(\mathrm{Sym}^n(X))$ factors via $M(X^n)^{S_n} \to M(\mathrm{Sym}^n(X))$. Since $S_n$ acts transitively on the geometric fibers of $f$, this second morphism is an isomorphism $\mathrm{Sym}^n(M(X)) \simeq M(X^n)^{S_n} \simeq M(\mathrm{Sym}^n(X))$ by [4, Corollaire 2.1.166]. More generally, if $G$ is a finite group which acts on the source $X$ of a finite surjective morphism $f : X \to Y$ between quasi-projective varieties, in such a way that $G$ acts transitively on the geometric fibers of $f$, then $M(Y) \simeq M(X)^G$, again by [4, Corollaire 2.1.166].

The main result of this section is a generalisation of the previous example where we do not have a global action on $X$ and $f$ is not necessarily finite but only small.

**Theorem 2.13.** *Let $f : X \to Y$ be a small surjective proper morphism between smooth connected varieties and $G$ be a finite group. Assume that the restriction $f^\circ : X^\circ \to Y^\circ$ to the locus* *with finite fibers is a principal $G$-bundle.*[3] *Then the action of $G$ on $M(X^\circ)$ extends to an action on $M(X)$ which induces an isomorphism $M(X)^G \simeq M(Y)$; we have a commutative diagram*

$$\begin{tikzcd} M(X^\circ) \arrow[r] \arrow[d] & M(X^\circ)^G \arrow[r, "\sim"] \arrow[d] & M(Y^\circ) \arrow[d] \\ M(X) \arrow[r] & M(X)^G \arrow[r, "\sim"] & M(Y). \end{tikzcd}$$

*where the top isomorphism comes from* [4, Corollaire 2.1.166].

*Proof.* Write $d = \dim(X) = \dim(Y)$. By Lemma 2.5, $Y^\circ$ is open and dense in $Y$. Since $f^\circ : X^\circ \to Y^\circ$ is a principal $G$-bundle, we have a morphism of groups $G \to \mathrm{Aut}_{Y^\circ}(X^\circ)$. We deduce a morphism of groups $G \to \mathrm{Aut}_{\mathrm{DM}(Y^\circ)}(f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ})$. By Proposition 2.10, this yields a morphism of groups $G \to \mathrm{Aut}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$. We compose with the morphism $\psi_f$ of Lemma 2.11 and get a morphism of groups $G \to \mathrm{Aut}_{\mathrm{DM}(k)}(M(X))$, which is the required action.

Let us check that the morphism $M(f) : M(X) \to M(Y)$ factors through $M(X)^G$. Given its construction, it suffices to show that the counit morphism $f_! f^! \mathbb{Q}_Y \to \mathbb{Q}_Y$ factors through $(f_! f^! \mathbb{Q}_Y)^G$. For this, it suffices to show that, for any $g \in G$, the composition $f_! f^! \mathbb{Q}_Y \xrightarrow{g} f_! f^! \mathbb{Q}_Y \to \mathbb{Q}_Y$ coincides with the counit of the adjunction $(f_!, f^!)$. By the same adjunction, this amounts to comparing two maps $f^! \mathbb{Q}_Y \to f^! \mathbb{Q}_Y$. By equation (2), we have $f^! \mathbb{Q}_Y \simeq \mathbb{Q}_X$. By [6, Proposition 11.1], there is for any $k$-variety $S$ an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Q}_S, \mathbb{Q}_S) \simeq \mathbb{Q}^{\pi_0(S)}.$$

The map from right to left is defined by sending endomorphisms of the constant étale sheaf $\mathbb{Q}_{S_{\text{ét}}}$ (which are exactly given by $\mathbb{Q}^{\pi_0(S)}$) to morphisms in $\mathrm{DM}(S)$. In particular, this isomorphism is contravariantly natural in $S$.

Combining this with the fact that $X^\circ$ is dense in $X$ and hence that $\pi_0(X) \simeq \pi_0(X^\circ)$, we see that the morphism

$$\mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X) \simeq \mathbb{Q}^{\pi_0(X)} \hookrightarrow \mathbb{Q}^{\pi_0(X^\circ)} \simeq \mathrm{Hom}_{\mathrm{DM}(X^\circ)}(\mathbb{Q}_{X^\circ}, \mathbb{Q}_{X^\circ})$$

is injective. Thus we can check the required equality after restriction to $X^\circ$; that is, we must show that for any $g \in G$, the composition $f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ} \xrightarrow{g} f^\circ_! f^{\circ !}\mathbb{Q}_{Y^\circ} \to \mathbb{Q}_{Y^\circ}$ coincides with the counit of the adjunction $(f^\circ_!, f^{\circ !})$. This is clear since $G$ acts through $\mathrm{Aut}_{Y^\circ}(X^\circ)$.

By construction, to show that the induced map $M(X)^G \to M(Y)$ is an isomorphism, it suffices to show that the morphism $(f_! f^! \mathbb{Q}_Y)^G \to \mathbb{Q}_Y$ is an isomorphism. Let $\Pi_G \in \mathrm{End}_{\mathrm{DM}(Y)}(f_! f^! \mathbb{Q}_Y)$ be the projector onto $(f_! f^! \mathbb{Q}_Y)^G$. Since $X$ and $Y$ are smooth of the same dimension $d$, the purity isomorphisms yield an isomorphism $f^! \mathbb{Q}_Y \simeq \mathbb{Q}_X$ (equation (2)). Moreover, this isomorphism is compatible with restriction to $Y^\circ$, in the sense that after applying $\tilde{j}^! = \tilde{j}^*$ for $\tilde{j} : X^\circ \to X$, it coincides with the simpler isomorphism $f^{\circ !}\mathbb{Q}_{Y^\circ} \simeq f^{\circ *}\mathbb{Q}_{Y^\circ} \simeq \mathbb{Q}_{X^\circ}$ (using that $f^\circ$ is étale).

Consider the composition

$$\Pi' : f_! f^! \mathbb{Q}_Y \xrightarrow{\eta_!} \mathbb{Q}_Y \xrightarrow{\frac{1}{|G|}} \mathbb{Q}_Y \xrightarrow{\epsilon_*} f_* \mathbb{Q}_X \simeq f_! f^! \mathbb{Q}_Y$$

where $\epsilon_*$ is the unit for the adjunction $(f^*, f_*)$ and $\eta_!$ is the counit for the adjunction $(f_!, f^!)$. By [4, Lemme 2.1.165], we see that $j^*\Pi'$ is a projector which coincides with $j^*\Pi_G$. By the injectivity of $j^*$ (Proposition 2.10), this implies that $\Pi' = \Pi_G$, thus $\Pi'$ is a projector, and to conclude it remains to identify the morphism $f_! f^! \mathbb{Q}_Y \to \mathrm{Im}(\Pi')$ with the morphism $f_! f^! \mathbb{Q}_Y \to \mathbb{Q}_Y$.

For this, it is clearly enough to show that the composition

$$\mathbb{Q}_Y \xrightarrow{\epsilon_*} f_* \mathbb{Q}_X \simeq f_! f^! \mathbb{Q}_Y \xrightarrow{\eta_!} \mathbb{Q}_Y$$

coincides with the multiplication by $|G|$.

[3]This assumption can be weakened to suppose that the restriction of $f$ to a dense open set $U \subset Y$ is a principal $G$-bundle, as Lemma 2.7 (iii) holds on replacing $Y_{f,0}$ with $U$.

By [6, Proposition 11.1], we have for any $k$-variety $S$ an isomorphism $\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Q}_S, \mathbb{Q}_S) \simeq \mathbb{Q}^{\pi_0(S)}$ which, as we explained above, is contravariantly natural in $S$. Since $Y$ and $Y^\circ$ are connected, we deduce that

$$\mathrm{Hom}_{\mathrm{DM}(Y)}(\mathbb{Q}_Y, \mathbb{Q}_Y) \simeq \mathbb{Q}_{\pi_0(Y)} \simeq \mathbb{Q}_{\pi_0(Y^\circ)} \simeq \mathrm{Hom}_{\mathrm{DM}(Y^\circ)}(\mathbb{Q}_{Y^\circ}, \mathbb{Q}_{Y^\circ})$$

hence it is enough to show this after restriction to $Y^\circ$. The corresponding composition is

$$\mathbb{Q}_{Y^\circ} \xrightarrow{\epsilon_*} f^\circ_* \mathbb{Q}_{X^\circ} \simeq f^\circ_! f^{\circ !} \mathbb{Q}_{Y^\circ} \xrightarrow{\eta_!} \mathbb{Q}_{Y^\circ}$$

which coincides with multiplication by $|G|$ by [4, Lemme 2.1.165]. □

**Remark 2.14.** Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

with $f$ and $f'$ satisfying the assumptions of Theorem 2.13 with groups $G$ and $G'$. If $g$ does not send the locus $Y^0$ into $(Y^0)'$, it is not clear how to formulate conditions which make the morphism $M(X) \to M(X')$ equivariant with respect to some given homomorphism $G \to G'$. However in the application in §4, we have an alternative description of the actions which make a certain equivariance property clear (see Proposition 4.2).

## 3. Motives of schemes of Hecke correspondences

In this section, we introduce some generalisations of the schemes of matrix divisors $\mathrm{Div}_{n,d}(D)$ and the flag-generalisation $\mathrm{FDiv}_{n,d}(D)$ and study their motives. The main result in this section is inspired by work of Laumon [30] and Heinloth [23].

3.1. **Definitions and basic properties.** For a family $\mathcal{E}$ of vector bundles on $C$ parametrised by a $k$-scheme $T$, we write $\mathrm{rk}(\mathcal{E}) = n$ and $\deg(\mathcal{E}) = d$ if the fibrewise rank and degree of this family are $n$ and $d$ respectively.

**Definition 3.1.** For $l \in \mathbb{N}$ and a family $\mathcal{E}$ of rank $n$ degree $d$ vector bundles over $C$ parametrised by $k$-scheme $T$, we define two $T$-schemes $\mathcal{H}^l_{\mathcal{E}/T}$ and $\mathcal{FH}^l_{\mathcal{E}/T}$ as follows: over $g : S \to T$, the points of these schemes parametrise the following injection of locally free sheaves

$$\mathcal{H}^l_{\mathcal{E}/T}(S) := \left\{ \phi : \mathcal{F} \hookrightarrow (g \times \mathrm{id}_C)^*\mathcal{E} : \begin{array}{c} \mathcal{F} \to S \times C \text{ family of vector bundles on } C \\ \mathrm{rk}(\mathcal{F}) = n, \deg(\mathcal{F}) = d - l \end{array} \right\}$$

and chains of vector bundle injections respectively

$$\mathcal{FH}^l_{\mathcal{E}/T}(S) := \left\{ \mathcal{F}_l \hookrightarrow \mathcal{F}_{l-1} \cdots \hookrightarrow \mathcal{F}_0 := (g \times \mathrm{id}_C)^*\mathcal{E} : \begin{array}{c} \mathcal{F}_i \to S \times C \text{ family of vector bundles} \\ \mathrm{rk}(\mathcal{F}_i) = n, \deg(\mathcal{F}_i) = d - i \text{ for } 1 \leq i \leq l \end{array} \right\}.$$

We refer to $\mathcal{H}^l_{\mathcal{E}/T}$ as the $T$-scheme of length $l$ Hecke correspondences of $\mathcal{E}$ and the $\mathcal{FH}^l_{\mathcal{E}/T}$ as the $T$-scheme of $l$-iterated Hecke correspondences of $\mathcal{E}$.

Let us first explain why these are both schemes over $T$. The scheme of length $l$ Hecke correspondences $\mathcal{H}^l_{\mathcal{E}/T}$ is the Quot scheme over $T$

$$\mathcal{H}^l_{\mathcal{E}/T} = \mathrm{Quot}^{(0,l)}_{T \times C/T}(\mathcal{E})$$

parametrising quotient families of $\mathcal{E}$ of rank 0 and degree $l$, which is a projective $T$-scheme. Similarly $\mathcal{FH}^l_{\mathcal{E}/T}$ is a generalisation of Quot schemes to allow flags of arbitrary length, called a Flag-Quot or Drap scheme (see [27, Appendix 2A]); thus $\mathcal{FH}^l_{\mathcal{E}/T}$ is also projective over $T$. In fact, as we are considering a Quot scheme of torsion sheaves of a smooth projective curve, both $\mathcal{H}^l_{\mathcal{E}/T}$ and $\mathcal{FH}^l_{\mathcal{E}/T}$ are smooth $T$-schemes (see [27, Propositions 2.2.8 and 2.A.12]). In particular, if $T/k$ is smooth (resp. projective), then both these schemes are smooth (resp. projective) over $k$.

**Example 3.2.** Let $T = \mathrm{Spec}(k)$ and $\mathcal{E} = \mathcal{O}_C(D)^{\oplus n}$ for a divisor $D$ on $C$; then

$$\mathcal{H}^{n\deg(D)-d}_{\mathcal{E}/T} = \mathrm{Div}_{n,d}(D) \quad \text{and} \quad \mathcal{FH}^{n\deg(D)-d}_{\mathcal{E}/T} = \mathrm{FDiv}_{n,d}(D),$$

where $\mathrm{FDiv}_{n,d}(D) = \{\mathcal{E}_{n\deg(D)-d} \subsetneq \cdots \subsetneq \mathcal{E}_0 = \mathcal{O}_C(D)^{\oplus n} : (\mathrm{rk}, \deg)(\mathcal{E}_i) = (n, n\deg(D) - i)\}$ and these schemes of (iterated) Hecke correspondences are both smooth and projective.

We introduce some notation and properties of these Hecke schemes in the following remark.

**Remark 3.3.** Let $\mathcal{E}$ be a family of rank $n$ degree $d$ vector bundles over $C$ parametrised by $T$.

(i) For $l = 0$, we note that $\mathcal{FH}^0_{\mathcal{E}/T} = \mathcal{H}^0_{\mathcal{E}/T} = T$ and for $l = 1$, we have

$$\mathcal{FH}^1_{\mathcal{E}/T} = \mathcal{H}^1_{\mathcal{E}/T} \to T \times C$$

where this projection is given by taking the support of the family of degree 1 torsion sheaves. Moreover, we have an isomorphism of $T$-schemes

$$\mathcal{H}^1_{\mathcal{E}/T} \cong \mathbb{P}(\mathcal{E})$$

where $\mathbb{P}(-)$ denotes the moduli space of 1-dimensional quotients of a coherent sheaf (i.e. following Grothendieck's convention). Indeed, first of all pointwise this is explained by the fact that an elementary lower modification[4] of a vector bundle $E \to C$ at $x \in C$ is equivalent to a surjection $E_x \twoheadrightarrow \kappa(x)$ (up to scalar multiplication). More precisely one constructs the morphism $\mathbb{P}(\mathcal{E}) \to \mathcal{H}^1_{\mathcal{E}/T}$ by constructing a family of degree 1 torsion quotients of $\mathcal{E}$ parametrised by $\mathbb{P}(\mathcal{E})$ as follows. Let $r : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}) \times_T \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}) \times_T (T \times C)$ be the composition of the diagonal with $\mathrm{id}_{\mathbb{P}(\mathcal{E})} \times_T \pi_{\mathbb{P}}$ for the projection $\pi_{\mathbb{P}} : \mathbb{P}(\mathcal{E}) \to T \times C$. We pushforward the universal quotient $\pi_{\mathbb{P}}^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ along $r$ and consider the composition

$$q^*\mathcal{E} \to r_*r^*q^*\mathcal{E} \cong r_*\pi_{\mathbb{P}}^*\mathcal{E} \to r_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

which is the desired family of quotient sheaves, where $q : \mathbb{P}(\mathcal{E}) \times_T (T \times C) \to T \times C$ is the projection.

(ii) Since $\mathcal{FH}^l_{\mathcal{E}/T}$ is a Flag-Quot scheme there is a universal flag of vector bundles

$$\mathcal{U}^l_l \hookrightarrow \mathcal{U}^l_{l-1} \hookrightarrow \cdots \hookrightarrow \mathcal{U}^l_1 \hookrightarrow \mathcal{U}^l_0 := p_2^*\mathcal{E}$$

over $\mathcal{FH}^l_{\mathcal{E}/T} \times_T (T \times C) \cong \mathcal{FH}^l_{\mathcal{E}/T} \times C$. In fact, Flag-Quot schemes, and in particular schemes of iterated Hecke correspondences, are constructed as iterated relative Quot schemes. More precisely, we have

$$\pi_l : \mathcal{FH}^l_{\mathcal{E}/T} \cong \mathcal{H}^1_{\mathcal{U}^{l-1}_{l-1}/\mathcal{FH}^{l-1}_{\mathcal{E}/T}} \cong \mathbb{P}(\mathcal{U}^{l-1}_{l-1}) \to \mathcal{FH}^{l-1}_{\mathcal{E}/T} \times_T (T \times C) \cong \mathcal{FH}^{l-1}_{\mathcal{E}/T} \times C.$$

where $\pi_l(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) := (\mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0, \mathrm{supp}(\mathcal{F}_{l-1}/\mathcal{F}_l))$

(iii) There is a map $P_l : \mathcal{FH}^l_{\mathcal{E}/T} \to T \times C^l$ obtained by composing the maps $\pi_j$ for $1 \leq j \leq l$

$$P_l : \mathcal{FH}^l_{\mathcal{E}/T} \to \mathcal{FH}^{l-1}_{\mathcal{E}/T} \times_T (T \times C) \to \mathcal{FH}^{l-2}_{\mathcal{E}/T} \times_T (T \times C)^{\times_T 2} \cdots \to T \times_T (T \times C)^{\times_T l}.$$

Explicitly, we have $P_l(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) = (\mathrm{supp}(\mathcal{F}_0/\mathcal{F}_1), \dots, \mathrm{supp}(\mathcal{F}_{l-1}/\mathcal{F}_l))$.

(iv) For $1 \leq j \leq l$, we let $\mathrm{pr}^l_j : \mathcal{FH}^l_{\mathcal{E}/T} \to T \times C$ denote the composition of $P_l$ with the projection onto the $j$th copy of $T \times C$; that is,

$$\mathrm{pr}^l_j(\mathcal{F}_l \subsetneq \mathcal{F}_{l-1} \subsetneq \cdots \subsetneq \mathcal{F}_0) = \mathrm{supp}(\mathcal{F}_{j-1}/\mathcal{F}_j).$$

(v) Let $p_l : \mathcal{FH}^l_{\mathcal{E}/T} \to \mathcal{FH}^{l-1}_{\mathcal{E}/T}$ denote the composition of $\pi_l$ with the projection to the first factor; then for $1 \leq j \leq l-1$, we have $(p_l \times \mathrm{id}_C)^*\mathcal{U}^{l-1}_j = \mathcal{U}^l_j$.

[4]That is, a Hecke modification $E' \subset E$ whose quotient is supported at $x$ and has length 1.

**Lemma 3.4.** *Let $\mathcal{E}$ be a family of rank $n$ degree $d$ vector bundles over $C$ parametrised by a scheme $T$; then the scheme $\mathcal{FH}^l_{\mathcal{E}/T}$ is an $l$-iterated $\mathbb{P}^{n-1}$-bundle over $T \times C^l$. More precisely, we have the following sequence of projective bundles*

$$\mathcal{FH}^l_{\mathcal{E}/T} \cong \mathbb{P}(\mathcal{U}^{l-1}_{l-1}) \to \mathcal{FH}^{l-1}_{\mathcal{E}/T} \times C \cong \mathbb{P}(\mathcal{U}^{l-2}_{l-2}) \times C \to \cdots \to \mathcal{FH}^1_{\mathcal{E}/T} \times C^{l-1} \cong \mathbb{P}(\mathcal{E}) \times C^{l-1} \to T \times C^l.$$

*Proof.* This follows by induction from Remark 3.3 (i) and (ii). □

By repeatedly applying the projective bundle formula, we obtain the following corollary.

**Corollary 3.5.** *Let $\mathcal{E}$ be family of rank $n$ degree $d$ vector bundles over $C$ parametrised by a scheme $T$. Then*

$$M(\mathcal{FH}^l_{\mathcal{E}/T}) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}.$$

In fact, we will need to explicitly identify this isomorphism. For a rank $n$ vector bundle $\mathcal{V}$ over a scheme $X$, the projective bundle $\pi : \mathbb{P}(\mathcal{V}) \to X$ is equipped with a line bundle $\mathcal{L} := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. The first chern class of this line bundle defines a map $c_1(\mathcal{L}) : M(\mathbb{P}(\mathcal{V})) \to \mathbb{Q}\{1\}$ and for $i \geq 0$, by composing with the diagonal $\Delta : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})^{\times i}$ we obtain maps

$$c_1(\mathcal{L})^{\otimes i} : M(\mathbb{P}(\mathcal{V})) \xrightarrow{M(\Delta)} M(\mathbb{P}(\mathcal{V}))^{\otimes i} \xrightarrow{c_1(\mathcal{L})^{\otimes i}} \mathbb{Q}\{i\}$$

which together define a map $[c_1(\mathcal{L})] := \oplus_{i=0}^{n-1} c_1(\mathcal{L})^{\otimes i} : M(\mathbb{P}(\mathcal{V})) \to \oplus_{i=-0}^{n-1} \mathbb{Q}\{i\} \simeq M(\mathbb{P}^{n-1})$. Then the projective bundle formula isomorphism can be explicitly written as the composition

$$\mathrm{PB}(\mathcal{L}) : M(\mathbb{P}(\mathcal{V})) \xrightarrow{M(\Delta)} M(\mathbb{P}(\mathcal{V}))^{\otimes 2} \xrightarrow{M(\pi) \otimes [c_1(\mathcal{L})]} M(X) \otimes M(\mathbb{P}^{n-1}).$$

**Remark 3.6.** On $\mathcal{FH}^l := \mathcal{FH}^l_{\mathcal{E}/T}$, we can inductively define $l$ line bundles $\mathcal{L}^l_1, \cdots, \mathcal{L}^l_l$ by

(i) $\mathcal{L}^l_l := \mathcal{O}(1) \to \mathbb{P}(\mathcal{U}^{l-1}_{l-1})$,
(ii) $\mathcal{L}^l_j := p_l^* \mathcal{L}^{l-1}_j$ for $1 \leq j \leq l-1$, where $p_l : \mathcal{FH}^l \to \mathcal{FH}^{l-1}$.

These $l$ line bundles on $\mathcal{FH}^l$ induce a morphism

$$\mathrm{PB}(\mathcal{L}^l_\bullet) : M(\mathcal{FH}^l_{\mathcal{E}/T}) \xrightarrow{M(\Delta)} M(\mathcal{FH}^l_{\mathcal{E}/T})^{\otimes l+1} \xrightarrow{M(P_l) \otimes [c_1(\mathcal{L}^l_\bullet)]} M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l},$$

where $[c_1(\mathcal{L}^l_\bullet)] = \otimes_{i=1}^l [c_1(\mathcal{L}^l_i)]$. Furthermore, on $\mathcal{FH}^l$ we have two universal objects:

(i) a surjection $\pi_l^* \mathcal{U}^{l-1}_{l-1} \twoheadrightarrow \mathcal{L}^l_l$ over $\mathcal{FH}^l$ (as $\mathcal{FH}^l \cong \mathbb{P}(\mathcal{U}^{l-1}_{l-1})$ by Remark 3.3),
(ii) a short exact sequence $0 \to \mathcal{U}^l_l \to \mathcal{U}^l_{l-1} \to \mathcal{T}^l_l \to 0$ over $\mathcal{FH}^l \times C$.

Since $\mathcal{U}^l_{l-1} = (p_l \times \mathrm{id}_C)^* \mathcal{U}^{l-1}_{l-1}$, the relationship between the line bundle $\mathcal{L}^l_l \to \mathcal{FH}^l$ and the family of degree 1 torsion sheaves $\mathcal{T}^l_l$ on $C$ parametrised by $\mathcal{FH}^l$ is

$$\mathcal{L}^l_l \cong ((\mathrm{id}_{\mathcal{FH}^l} \times \pi_l) \circ \Delta_{\mathcal{FH}^l})^* \mathcal{T}^l_l,$$

where $(\mathrm{id}_{\mathcal{FH}^l} \times \pi_l) \circ \Delta_{\mathcal{FH}^l} : \mathcal{FH}^l \xrightarrow{\Delta_{\mathcal{FH}^l}} \mathcal{FH}^l \times_{\mathcal{FH}^{l-1}} \mathcal{FH}^l \xrightarrow{\mathrm{id} \times \pi_l} \mathcal{FH}^l \times_{\mathcal{FH}^{l-1}} (\mathcal{FH}^{l-1} \times C) \simeq \mathcal{FH}^l \times C$. In fact, for $1 \leq j \leq l$, we can define maps

$$r^l_j = (\mathrm{id}_{\mathcal{FH}^l} \times \mathrm{pr}^l_j) \circ \Delta_{\mathcal{FH}^l} : \mathcal{FH}^l \to \mathcal{FH}^l \times_T \mathcal{FH}^l \to \mathcal{FH}^l \times_T (T \times C) \cong \mathcal{FH}^l \times C$$

such that $r^l_l = (\mathrm{id}_{\mathcal{FH}^l} \times \pi_l) \circ \Delta_{\mathcal{FH}^l}$. For $j < l$, the family of degree 1 torsion sheaves $\mathcal{T}^l_j := \mathcal{U}^l_{j-1}/\mathcal{U}^l_j$ on $C$ parametrised by $\mathcal{FH}^l$ is obtained as a pullback of $\mathcal{T}^{l-1}_j$ via the map $p_l \times \mathrm{id}_C$. Hence, for $1 \leq j \leq l$, we have isomorphisms relating the line bundles and families of torsion sheaves

$$\mathcal{L}^l_j \cong (r^l_j)^* \mathcal{T}^l_j. \tag{3}$$

We can now give a precise description of the isomorphism in Corollary 3.5.

**Lemma 3.7.** *The tuple $\mathcal{L}^l_\bullet = (\mathcal{L}^l_1, \cdots, \mathcal{L}^l_l)$ of line bundles on $\mathcal{FH}^l_{\mathcal{E}/T}$ induces a morphism*

$$\mathrm{PB}(\mathcal{L}^l_\bullet) : M(\mathcal{FH}^l_{\mathcal{E}/T}) \xrightarrow{M(\Delta)} M(\mathcal{FH}^l_{\mathcal{E}/T})^{\otimes l+1} \xrightarrow{M(P_l) \otimes [c_1(\mathcal{L}^l_\bullet)]} M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l},$$

*which coincides with the composition*

$$M(\mathcal{FH}^l_{\mathcal{E}/T}) \overset{\mathrm{PB}(\mathcal{L}^l_l)}{\longrightarrow} M(\mathcal{FH}^{l-1}_{\mathcal{E}/T}) \otimes M(C \times \mathbb{P}^{n-1}) \overset{\mathrm{PB}(\mathcal{L}^{l-1}_{l-1})\otimes M(\mathrm{id})}{\longrightarrow} \cdots \longrightarrow M(T \times C^l) \otimes M(\mathbb{P}^{n-1})^{\otimes l}$$

*and thus is an isomorphism.*

*Proof.* For this one uses that Chern classes are compatible with pullbacks, so that $c_1(\mathcal{L}^{l-1}_j) \circ M(p_l) = c_1(\mathcal{L}^l_j)$ for $1 \leq j \leq l-1$, as $p_l^*(\mathcal{L}^{l-1}_j) = \mathcal{L}^l_j$. Then one uses that $P_l$ is defined as the composition of the maps $\pi_i$ for $i \leq l$ together with the fact that for any morphism $f : X \to Y_1 \times Y_2$, we have the following commutative diagram

$$\begin{tikzcd} M(X) \arrow[r, "M(\Delta)"] \arrow[d, "M(f)"'] & M(X) \otimes M(X) \arrow[d, "M(f_1)\otimes M(f_2)"] \\ M(Y_1 \times Y_2) \arrow[r, "\simeq"] & M(Y_1) \otimes M(Y_2), \end{tikzcd}$$

where $f_i := \mathrm{pr}_i \circ f : X \to Y_i$ and the horizontal maps use Künneth isomorphisms. □

3.2. **The motive of the scheme of Hecke correspondences.** There is a forgetful map

$$f : \mathcal{FH}^l_{\mathcal{E}/T} \to \mathcal{H}^l_{\mathcal{E}/T}$$

that we will use to relate the motive of $\mathcal{H}^l_{\mathcal{E}/T}$ to that of $\mathcal{FH}^l_{\mathcal{E}/T}$, which we computed above. In fact, we plan to use the previous section to compare these motives, as the map $f$ is small. To prove that $f$ is a small map, we will describe it as the pullback of a small map along a flat morphism by generalising an argument of Heinloth [23, Proposition 11].

Let $\mathrm{Coh}_{0,l}$ denote the stack of rank 0 degree $l$ coherent sheaves on $C$ and let $\widetilde{\mathrm{Coh}}_{0,l}$ denote the stack which associates to a scheme $S$ the groupoid

$$\widetilde{\mathrm{Coh}}_{0,l}(S) = \langle \mathcal{T}_1 \hookrightarrow \mathcal{T}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{T}_l : \mathcal{T}_i \in \mathrm{Coh}_{0,i}(S) \rangle.$$

The forgetful map $f' : \widetilde{\mathrm{Coh}}_{0,l} \to \mathrm{Coh}_{0,l}$ fits into the following commutative diagram

$$\begin{tikzcd} \mathcal{FH}^l_{\mathcal{E}/T} \arrow[r, "\tilde{\mathrm{gr}}"] \arrow[d, "f"] & T \times \widetilde{\mathrm{Coh}}_{0,l} \arrow[r] \arrow[d, "\mathrm{id}_T \times f'"] & T \times C^l \arrow[d] \\ \mathcal{H}^l_{\mathcal{E}/T} \arrow[r, "\mathrm{gr}"] & T \times \mathrm{Coh}_{0,l} \arrow[r] & T \times \mathrm{Sym}^l(C) \end{tikzcd} \tag{4}$$

where the morphisms gr and $\tilde{\mathrm{gr}}$ are given by taking the associated graded sheaves of the (flags) of vector bundle injections and the left square in this diagram is Cartesian. Furthermore, by [30] paragraph (3.2) and the proof of Theorem 3.3.1, the map $f'$ of algebraic stacks is small and representable, and its restriction to the locus of torsion sheaves with support consisting of $l$ distinct points is an $S_l$-covering. By Lemma 2.6, $\mathrm{id}_T \times f'$ is small and a $S_l$-covering on this dense open. Since the morphism gr is smooth and thus flat (see the proof of [23, Proposition 11]), we deduce by Lemma 2.6 that $f$ is small and generically a $S_l$-covering. By Theorem 2.13, there is an induced $S_l$-action on $M(\mathcal{FH}^l_{\mathcal{E}/T})$ and we can now prove the following result.

**Theorem 3.8.** *Let $\mathcal{E}$ be family of rank $n$ degree $d$ vector bundles over $C$ parametrised by a smooth $k$-scheme $T$. Then via the isomorphism $M(\mathcal{FH}^l_{\mathcal{E}/T}) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}$ of Corollary 3.5, the $S_l$-action permutes the $l$-copies of $M(C \times \mathbb{P}^{n-1})$. Moreover, we have*

$$M(\mathcal{H}^l_{\mathcal{E}/T}) \cong M(T) \otimes M(\mathrm{Sym}^l(C \times \mathbb{P}^{n-1})).$$

*Proof.* We note that as $T$ is smooth, both $\mathcal{H}^l_{\mathcal{E}/T}$ and $\mathcal{FH}^l_{\mathcal{E}/T}$ are smooth over $k$. By Lemma 3.7, there is an isomorphism

$$M(\mathcal{FH}^l_{\mathcal{E}/T}) \cong M(T) \otimes M(C \times \mathbb{P}^{n-1})^{\otimes l}$$

induced by $l$ line bundles $\mathcal{L}_1^l, \dots, \mathcal{L}_l^l$ on $\mathcal{FH}^l_{\mathcal{E}/T}$ (which are the pullbacks of the ample bundles on each projective bundle) and the projection $P_l : \mathcal{FH}^l_{\mathcal{E}/T} \to T \times C^l$. The $S_l$-action on $M(\mathcal{FH}^l_{\mathcal{E}/T})$ from Theorem 2.13 (including the footnote) is induced by the $S_l$-action on the open subset $\mathcal{F}U := p^{-1}(U)$, where $U \subset \mathcal{H}^l_{\mathcal{E}/T}$ parametrises length $l$ Hecke correspondences whose degree $l$ torsion quotient has support consisting of $l$ distinct points. The $S_l$-action on $\mathcal{F}U$ corresponds to permuting the $l$ universal degree 1 torsion quotients $\mathcal{T}_1^l, \dots, \mathcal{T}_l^l$. By Remark 3.6, this corresponds to permuting the $l$ line bundles $\mathcal{L}_i^l$ on $\mathcal{FH}^l_{\mathcal{E}/T}$ (see equation (3)). Therefore, the induced $S_l$-action on $M(\mathcal{FH}^l_{\mathcal{E}/T})$ permutes the $l$-copies of $M(C \times \mathbb{P}^{n-1})$. As $f$ is a small proper surjective map of smooth varieties, Theorem 2.13 yields an isomorphism

$$M(\mathcal{FH}^l_{\mathcal{E}/T})^{S_l} \cong M(\mathcal{H}^l_{\mathcal{E}/T}).$$

Finally, by Example 2.12 we have $\mathrm{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1}) \simeq M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1}))$. □

In particular, if we apply this to $T = \operatorname{Spec} k$ and $\mathcal{E} = \mathcal{O}_C(D)^{\oplus n}$ for a divisor $D$ on $C$ (see Example 3.2), we obtain Theorem 1.3 as a special case of this result. Furthermore, the motive of the Quot scheme of length $l$ torsion quotients of a locally free sheaf $\mathcal{E}$ over $T \times C/T$ only depends on the rank of $\mathcal{E}$; we explicitly state this as a corollary for $T = \operatorname{Spec} k$, as there are similar recent results concerning the class of such Quot schemes in the Grothendieck ring of varieties [7, 34].

**Corollary 3.9.** *Let $\mathcal{E}$ be a rank $n$ locally free sheaf on $C$ and $l \in \mathbb{N}$. Then the motive of the Quot scheme $\mathrm{Quot}^{(0,l)}_{C/k}(\mathcal{E})$ parametrising length $l$ torsion quotients of $\mathcal{E}$ is*

$$M(\mathrm{Quot}^{(0,l)}_{C/k}(\mathcal{E})) \cong M(\mathrm{Sym}^l(C \times \mathbb{P}^{n-1})).$$

*In particular, this motive only depends on the rank $n$ of $\mathcal{E}$.*

## 4. The formula for the motive of the stack of vector bundles

4.1. **The transition maps in the inductive system.** Throughout this section we fix $x \in C(k)$ and let $s_x : \operatorname{Spec} k \to C$ be the inclusion of $x$. The inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(x)$ defines an inductive sequence of morphisms $i_l : \mathrm{Div}_{n,d}(l) \to \mathrm{Div}_{n,d}(l+1)$ indexed by $l \in \mathbb{N}$. In this section, we will lift the maps $i_l : \mathrm{Div}_{n,d}(l) \to \mathrm{Div}_{n,d}(l+1)$ to the schemes of iterated Hecke correspondences and compute the induced maps of motives. We recall that

$$\mathrm{Div}_{n,d}(l) = \mathcal{H}^{nl-d}_{\mathcal{O}_C(lx)^{\oplus n}/\operatorname{Spec} k} \quad \text{and} \quad \mathrm{FDiv}_{n,d}(l) = \mathcal{FH}^{nl-d}_{\mathcal{O}_C(lx)^{\oplus n}/\operatorname{Spec} k}$$

and we will drop the subscripts for Hecke schemes throughout the rest of this section.

The inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(x)$ induces an inclusion $\mathcal{O}_C^{\oplus n} \hookrightarrow \mathcal{O}_C(x)^{\oplus n}$. Let $\mathcal{F}_\bullet$ be a flag of subsheaves

$$\mathcal{F}_\bullet = (\mathcal{O}_C^{\oplus n} = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = \mathcal{O}_C(x)^{\oplus n})$$

with the property that $\deg(\mathcal{F}_i) = \deg(\mathcal{F}_{i-1}) + 1$ for $1 \leq i \leq n$. The flag $\mathcal{F}_\bullet$ determines, for all $l \in \mathbb{N}$, a morphism $A_l(\mathcal{F}_\bullet) : \mathrm{FDiv}_{n,d}(l) \to \mathrm{FDiv}_{n,d}(l+1)$ defined by concatenating the universal flag on $\mathrm{FDiv}_{n,d}(l) \times C$ with the pullback of the flag $\mathcal{F}_\bullet \otimes \mathcal{O}_C(lx)$ to $\mathrm{FDiv}_{n,d}(l) \times C$. Moreover $A_l(\mathcal{F}_\bullet)$ lifts the morphism $i_l : \mathrm{Div}_{n,d}(l) \to \mathrm{Div}_{n,d}(l+1)$.

Recall that we have maps $P_{nl-d} : \mathrm{FDiv}_{n,d}(l) = \mathcal{FH}^{nl-d} \to C^{nl-d}$ defined in Remark 3.3. The morphism $A_l(\mathcal{F}_\bullet)$ sits in a commutative diagram

$$\begin{tikzcd} \mathrm{FDiv}_{n,d}(l) \arrow[r, "A_l(\mathcal{F}_\bullet)"] \arrow[d, "P_{nl-d}"'] & \mathrm{FDiv}_{n,d}(l+1) \arrow[d, "P_{n(l+1)-d}"] \\ C^{nl-d} \arrow[r, "c_l"] & C^{n(l+1)-d}. \end{tikzcd} \tag{5}$$

where $c_l := s_x^n \times \mathrm{id}_{C^{nl-d}}$. Recall that $\mathrm{pr}_j^{nl-d} : \mathrm{FDiv}_{n,d}(l) \to C^{nl-d} \to C$ denotes the composition of $P_{nl-d}$ with the projection onto the $j$th factor. We have

$$\mathrm{pr}_j^{n(l+1)-d} \circ A_l(\mathcal{F}_\bullet) = \begin{cases} t_x & \text{if } 1 \leq j \leq n \\ \mathrm{pr}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d, \end{cases} \tag{6}$$

where $t_x : \mathrm{FDiv}_{n,d}(l) \to \operatorname{Spec} k \to C$ is the composition of the structure map with $s_x$.

Similarly, a tuple $p := (p_1, \dots, p_n) \in (\mathbb{P}^{n-1})^n$ induces $b_l(p) : (\mathbb{P}^{n-1})^{nl-d} \to (\mathbb{P}^{n-1})^{n(l+1)-d}$ which is the identity on the last $nl-d$ factors. We define

$$a_l(p) := c_l \times b_l(p) : (C \times \mathbb{P}^{n-1})^{nl-d} \to (C \times \mathbb{P}^{n-1})^{n(l+1)-d}.$$

**Lemma 4.1.** *Every choice of flag $\mathcal{F}_\bullet$ as above induces the same map of motives*

$$M(A_l) := M(A_l(\mathcal{F}_\bullet)) : M(\mathrm{FDiv}_{n,d}(l)) \to M(\mathrm{FDiv}_{n,d}(l+1))$$

*and every choice of tuple $p \in (\mathbb{P}^{n-1})^n$ induces the same map of motives*

$$M(b_l) = M(b_l(p)) : M(\mathbb{P}^{n-1})^{\otimes nl-d} \to M(\mathbb{P}^{n-1})^{\otimes n(l+1)-d}.$$

*Proof.* The flag $\mathcal{F}_\bullet$ is uniquely specified by its image $\overline{\mathcal{F}}_\bullet$ in the quotient $\mathcal{O}_C(x)^{\oplus n}/\mathcal{O}_C^{\oplus n}$, which is a skycraper sheaf with fibre at $x$ isomorphic to $k^n$. Flags in $k^n$ are parametrised by $k$-points of the flag variety $\mathrm{GL}_n/B$, and we can construct in this way a map

$$A_l : \mathrm{GL}_n/B \times \mathrm{FDiv}_{n,d}(l) \to \mathrm{FDiv}_{n,d}(l+1))$$

whose restriction to $\{\overline{\mathcal{F}}_\bullet\} \times \mathrm{FDiv}_{n,d}(l)$ is $A_l(\mathcal{F}_\bullet)$. It is well-known that $\mathrm{Hom}(\mathbb{Q}_k, M(\mathrm{GL}_n/B)) \simeq \mathrm{CH}_0(\mathrm{GL}_n/B)_{\mathbb{Q}}$ is isomorphic to $\mathbb{Q}$ via the degree map (this follows for instance from the cellular structure given by the Bruhat decomposition, see [18, Example 19.1.11]). Together with the Künneth isomorphism for $M(-)$, this implies that for any two flags $\mathcal{F}_\bullet$ and $\mathcal{F}'_\bullet$, we have $[\mathcal{F}_\bullet] = [\mathcal{F}'_\bullet] \in \mathrm{Hom}(\mathbb{Q}_k, M(\mathrm{GL}_n/B))$ and that $M(A_l(\mathcal{F}_\bullet)) = M(A_l(\mathcal{F}'_\bullet)$ as claimed. The second statement follows from a similar argument as $\mathrm{CH}_0(\mathbb{P}^{n-1})_{\mathbb{Q}} \simeq \mathbb{Q}$. ☐

As we are only interested in studying these maps motivically, we will drop the choice of flag $\mathcal{F}_\bullet$ and tuple $p$ from the notation and simply write $A_l$, $b_l$ and $a_l$ for these morphisms.

By Lemma 3.7, there is an $S_{nl-d}$-equivariant isomorphism

$$\mathrm{PB}(\mathcal{L}_\bullet^{nl-d}) : M(\mathrm{FDiv}_{n,d}(l)) = M(\mathcal{FH}^{nl-d}) \to M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$$

determined by line bundles $\mathcal{L}_j^{nl-d}$ on $\mathcal{FH}^{nl-d}$ for $1 \leq j \leq nl-d$. Moreover, we have homomorphisms $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$ such that the maps $c_l : C^{nl-d} \to C^{n(l+1)-d}$ are equivariant.

**Proposition 4.2.** *For each $l$, we have a commutative diagram*

$$\begin{tikzcd} M(\mathrm{FDiv}_{n,d}(l)) \arrow[r, "M(A_l)"] \arrow[d, "\mathrm{PB}(\mathcal{L}_\bullet^{nl-d})"', "\wr"] & M(\mathrm{FDiv}_{n,d}(l+1)) \arrow[d, "\wr"', "\mathrm{PB}(\mathcal{L}_\bullet^{n(l+1)-d})"] \\ M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \arrow[r, "M(a_l)"] & M(C \times \mathbb{P}^{n-1})^{\otimes n(l+1)-d} \end{tikzcd} \tag{7}$$

*such that the horizontal maps are equivariant with respect to $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$.*

*Proof.* We claim that the pullbacks via $A_l$ (for any flag $\mathcal{F}_\bullet$) of the line bundles $\mathcal{L}_j^{n(l+1)-d}$ satisfy

$$A_l^* \mathcal{L}_j^{n(l+1)-d} = \begin{cases} \mathcal{O}_{\mathcal{FH}^{nl-d}} & \text{if } 1 \leq j \leq n \\ \mathcal{L}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d. \end{cases} \tag{8}$$

We recall that we have $n(l+1)-d$ families of degree 1 torsion sheaves on $C$ parametrised by $\mathrm{FDiv}_{n,d}(l+1) = \mathcal{FH}^{n(l+1)-d}$ given by the successive quotients of the universal flag of vector bundles on $\mathcal{FH}^{n(l+1)-d} \times C$; these families of torsion sheaves are denoted by

$$\mathcal{T}_j^{n(l+1)-d} := \mathcal{U}_{j-1}^{n(l+1)-d}/\mathcal{U}_j^{n(l+1)-d} \quad \text{for } 1 \leq j \leq n(l+1)-d.$$

The pullbacks of these families of torsion sheaves along $A_l$ (for any flag $\mathcal{F}_\bullet$) are as follows:

$$(A_l \times \mathrm{id}_C)^* \mathcal{T}_j^{n(l+1)-d} = \begin{cases} p_C^* k_x & \text{if } 1 \leq j \leq n \\ \mathcal{T}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d, \end{cases} \tag{9}$$

where $p_C : \mathcal{FH}^{n(l+1)-d} \times C \to C$ denotes the projection and $k_x$ is the skyscraper sheaf at $x$. Consequently, Claim (8) follows from equations (3), (6) and (9).

Similarly, if we let $\mathcal{M}_j^{nl-d}$ denote the line bundle on $(C \times \mathbb{P}^{n-1})^{nl-d}$ obtained by pulling back $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ via the $j$th projection, we have

$$a_l^* \mathcal{M}_j^{n(l+1)-d} = \begin{cases} \mathcal{O}_{(C\times\mathbb{P}^{n-1})^{nl-d}} & \text{if } 1 \leq j \leq n \\ \mathcal{M}_{j-n}^{nl-d} & \text{if } n+1 \leq j \leq n(l+1)-d. \end{cases}$$

Since the action of the symmetric groups on these motives corresponds to permuting the order of these line bundles, we see that $M(A_l)$ and $M(a_l)$ are both equivariant with respect to $\varphi_l$.

Finally let us prove the commutativity of the square (7). For this we require the explicit formula for the iterated projective bundle isomorphisms given in Lemma 3.7:

$$\mathrm{PB}(\mathcal{L}_\bullet^{nl-d}) = (M(P_{nl-d}) \otimes [c_1(\mathcal{L}_\bullet^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}),$$

where $[c_1(\mathcal{L}_\bullet^{nl-d})] : M(\mathcal{FH}^{nl-d}) \to M(\mathbb{P}^{n-1})^{nl-d}$ is the map induced by powers of the first chern classes of the line bundles $\mathcal{L}_j^{nl-d}$ for $1 \leq j \leq nl-d$. If we insert $n$ copies of the structure sheaf on $\mathcal{FH}^{nl-d}$ into this family, we obtain a map

$$[c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_\bullet^{nl-d})] : M(\mathcal{FH}^{nl-d}) \to M(\mathbb{P}^{n-1})^{n(l+1)-d}.$$

In fact, since $c_1(\mathcal{O})$ is the zero map, we see that $[c_1(\mathcal{O})] : M(\mathcal{FH}^{nl-d}) \to M(\mathbb{P}^{n-1})$ is the composition of the structure map $M(\mathcal{FH}^{nl-d}) \to \mathbb{Q}\{0\}$ with the inclusion $\mathbb{Q}\{0\} \hookrightarrow M(\mathbb{P}^{n-1})$ of any point in $\mathbb{P}^{n-1}$. Therefore, we can write the lower diagonal composition in (7) as

$$M(a_l) \circ \mathrm{PB}(\mathcal{L}_\bullet^{nl-d}) = (M(c_l \circ P_{nl-d}) \otimes [c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_\bullet^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}).$$

Then by (8), we have

$$[c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_\bullet^{nl-d})] = [c_1(\mathcal{L}_\bullet^{n(l+1)-d})] \circ M(A_l)$$

and as diagram (5) commutes, we deduce that

$$\mathrm{PB}(\mathcal{L}_\bullet^{n(l+1)-d}) \circ M(A_l) = (M(c_l \circ P_{nl-d}) \otimes [c_1(\mathcal{O}, \dots, \mathcal{O}, \mathcal{L}_\bullet^{nl-d})]) \circ M(\Delta_{\mathcal{FH}^{nl-d}}),$$

which completes the proof that the square (7) commutes. ☐

Since $a_l : (C \times \mathbb{P}^{n-1})^{nl-d} \to (C \times \mathbb{P}^{n-1})^{n(l+1)-d}$ is equivariant with respect to $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$, we obtain an induced map between the associated symmetric products

$$\begin{tikzcd} (C \times \mathbb{P}^{n-1})^{nl-d} \arrow[r, "a_l"] \arrow[d] & (C \times \mathbb{P}^{n-1})^{n(l+1)-d} \arrow[d] \\ \mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1}) \arrow[r, "\mathrm{Sym}(a_l)"] & \mathrm{Sym}^{n(l+1)-d}(C \times \mathbb{P}^{n-1}). \end{tikzcd}$$

By Theorem 1.3, there is an isomorphism

$$e_l : M(\mathrm{Div}_{n,d}(l)) \cong M(\mathrm{FDiv}_{n,d}(l))^{S_{nl-d}} \cong \mathrm{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1})$$

where the second isomorphism is induced by the $S_{nl-d}$-equivariant isomorphism $\mathrm{PB}(\mathcal{L}_\bullet^{nl-d})$.

**Corollary 4.3.** *The following diagram commutes*

$$\begin{tikzcd} M(\mathrm{Div}_{n,d}(l)) \arrow[r, "M(i_l)"] \arrow[d, "e_l"', "\wr"] & M(\mathrm{Div}_{n,d}(l+1)) \arrow[d, "e_{l+1}", "\wr"'] \\ \mathrm{Sym}^{nl-d} M(C \times \mathbb{P}^{n-1}) \arrow[r, "M(\mathrm{Sym}(a_l))"] & \mathrm{Sym}^{n(l+1)-d} M(C \times \mathbb{P}^{n-1}). \end{tikzcd}$$

*Proof.* By the equivariance property of $M(A_l)$ observed in Proposition 4.2 and the fact that $A_l$ lifts $i_l$, the isomorphisms of Theorem 1.3 fit in a commutative diagram

$$\begin{tikzcd} M(\mathrm{FDiv}_{n,d}(l))^{S_{nl-d}} \arrow[r, "M(A_l)"] \arrow[d, "\wr"'] & M(\mathrm{FDiv}_{n,d}(l+1))^{S_{n(l+1)-d}} \arrow[d, "\wr"'] \\ M(\mathrm{Div}_{n,d}(l)) \arrow[r, "M(i_l)"] & M(\mathrm{Div}_{n,d}(l+1)) \end{tikzcd}$$

The corollary then follows from combining this diagram with the diagram of Proposition 4.2. $\square$

4.2. **A proof of the formula.** Given a smooth commutative connected algebraic group $G$ over $k$, one can associate a motive $M_1(G)$ which, under realisation functors, maps to the first homology group of $G$. This is constructed in [2, Definition 3.3], modeled on the similar construction in [1, Definition 2.1.4]. Note that, in the case where $G$ is an abelian variety, the (Chow) motive $M(G)$ has been studied by Deninger-Murre [15] and Künnemann [29], and their results cover essentially what we need, except the precise connection between the motive of a curve and its Jacobian. This, together with the convenience of quoting results directly in the language of Voevdosky motives, is the reason why we appeal to later references ([1], [31] and [2]) where the results of [15, 29] have been extended to other smooth commutative group schemes.

The rational point $x \in C(k)$ gives rise to a decomposition $M(C) = \mathbb{Q}\{0\} \oplus \overline{M}(C)$, where $\overline{M}(C) = M_1(\mathrm{Jac}(C)) \oplus \mathbb{Q}\{1\}$, see [31, Proposition 2.9]. The motive of $\mathrm{Jac}(C)$ can be recovered from the motive $M_1(\mathrm{Jac}(C))$ (see [2, Theorem 4.3.(i)-(ii)]):

$$M(\mathrm{Jac}(C)) = \bigoplus_{i=0}^{2g} \mathrm{Sym}^i(M_1(\mathrm{Jac}(C))) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M_1(\mathrm{Jac}(C))).$$

We can then write

$$M(C \times \mathbb{P}^{n-1}) = M(C) \otimes \left( \bigoplus_{i=0}^{n-1} \mathbb{Q}\{i\} \right) = \mathbb{Q}\{0\} \oplus \overline{M}(C) \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\}.$$

Let $M_{C,n} := \overline{M}(C) \oplus \bigoplus_{i=1}^{n-1} M(C)\{i\}$; then (for example, by [1, Lemma B.3.1])

$$\mathrm{Sym}^{nl-d}(M(C \times \mathbb{P}^{n-1})) = \mathrm{Sym}^{nl-d}(\mathbb{Q}\{0\} \oplus M_{C,n}) = \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}).$$

**Lemma 4.4.** *There is a commutative diagram*

$$\begin{tikzcd} M(\mathrm{Div}_{n,d}(l)) \arrow[r, "M(i_l)"] \arrow[d, "\wr"] & M(\mathrm{Div}_{n,d}(l+1)) \arrow[d, "\wr"'] \\ \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}) \arrow[r] & \bigoplus_{i=0}^{n(l+1)-d} \mathrm{Sym}^i(M_{C,n}) \end{tikzcd}$$

*where the lower map is the obvious inclusion.*

*Proof.* Let us start with the description of the transition map given in Corollary 4.3. We see that the map $a_l : (C \times \mathbb{P}^{n-1})^{nl-d} \to (C \times \mathbb{P}^{n-1})^{n(l+1)-d}$ can be described motivically as

$$M(a_l) : M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \cong \mathbb{Q}\{0\}^{\otimes n} \otimes M(C \times \mathbb{P}^{n-1})^{\otimes (nl-d)} \xrightarrow{\iota^{\otimes n} \otimes M(\mathrm{id})} M(C \times \mathbb{P}^{n-1})^{\otimes n(l+1)-d}$$

where $\iota : \mathbb{Q}\{0\} \to M(C \times \mathbb{P}^{n-1}) = \mathbb{Q}\{0\} \oplus M_{C,n}$ is the natural inclusion of this direct factor. It thus follows that the symmetrised map $M(\mathrm{Sym}(a_l))$ is the claimed inclusion. $\square$

**Theorem 4.5.** *If $C(k) \neq \emptyset$, then the motive of* $\mathrm{Bun}_{n,d}$ *satisfies*

$$M(\mathrm{Bun}_{n,d}) \simeq \operatorname*{hocolim}_{l} \left( \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{C,n}) \right) \simeq \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M_{C,n}).$$

*More precisely, we have*

$$M(\mathrm{Bun}_{n,d}) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}).$$

*Proof.* The first claim follows from Lemma 4.4 and Theorem 1.2. For the second claim, we introduce the notation $\mathrm{Sym}^*(M) := \oplus_{i=0}^{\infty} \mathrm{Sym}^i(M)$ for any motive $M$; then

(i) $\mathrm{Sym}^*(M_1 \oplus M_2) = \mathrm{Sym}^*(M_1) \otimes \mathrm{Sym}^*(M_2)$ (by [1, Lemma B.3.1]),
(ii) $Z(C, \mathbb{Q}\{i\}) = \mathrm{Sym}^*(M(C)\{i\})$ (by definition of motivic zeta values),
(iii) $\mathrm{Sym}^*(\mathbb{Q}\{1\}) = M(B\mathbb{G}_m)$ (see [26, Example 2.21] based on [37, Lemma 8.7]),
(iv) $\mathrm{Sym}^*(M_1(\mathrm{Jac}(C))) = M(\mathrm{Jac}(C))$ (by [1, Proposition 4.3.5]),

and the formula follows from these observations. □

4.3. **An alternative proof using previous results.** We will give a second proof of this formula for $M(\mathrm{Bun}_{n,d})$, also based on Corollary 4.3 but which follows more closely our previous work [26]. The idea is to describe the unsymmetrised transition maps $M(a_l)$ by decomposing the motives $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ using $M(\mathbb{P}^{n-1}) = \oplus_{i=0}^{n-1} \mathbb{Q}\{i\}$.

**Remark 4.6.** By returning to the decomposition $M(\mathbb{P}^{n-1}) = \oplus_{i=0}^{n-1} \mathbb{Q}\{i\}$, we can describe the maps $M(a_l)$ explicitly. Indeed we have a decomposition $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ indexed by ordered tuples $I = (i_1, \cdots, i_{nl-d}) \in \mathcal{I}_l := \{0, \cdots, n-1\}^{\times\, nl-d}$ of the form

$$M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} = \bigoplus_{I \in \mathcal{I}_l} \bigotimes_{j=1}^{nl-d} M(C)\{i_j\} = \bigoplus_{I \in \mathcal{I}_l} M(C^{nl-d})\{|I|\},$$

where $|I| = \sum_{j=1}^{nl-d} i_j$.

There is a map $h_l : \mathcal{I}_l \to \mathcal{I}_{l+1}$ given by $I \mapsto (0, \ldots, 0, I)$ (inserting $n$ zeros) such that the map $M(a_l) : M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \to M(C \times \mathbb{P}^{n-1})^{\otimes n(l+1)-d}$ sends the direct summand indexed by $I \in \mathcal{I}_l$ to the direct summand indexed by the tuple $h_l(I) \in \mathcal{I}_{l+1}$ via the map

$$M(c_l)\{|I|\} : M(C^{nl-d})\{|I|\} \to M(C^{n(l+1)-d})\{|I|\} = M(C^{n(l+1)-d})\{|(0, \ldots, 0, I)|\}. \tag{10}$$

The $S_{nl-d}$-action on $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ permutes these direct summands via the obvious action of $S_{nl-d}$ on $\mathcal{I}_l$. By Example 2.12, the invariant part is the motive of $\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})$ which has an associated decomposition. The index set for this decomposition is

$$\mathcal{B}_l := \left\{ m = (m_0, \ldots, m_{n-1}) \in \mathbb{N}^n : \sum_{i=0}^{n-1} m_i = nl - d \right\}.$$

Moreover, for $I \in \mathcal{I}_l$, we let $\tau_l(I)_r = \#\{i_j : i_j = r\}$, then $\tau_l(I) = (\tau_l(I)_0, \ldots, \tau_l(I)_{n-1}) \in \mathcal{B}_l$ and the map $\tau_l : \mathcal{I}_l \to \mathcal{B}_l$ is $S_{nl-d}$-invariant with $|I| = \sum_{i=0}^{n-1} i\tau_l(I)_i$. By grouping together the factors with the same values of $i_j$, there is a map

$$C^{nl-d} \to \prod_{i=0}^{n-1} \mathrm{Sym}^{\tau_l(I)_i}(C) \tag{11}$$

which is the quotient of the natural action of $\mathrm{Stab}(I) \cong \prod_{i=0}^{n-1} S_{\tau_l(I)_i}$.

**Lemma 4.7.** *For each $l$, we have a decomposition*

$$M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})) = \bigoplus_{m \in \mathcal{B}_l} \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\}$$

*such that the following statements hold.*

(i) *For each $m \in \mathcal{B}_l$, we have a commutative diagram*

$$\begin{tikzcd} M(C \times \mathbb{P}^{n-1})^{\otimes nl-d} \arrow[r] \arrow[d] & M(\mathrm{Sym}^{nl-d}(C \times \mathbb{P}^{n-1})) \arrow[d] \\ \bigoplus_{I \in \tau_l^{-1}(m)} M(C^{nl-d})\{|I|\} \arrow[r] & \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\} \end{tikzcd}$$

*where the lower maps are induced by the maps* (11).

(ii) *The transition maps $M(\mathrm{Sym}(a_l))$ in Corollary 4.3 decompose as maps*

$$\kappa_{m,m'} : \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\} \to \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m'_i}(M(C))\{im'_i\}$$

*for $m \in \mathcal{B}_l$ and $m' \in \mathcal{B}_{l+1}$ with $\kappa_{m,m'} = 0$ unless $m' = m + (n, 0, \dots 0)$, in which case this map is induced by the morphism of varieties*

$$\prod_{i=0}^{n-1} \mathrm{Sym}^{m_i}(C) \to \prod_{i=0}^{n-1} \mathrm{Sym}^{m'_i}(C)$$

*which is the map $\mathrm{Sym}(s_x^n \times \mathrm{id}_{C^{m_i}})$ on the $0$th factor and the identity on all other factors*[5].

*Proof.* We will give the decomposition and the proof of (i) simultaneously, by collecting the direct summands in the decomposition of $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ which are preserved by the $S_{nl-d}$-action and taking their invariant parts. For this, we recall that there is a $S_{nl-d}$-action on $\mathcal{I}_l$ and the map $\tau_l : \mathcal{I}_l \to \mathcal{B}_l$ is $S_{nl-d}$-invariant and the fibres consist of single orbits. For $I \in \mathcal{I}_l$ with $m = \tau_l(I)$, we note that the quotient of the associated action of $\mathrm{Stab}(I) = \prod_{i=0}^{n-1} S_{m_i}$ on $C^{nl-d}$ is isomorphic to $\prod_{i=0}^{n-1} \mathrm{Sym}^{m_i}(C)$. Therefore, the motive appearing in the left lower corner of the diagram in statement (i) is a direct summand of $M(C \times \mathbb{P}^{n-1})^{\otimes nl-d}$ that is preserved by the $S_{nl-d}$-action and its $S_{nl-d}$-invariant piece is precisely the motive appearing in the lower right corner. This proves the first statement and the decomposition.

To describe the behaviour of the symmetrised transition maps with respect to this decomposition, we recall that the unsymmetrised transition maps send the direct summand indexed by $I \in \mathcal{I}_l$ to $h_l(I) = (0, ...0, I) \in \mathcal{I}_{l+1}$. The unsymmetrised transition maps on these direct summands are described by (10) and so it remains to describe the induced map on the invariant parts for the actions of the symmetric groups. Since $h_l : \mathcal{I}_l \to \mathcal{I}_{l+1}$ is equivariant for the actions of the symmetric groups via the homomorphism $\varphi_l : S_{nl-d} \hookrightarrow S_{n(l+1)-d}$, it descends to map

$$\overline{h} : \mathcal{B}_l \to \mathcal{B}_{l+1} \quad \text{where} \quad \overline{h}(m) = m + (n, 0, \dots, 0).$$

Thus, $\kappa_{m,m'}$ is zero unless $m' = \overline{h}(m)$. For $m' = \overline{h}(m)$, $I \in \tau_l^{-1}(m)$ and $I' \in \tau_{l+1}^{-1}(m')$ note that

$$|I| = |I'| = \sum_{i=0}^{n-1} im_i = \sum_{i=0}^{n-1} im'_i$$

and

$$\mathrm{Stab}(I) = \prod_{i=0}^{n-1} S_{m_i} \quad \text{and} \quad \mathrm{Stab}(I') = \prod_{i=0}^{n-1} S_{m'_i} = S_{m_0+n} \times \prod_{i=1}^{n-1} S_{m_i}.$$

[5]We recall that $s_x : \mathrm{Spec}\, k \to C$ denotes the inclusion of the point $x \in C$.

In particular, the map $c_l = s_x^n \times \mathrm{id} : C^{nl-d} \to C^{n(l+1)-d}$ is equivariant for the induced actions of $\mathrm{Stab}(I)$ and $\mathrm{Stab}(I')$ and there is a map between the quotients

$$\begin{tikzcd} C^{nl-d} \arrow[r, "c_l"] \arrow[d] & C^{n(l+1)-d} \arrow[d] \\ \prod_{i=0}^{n-1} \mathrm{Sym}^{m_i}(C) \arrow[r] & \prod_{i=0}^{n-1} \mathrm{Sym}^{m'_i}(C) \end{tikzcd}$$

which is $\mathrm{Sym}(s_x^n \times \mathrm{id}_{C^{m_i}})$ on the 0th factor and the identity on the other factors. Combined with (i), this concludes the proof of (ii). □

**Corollary 4.8.** *The transition maps $M(i_l) : M(\mathrm{Div}_{n,d}(l)) \to M(\mathrm{Div}_{n,d}(l+1))$ fit in the following commutative diagram*

$$\begin{tikzcd} M(\mathrm{Div}_{n,d}(l)) \arrow[r, "M(i_l)"] \arrow[d, "\wr"] & M(\mathrm{Div}_{n,d}(l+1)) \arrow[d, "\wr"'] \\ \bigoplus_{m\in\mathcal{B}_l} \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m_i}(M(C))\{im_i\} \arrow[r, "\bigoplus_{m,m'} \kappa_{m,m'}"] & \bigoplus_{m'\in\mathcal{B}_{l+1}} \bigotimes_{i=0}^{n-1} \mathrm{Sym}^{m'_i}(M(C))\{im'_i\}. \end{tikzcd}$$

*where the maps $\kappa_{m,m'}$ are as in Lemma 4.7.*

*Proof.* This follows from Lemma 4.7 and Corollary 4.3. □

This looks very similar to [26, Conjecture 3.9], except we do not know whether the vertical maps in this commutative diagram coincide with the maps given by the Białynicki-Birula decompositions used in the formulation of this conjecture. Nevertheless, with the description of the transition maps in Corollary 4.8, one can apply the proof of [26, Theorem 3.18] to obtain an alternative proof of the formula for the motive of $\mathrm{Bun}_{n,d}$ appearing in Theorem 1.1.

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